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A Class of Nonlinear Elasticity Problems with No Local but Many Global Minimizers

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Dedicated to 85th birthday of R. Fosdick, a teacher and a friend.

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Abstract
We present a class of models of elastic phase transitions with incompatible energy wells in an arbitrary space dimension, where in a hard device an abundance of Lipschitz global minimizers coexists with a complete lack of strong local minimizers. The analysis is based on the proof that every strong local minimizer in a hard device is also a global minimizer which is applicable much beyond the chosen class of models. Along the way we show that a new demonstration of sufficiency for a subclass of affine boundary conditions can be built around a novel nonlinear generalization of the classical Clapeyron theorem.

Keywords Nonlinear elasticity · Elastic stability · Elastic phase transitions · Jump set · Binodal · Laminates · Local minimizers · Metastability · Hysteresis

Mathematics Subject Classification 74A50 · 74G65 · 49K40 · 49S05

1 Introduction

The phenomenon of *metastability* in elastostatics, manifesting itself through the existence of strong local minimizers that are not global,[1] is usually associated with Neumann boundary conditions (soft device) and linked to the incompatibility of the energy wells [5]. In Dirichlet problem (hard device) the situation is different and the incompatibility of the energy wells is not sufficient for metastability. To corroborate this statement we present here a class of hyperelastic materials with incompatible energy wells for which one can prove the absence of metastability for any Dirichlet boundary conditions (hard device).

[1] In physics literature the term metastability usually refers to the presence of configurations that are stable only under sufficiently small perturbations. Here we specify the notion of "small" and interpret strong local minimizers as configurations minimizing among competitors with deformation fields that are uniformly close. A more conventional weak local minimizers are configurations minimizing among the competitors with not only deformations but also deformation gradients that are uniformly close.

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More specifically, we prove for the associated class of vectorial variational problems the absence of strong local minimizers which are not global on any domain and in any number of dimensions. For a subclass of such problems, amenable to an especially transparent analytical treatment, we exhibit a concurrent nonuniqueness of global minimizers. While the simplest elasticity problem, exhibiting in a hard device both the abundance of global minimizers and the lack of strong local minimizers, is a scalar problem in one dimension [15], the question whether the effect survives in a multidimensional vectorial setting has been so far open. The general fact that the elastic energy relaxation can in principle produce non-uniqueness of optimal microstructures is well-known [22, 38], however, whether such non-uniqueness extends to strong local minimizers is usually difficult to check.

We recall that to ensure uniqueness in nonlinear elasticity, the use of Dirichlet boundary conditions and the presence of topological, or even geometric simplicity of the domain, are essential [50, 59, 61]. Uniqueness has been established for star-shaped domains, affine displacement boundary conditions, and strictly quasiconvex stored energy functions [36, 60]. It was also understood that whether uniqueness holds may depend on the regularity class in which one looks for a minimizer [37], as well as its integrability class [4, 29, 57]. Instead, for mixed, Dirichlet-Neumann boundary-value problems of nonlinear elasticity, nonuniqueness has been found ubiquitous with the most familiar examples being those associated with buckling and related to the emergence of multiple symmetry-related energy minima [18], but not only, [23]. The possibility of nonuniqueness with purely Dirichlet boundary conditions is usually associated with quenched inhomogeneity as in problems with residual stresses or non-zero body tractions [14, 42, 63].

The interest of a simple example considered in this paper stems from the fact that it exhibits multiplicity of global minimizers in a hard device problem in the absence of any geometrical complexity of the domain and does not require quenched inhomogeneity. Furthermore, the obtained non-uniqueness is not due to the scalar nature of the problem (absence of compatibility constraint) and is unrelated to symmetry related degeneracy as in the case of Euler buckling. Instead, it is evocative of nonuniqueness in the problems of microstructure optimization in composites [3, 22, 43, 47].

In this paper we limit our attention to the class of "geometrically linearized" Hadamard materials which can be viewed as simplified analogs of the fully nonlinear Hadamard materials [27, 31]. The energy density functions of the latter are of the form

$$W(\boldsymbol{F}) = \mu|\boldsymbol{F}|^2 + h(\det \boldsymbol{F}). \tag{1.1}$$

Here $\boldsymbol{F}$ is the deformation gradient and μ is the measure of rigidity. The non-negative function $h(d)$ is defined on $(0, +\infty)$ and has the property that $h(d) \to \infty$, as $d \to 0^+$. If we now formally use for the materials (1.1) the "geometric" approximation $\det \boldsymbol{F} \approx 1 + \mathrm{Tr}\,(\boldsymbol{F} - \boldsymbol{I})$, valid in the limit $\boldsymbol{F} \to \boldsymbol{I}$, a formal asymptotic expansion with respect to a small parameter would necessarily also imply the "physical" linearization which trivializes the problem. Since our goal is to retain and exploit physical non-linearity, we consider instead the energy

$$W(\boldsymbol{H}) = g(\mathrm{Tr}\,\boldsymbol{H}) + \mu|\boldsymbol{H}|^2, \qquad \boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}. \tag{1.2}$$

which is formally unrelated to (1.1). However, following [1, 2, 10, 32, 33], we can perform geometric linearization in the second term in (1.2), replacing $\boldsymbol{H}$ with its symmetric part $\boldsymbol{\varepsilon} = (\boldsymbol{H} + \boldsymbol{H}^t)/2$ and, to emphasize the isotropic nature of the resulting energy density, write the result in the form

$$W_0(\boldsymbol{H}) = f(\mathrm{Tr}\,\boldsymbol{\varepsilon}) + \mu|\mathrm{dev}(\boldsymbol{\varepsilon})|^2, \qquad \mathrm{dev}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} - \frac{1}{d}\mathrm{Tr}\,(\boldsymbol{\varepsilon})\boldsymbol{I}. \tag{1.3}$$

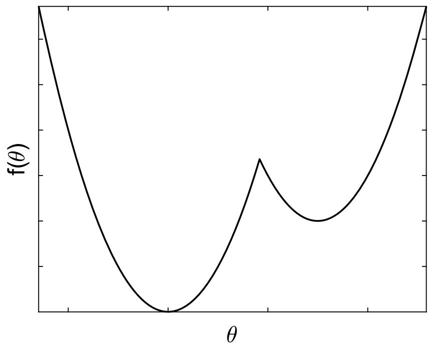

Fig. 1 Double-well nonlinearity in a geometrically linear bi-quadratic Hadamard material

Despite the achieved formal simplification, the mathematical features of interest are similar in the equilibrium problems for materials with the energies (1.2) and (1.3). We will therefore, focus on (1.3), knowing that it also preserves at least the linearized version of the frame indifference property.

We assume that $\mu > 0$ and choose the function f to be nonconvex and have a "double-well" shape. For additional analytical transparency we will use in our explicit constructions the bi-quadratic potential

$$f(\theta) = \min\{\kappa_0\theta^2, \kappa_0(\theta - \theta_p)^2 + f_0\}, \tag{1.4}$$

illustrated in Fig. 1. The ensuing model describes a material capable of undergoing a purely dilatational phase transformation between elastic phases which are both linearly elastic and have the same elastic moduli but different reference (chemical) energies. The relaxation in such elasticity problems is nontrivial and historically has been viewed as a benchmark case for various mathematical theories of elastic phase transitions [33, 38, 49].

If the energy wells are rank-one connected, the quasiconvexification of a non-convex energy can be expected to coincide with rank-one convexification and in many cases even with convexification. Contrariwise, the incompatibility of the wells can be responsible for a nontrivial structure of the quasiconvex envelope. In this respect the material (1.2) is of interest because, even though the non-convexity enters (1.3) only through a scalar potential, the energy wells remain not rank-one connected. This means, for instance, that even if the energy wells in (1.2) are both at zero energy, not only the relaxed energy is nonzero but also the optimal microstructures are nontrivial.

The energy density (1.3) is non rank-one convex for all $\mu > 0$, as long as $f(\theta)$ is given by (1.4) [38]. Materials with such energies, loaded in a soft device, are expected to exhibit constitutive hysteresis, since in this case, due to a weaker constraint imposed by the Neumann boundary conditions, a broad class of strong local minima should exist [7, 30, 51–53, 56, 67]. Moreover, under the soft device loading the class of global minimizers can be expected to be very narrow as homogeneous configurations would outperform all other competitors. There have been many attempts to solve the soft device problem in physics literature, see for instance [11] and the references cited therein, however the rigorous mathematical analysis is still lacking. Setting this problem temporarily aside, we focus in this paper on the hard device loading and show that in this case the strong local minima in the model (1.3) are absent, the quasistatic mechanical response is defined uniquely, and the deformation path is reversible as it corresponds to global minimization of the total energy at each value of the loading parameter. The associated stress-strain relation can be obtained from the knowledge of the quasiconvex envelope of the energy (1.3); the latter can be characterized explicitly in the whole range of parameters for all double-well shaped potentials[2] $f(\theta)$. Here, to make the paper self-contained, we compute the relaxation of the energy (1.3) using simple laminates as the optimal microstructure; a different proof of the same result, based on matching of upper and lower bounds, was previously given in [26].

The most important general (not energy-specific) result obtained in this paper is the formulation of the necessary and sufficient conditions for global minimizers in star-shaped domains with affine boundary conditions. It's proof contains new tools for characterizing energy minimizing configurations in the problems where the energy relaxation, or at least the elastic binodal, is explicitly known. An important technical lemma in the associated analysis is a fundamental nonlinear generalization of the classical Clapeyron theorem[3] which links the energy of the equilibrium configuration with the work of the (generalized) forces on the boundary of the body. Combining Clapeyron's theorem with affine boundary conditions and local material stability (quasiconvexity) we obtain the inequality establishing global minimality of all stationary locally stable configurations in star-shaped domains. These results may be viewed, in a more general perspective, as an extension of optimality conditions for extremal microstructures in composites (e.g. [22]).

To demonstrate the utility of the obtained necessary and sufficient conditions we show that one can reduce the problem of finding a global minimizer for the energy (1.3) in a finite domain, to the solution of a nonlinear free boundary problem involving a system of linear PDEs. It is remarkable that the solution of such a general problem can be found explicitly for any double-well scalar potential $f(\theta)$. The degeneracy in this problem, explained by the fact that the optimality conditions do not place very stringent constraints on the displacement gradient in an optimal configuration, is behind the multiplicity of global minimizers. We show however, in Appendix B, that topologically simple energy minimizers which maintain the symmetry of the domain may fail to exist in some domains with corners.

Our analysis of optimal microstructures suggests that, despite the unavailability of the constitutive hysteresis in the considered class of problems, the direct and the reverse transformation may follow morphologically different paths in the real physical space. Therefore, despite the absence of metastability, one may expect "morphological hysteresis", whereby the loading and unloading occur along different morphological paths. The fact, that along such paths the system traverses distinct but energetically equivalent configurations, may be of interest in applications if one could design a way of biasing one of the paths through, say, imposing additional control on the gradients.

The paper is organized as follows. In Sect. 2 we compute the quasiconvexification of the energy density (1.3), and use the result to prove the absence of strong local minimizers that are not global in any domain and for any Dirichlet type boundary conditions. The necessary and sufficient conditions for global minimizers are formulated in Sect. 3. The issue of attainment and the multiplicity of global minimizers in our model are discussed in Sect. 4. The last Sect. 5 summarizes the results and presents conclusions. Our Appendix A contains a technical discussion of the generic loss of strict ellipticity of the rank one convexification of an energy density. In Appendix B we show the non-existence of square-symmetric simply connected inclusion minimizing the energy in the square.

[2] In fact, the quasiconvex envelope of the energy can be characterized for all continuous potentials $f(\theta)$ that are bounded from below.

[3] The necessity of Clapeyron theorem-type formula in the study of uniqueness in the Dirichlet setting was first realized by Knops and Stuart in [36].

2 Local Minimizers

Given that the case of interest involves hard device loading, we study strong local and global minima of the functional

$$\mathcal{E}_0(\boldsymbol{u}) = \int_{\Omega} \left\{ f(\nabla \cdot \boldsymbol{u}) + \mu \left| e(\boldsymbol{u}) - \frac{1}{d}(\nabla \cdot \boldsymbol{u})\boldsymbol{I} \right|^2 \right\} d\boldsymbol{x} \tag{2.1}$$

among all Lipschitz displacement vector fields $\boldsymbol{u} \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ subject to the constraint

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega. \tag{2.2}$$

Here $\boldsymbol{u}_0(\boldsymbol{x})$ is a given Lipschitz function on $\partial\Omega$, and $\Omega \subset \mathbb{R}^d$ denotes a Lipschitz domain. Restricting attention to Lipschitz minimizers allows us to focus on instabilities caused by failure of quasiconvexity [12, 48] and exclude some other instabilities related, for instance, to the mismatch between the integrability of the minimizing sequences and the growth of the energy density at infinity [6].

First we recall that one of the necessary conditions of strong local minimality is stability with respect to nucleation of a coherent precipitate of the new phase in the interior of the old phase. That means that we must have $\nabla \boldsymbol{u}(\boldsymbol{x}) \in \boldsymbol{\mathcal{A}}$, for a.e. $\boldsymbol{x} \in \Omega$, [61, Proposition 4.1], where

$$\boldsymbol{\mathcal{A}} = \{\boldsymbol{H} \in \mathbb{M} : W_0(\boldsymbol{H}) = QW_0(\boldsymbol{H})\}, \tag{2.3}$$

and where $\mathbb{M}$ denotes the set of all $d \times d$ matrices and $QW_0(\boldsymbol{H})$ is the quasiconvexification of W_0 [12].

In this paper we show that if $\boldsymbol{u}(\boldsymbol{x})$ is a weak solution of the Euler-Lagrange equation for the energy (2.1) with the boundary conditions (2.2) and satisfies $\nabla \boldsymbol{u}(\boldsymbol{x}) \in \boldsymbol{\mathcal{A}}$ for a.e. $\boldsymbol{x} \in \Omega$ then it is necessarily a global minimizer for $\mathcal{E}_0(\boldsymbol{u})$.

Before we move to the actual analysis, it is appropriate to mention that we are not aware of any examples of proper metastable states in a hard device in domains with trivial topology. Moreover it has been proved for general energies that every strong local minimizer is global in the hard device with affine Dirichlet boundary conditions in star-shaped domains [60]. In our special example, the same result turns out to be true for all Dirichlet boundary conditions in domains with piecewise smooth boundaries and arbitrary topology.

The first step of the analysis is to characterize the set $\boldsymbol{\mathcal{A}}$ of admissible displacement gradients for the chosen material model. It can result from the computation of $QW_0(\boldsymbol{H})$ which in our case can be done explicitly.

Theorem 2.1 *Suppose that $W_0(\boldsymbol{H})$ is given by (1.3). Then,*

$$QW_0(\boldsymbol{H}) = \Phi^{**}(\mathrm{Tr}\,\boldsymbol{H}) + \frac{\mu}{4}|\boldsymbol{H} - \boldsymbol{H}^t|^2 - 2\mu J_2(\boldsymbol{H}), \tag{2.4}$$

where

$$\Phi(\theta) = f(\theta) + \frac{d-1}{d}\mu\theta^2, \tag{2.5}$$

$$2J_2(\boldsymbol{H}) = (\mathrm{Tr}\,\boldsymbol{H})^2 - \mathrm{Tr}\,(\boldsymbol{H}^2) = 2\sum_{i<j} \begin{vmatrix} h_{ii} & h_{ij} \\ h_{ji} & h_{jj} \end{vmatrix} \tag{2.6}$$

is a quadratic null-Lagrangian, and Φ^{**} denotes the convexification of $\Phi(\theta)$.

Proof It will be convenient to start with the well known definition of quasiconvexification of an arbitrary energy density function $W(\boldsymbol{H})$ using periodic functions [12]:

$$QW(\boldsymbol{H})=\inf_{\boldsymbol{\phi}-Q\text{-periodic}}\fint_Q W(\boldsymbol{H}+\nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x}, \tag{2.7}$$

where $Q=[0,1]^d$. The computation of QW_0 is based on the following formula

$$\left|e(\boldsymbol{u})-\frac{1}{d}(\nabla\cdot\boldsymbol{u})\boldsymbol{I}\right|^2=\frac{d-1}{d}(\nabla\cdot\boldsymbol{u})^2+\frac{1}{4}|\nabla\boldsymbol{u}-(\nabla\boldsymbol{u})^t|^2-2J_2(\nabla\boldsymbol{u}). \tag{2.8}$$

Let $\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{H}\boldsymbol{x}+\boldsymbol{\phi}(\boldsymbol{x})$. Then

$$\fint_Q W_0(\nabla\boldsymbol{u})d\boldsymbol{x}=-2\mu J_2(\boldsymbol{H})+\fint_Q\left(\Phi(\nabla\cdot\boldsymbol{u})+\frac{\mu}{4}|\nabla\boldsymbol{u}-(\nabla\boldsymbol{u})^t|^2\right)d\boldsymbol{x}. \tag{2.9}$$

Since $\Phi^{**}(\theta)\le\Phi(\theta)$ for all $\theta\in\mathbb{R}$, we obtain the inequality

$$\fint_Q W_0(\nabla\boldsymbol{u})d\boldsymbol{x}\ge-2\mu J_2(\boldsymbol{H})+\fint_Q\left(\Phi^{**}(\nabla\cdot\boldsymbol{u})+\frac{\mu}{2}|\nabla\boldsymbol{u}-(\nabla\boldsymbol{u})^t|^2\right)d\boldsymbol{x}.$$

Now applying the Jensen's inequality we get:

$$\fint_Q W_0(\nabla\boldsymbol{u})d\boldsymbol{x}\ge-2\mu J_2(\boldsymbol{H})+\Phi^{**}(\mathrm{Tr}\,\boldsymbol{H})+\frac{\mu}{4}|\boldsymbol{H}-\boldsymbol{H}^t|^2. \tag{2.10}$$

Therefore,

$$QW_0(\boldsymbol{H})\ge-2\mu J_2(\boldsymbol{H})+\Phi^{**}(\mathrm{Tr}\,\boldsymbol{H})+\frac{\mu}{4}|\boldsymbol{H}-\boldsymbol{H}^t|^2. \tag{2.11}$$

In order to prove equality, we need to exhibit a periodic function $\boldsymbol{\phi}$ such that $\nabla\boldsymbol{\phi}=(\nabla\boldsymbol{\phi})^t$ and

$$\fint_Q \Phi(\nabla\cdot\boldsymbol{\phi}(z)+\mathrm{Tr}\,\boldsymbol{H})dz=\Phi^{**}(\mathrm{Tr}\,\boldsymbol{H}). \tag{2.12}$$

If $\Phi(\theta)$, given by (2.5) is convex,[4] then the original energy $W_0(\boldsymbol{H})$ is quasiconvex (polyconvex, actually), and formula (2.4) holds. The case of interest is therefore when $\Phi(\theta)$ retains the double-well shape, in which case $\Phi^{**}(\theta)=\Phi(\theta)$ if and only if $\theta\notin(\theta_1,\theta_2)$, as in Fig. 2. For example, for the potential (1.4) we find

$$\theta_1=\frac{1}{2}\left(\frac{f_0}{\kappa_0\theta_p}+\frac{(d-1)\mu\theta_p}{d\kappa_0+(d-1)\mu}\right),\qquad\theta_2=\theta_p+\frac{1}{2}\left(\frac{f_0}{\kappa_0\theta_p}-\frac{(d-1)\mu\theta_p}{d\kappa_0+(d-1)\mu}\right). \tag{2.13}$$

Thus, if $\mathrm{Tr}\,\boldsymbol{H}\notin(\theta_1,\theta_2)$, then $\boldsymbol{\phi}=\boldsymbol{0}$ attains equality in (2.10). If $\mathrm{Tr}\,\boldsymbol{H}\in(\theta_1,\theta_2)$, then there exists $\omega\in(0,1)$ such that

$$\mathrm{Tr}\,\boldsymbol{H}=\omega\theta_1+(1-\omega)\theta_2. \tag{2.14}$$

[4] This is never the case for the potential (1.4).

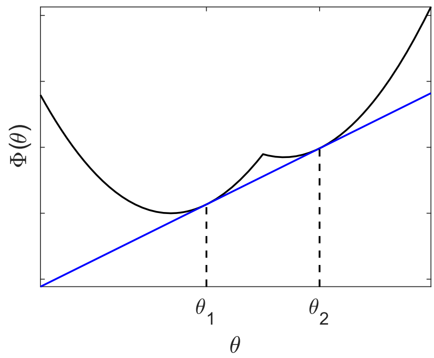

Fig. 2 Definition of $\theta_1 < \theta_2$ via the common tangent construction

In order to attain equality in (2.10) we need to split the period cell $Q = [0, 1]^d$ into two subsets A_1, of volume ω and A_2 of volume $1 - \omega$ such that

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = \theta_1 \chi_{A_1}(\boldsymbol{x}) + \theta_2 \chi_{A_2}(\boldsymbol{x}) \tag{2.15}$$

for all $\boldsymbol{x} \in Q$. This is easily achieved with a laminate construction with $A_1 = [0, \omega] \times [0, 1]^2$ and $A_2 = [1 - \omega, 1] \times [0, 1]^2$ and function $\nabla \boldsymbol{\phi}$ being symmetric and piecewise constant:

$$\nabla \boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{M}_1 \chi_{A_1}(\boldsymbol{x}) + \boldsymbol{M}_2 \chi_{A_2}(\boldsymbol{x}).$$

The symmetric matrices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are easily found from the periodicity of $\boldsymbol{\phi}(\boldsymbol{x})$, that is equivalent to $⨍_Q \nabla \boldsymbol{\phi}(\boldsymbol{x}) dx = \boldsymbol{0}$, and (2.15):

$$\boldsymbol{M}_1 = (1 - \omega)[\![\theta]\!] \boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \qquad \boldsymbol{M}_2 = -\omega [\![\theta]\!] \boldsymbol{e}_1 \otimes \boldsymbol{e}_1,$$

where $[\![\theta]\!] = \theta_1 - \theta_2$. These choices guarantee equality in (2.12). □

We remark that the set A_1 in the proof above can be completely arbitrary, aside from the volume constraint $|A_1| = \omega$. The corresponding periodic function $\boldsymbol{\phi}$ is then given in terms of its Fourier coefficients $\hat{\boldsymbol{\phi}}(\mathbf{k}) = [\![\theta]\!] \hat{\chi}_{A_1}(\mathbf{k}) \mathbf{k} / (2\pi i |\mathbf{k}|)$, if $\mathbf{k} \neq 0$, and $\hat{\boldsymbol{\phi}}(0) = 0$. The absence of any restrictions on the shape and location of the phases in this context is often referred to as Bitter-Crum theorem [17, 20, 38, 41]. For the case when $f(\theta)$ is a minimum of two quadratic functions the obtained expression for the relaxed energy can be deduced from the more general results of Khachaturyan [34, 54, 55, 64], Pipkin [49], and Kohn [38]. Our proof, however, which applies to arbitrary nonlinear choices of $f(\theta)$, is different from those presented before as it takes full advantage of the specifics of purely volumetric phase transitions. It relies on the universal extraction of the null Lagrangian part of the energy which is characteristic for the whole class of considered problems. The ideas behind our approach come from the very heart of classical calculus of variations [66]. A similar technique, called the "translation method" is also broadly used to obtain optimal bounds on effecitve moduli in the theory of composites [46].

Corollary 2.2

$$\boldsymbol{\mathcal{A}} = \{\boldsymbol{H} \in \mathbb{M} : \operatorname{Tr} \boldsymbol{H} \notin (\theta_1, \theta_2)\}. \tag{2.16}$$

The constructed energy density $QW_0(\boldsymbol{H})$ is obviously polyconvex, and coincides with $RW_0(\boldsymbol{H})$, the rank-1 convexification of $W_0(\boldsymbol{H})$. Indeed, as we see from the proof of Theorem 2.1, the infimum in (2.7) can be achieved by a simple laminate. In view of the simplicity of the optimal microstructures, one can *post factum* conclude that computing $QW_0(\boldsymbol{H})$ explicitly was not really necessary for the characterization of the set $\boldsymbol{\mathcal{A}}$. We could also delineate it by using inequality (2.10) and relying on the knowledge and properties of the jump set introduced in [24, 25]. Indeed, inequality (2.10) implies that all $\boldsymbol{H}$ satisfying $\Phi^{**}(\operatorname{Tr}\boldsymbol{H}) = \Phi(\operatorname{Tr}\boldsymbol{H})$ are admissible, i.e. $\{\boldsymbol{H} \in \mathbb{M} : \operatorname{Tr}\boldsymbol{H} \notin (\theta_1, \theta_2)\} \subset \boldsymbol{\mathcal{A}}$. We can then show that all $\boldsymbol{H}$ with $\theta_1 < \operatorname{Tr}\boldsymbol{H} < \theta_2$ fail rank-one convexity by looking for pairs $\boldsymbol{H}_\pm$ of the displacement gradient values that can occur on the two sides of a smooth jump discontinuity representing a phase boundary. We recall, [24], that these matrices satisfy the equations

$$[\![\boldsymbol{H}]\!] = \boldsymbol{a} \otimes \boldsymbol{n}, \quad [\![W_{\boldsymbol{H}}]\!]\boldsymbol{n} = \boldsymbol{0}, \quad [\![W_{\boldsymbol{H}}]\!]^t\boldsymbol{a} = \boldsymbol{0}, \quad [\![W]\!] = \langle\{W_{\boldsymbol{H}}\}, [\![\boldsymbol{H}]\!]\rangle, \tag{2.17}$$

where

$$[\![\boldsymbol{H}]\!] = \boldsymbol{H}_+ - \boldsymbol{H}_-, \qquad \{W_{\boldsymbol{H}}\} = \frac{1}{2}(W_{\boldsymbol{H}}(\boldsymbol{H}_+) + W_{\boldsymbol{H}}(\boldsymbol{H}_-)).$$

We have shown in [26] that the jump set for the energy (1.3) consists of the union of two hyperplanes

$$\mathfrak{J}_- = \{\boldsymbol{H} \in \mathbb{M} : \operatorname{Tr}\boldsymbol{H} = \theta_1\}, \qquad \mathfrak{J}_+ = \{\boldsymbol{H} \in \mathbb{M} : \operatorname{Tr}\boldsymbol{H} = \theta_2\},$$

so that if $\operatorname{Tr}\boldsymbol{H}_- = \theta_1$, then the set of corresponding $\boldsymbol{H}_+$ is a projective plane worth of points

$$\boldsymbol{H}_+ = \boldsymbol{H}_- + (\theta_2 - \theta_1)\boldsymbol{n} \otimes \boldsymbol{n}, \quad |\boldsymbol{n}| = 1.$$

As we have already remarked, inequality (2.10) implies that $\mathfrak{J}_\pm \subset \mathcal{A}$.

The lemma formulated below, which is valid for general energy densities $W(\boldsymbol{F})$, shows that our results from [25] imply that all matrices $\boldsymbol{H}_t = t\boldsymbol{H}_+ + (1-t)\boldsymbol{H}_-$ would also fail to be rank-one convex for all $t \in (0, 1)$, proving equality (2.16), as the line segments $\boldsymbol{H}_t$ cover all points $\boldsymbol{H}$, satisfying $\theta_1 < \operatorname{Tr}\boldsymbol{H} < \theta_2$.

Lemma 2.3 *Suppose $\boldsymbol{H}_\pm$ is a corresponding pair of points on the jump set $\mathfrak{J}$ of the energy density function $W(\boldsymbol{H})$. Suppose that*

(i) $W(\boldsymbol{H}) \geq W(\boldsymbol{H}_-) + \langle \boldsymbol{P}_-, \boldsymbol{H} - \boldsymbol{H}_-\rangle$ *for any matrix $\boldsymbol{H}$, such that* $\operatorname{rank}(\boldsymbol{H} - \boldsymbol{H}_-) = 1$, *where* $\boldsymbol{P}_- = W_{\boldsymbol{H}}(\boldsymbol{H}_-)$;
(ii) *None of the matrices* $\boldsymbol{H}_t = t\boldsymbol{H}_+ + (1-t)\boldsymbol{H}_-$, $t \in (0, 1)$ *satisfies the jump set equations* (2.17) *with $\boldsymbol{H}_-$ as the corresponding pair.*

Then all of the matrices $\boldsymbol{H}_t$, $t \in (0, 1)$ must fail rank-one convexity in the sense that they must fail to satisfy the inequality

$$W(\boldsymbol{H}) \geq W(\boldsymbol{H}_t) + \langle \boldsymbol{P}_t, \boldsymbol{H} - \boldsymbol{H}_t\rangle \quad \forall \boldsymbol{H}: \ \operatorname{rank}(\boldsymbol{H} - \boldsymbol{H}_t) = 1, \tag{2.18}$$

where $\boldsymbol{P}_t = W_{\boldsymbol{H}}(\boldsymbol{H}_t)$.

Proof Suppose, with the goal of getting a contradiction, that (2.18) holds. Then, since matrices $\boldsymbol{H}_-$, $\boldsymbol{H}_t$ and $\boldsymbol{H}_+$ all lie on the same rank-one line, we obtain

$$W(\boldsymbol{H}_-) + t\langle \boldsymbol{P}_-, [\![\boldsymbol{H}]\!]\rangle \leq W(\boldsymbol{H}_t) \leq W(\boldsymbol{H}_+) - (1-t)\langle \boldsymbol{P}_t, [\![\boldsymbol{H}]\!]\rangle.$$

Now, using the jump set equations (2.17), we obtain

$$(1-t)(\langle \boldsymbol{P}_-, [\![\boldsymbol{H}]\!]\rangle - \langle \boldsymbol{P}_t, [\![\boldsymbol{H}]\!]\rangle) \geq 0.$$

Using the fact that $\boldsymbol{H}_t - \boldsymbol{H}_- = t[\![\boldsymbol{H}]\!]$ we obtain

$$\langle \boldsymbol{P}_t - \boldsymbol{P}_-, \boldsymbol{H}_t - \boldsymbol{H}_-\rangle \leq 0. \tag{2.19}$$

The variational significance of inequality (2.19), called the phase interchange stability inequality, was understood in [25], where, according to [25, Lemma 4.1], the pair of rank-one related matrices $\boldsymbol{H}_-$ and $\boldsymbol{H}_t$ would have to satisfy the jump set equations (2.17), in contradiction with the assumption (ii) of the lemma. Thus, the assumption that (2.18) holds cannot be true, and all matrices $\boldsymbol{H}_t$, $t \in (0,1)$, must fail to be rank-one convex. □

Remark 2.4 The energy density $QW_0(\boldsymbol{H}) = RW_0(\boldsymbol{H})$ is non-convex as it has a double-well shape along the multiples of $\boldsymbol{I}$, specifically, $QW_0(\varepsilon\boldsymbol{I}) = \Phi^{**}(d\varepsilon) - d(d-1)\mu\varepsilon^2$. This shows that its bulk modulus is negative in the interval $\theta \in (\theta_1, \theta_2)$, where θ_1 and θ_2 are given by (2.13) for the bi-quadratic material model (1.4). Moreover, the value of this modulus is constant and equal to $-2(d-1)\mu/d$, cf. [10], which corresponds exactly to the threshold for the loss of strong ellipticity of the equilibrium equations (saturation of the Legendre-Hadamard conditions). This is associated with the generic degeneration of the acoustic tensor along rank one envelopes as we explain in our Appendix A. Note also that the vanishing of the velocity of the longitudinal waves turns our phase mixture into an elastic æther [9, 16, 21].

We are now ready to show that for a class of energies $W(\boldsymbol{H})$ to which our example belongs all strong local minimizers in a hard device (Dirichlet boundary conditions) must be global. The first observation, leading to this result is that any two energies W and W' that have one and the same quasiconvex envelope are equivalent as far as existence of metastable states are concerned. This is because for any metastable state $\boldsymbol{u}$

$$\int_\Omega W(\nabla \boldsymbol{u})d\boldsymbol{x} = \int_\Omega QW(\nabla \boldsymbol{u})d\boldsymbol{x}.$$

Thus, any metastable configuration for W is also metastable for QW. Theorem 2.1 shows that our energy has the property

$$QW(\boldsymbol{H}) = N(\boldsymbol{H}) + C(\boldsymbol{H}), \tag{2.20}$$

where $N(\boldsymbol{H})$ is a null-Lagrangian and $C(\boldsymbol{H})$ is convex and C^1 smooth.

Theorem 2.5 *Let $W(\boldsymbol{H})$ be the energy density satisfying (2.20). Then any Lipschitz strong local minimizer of $\int_\Omega W(\nabla \boldsymbol{u})d\boldsymbol{x}$ with prescribed Dirichlet boundary conditions is a global minimizer.*

Proof Let $\boldsymbol{v}$ be a Lipschitz competitor that agrees with $\boldsymbol{u}$ on $\partial\Omega$. Then

$$\int_\Omega N(\nabla \boldsymbol{v})d\boldsymbol{x} = \int_\Omega N(\nabla \boldsymbol{u})d\boldsymbol{x}, \qquad C(\nabla \boldsymbol{v}) \geq C(\nabla \boldsymbol{u}) + \langle C_{\boldsymbol{H}}(\nabla \boldsymbol{u}), \nabla(\boldsymbol{v}-\boldsymbol{u})\rangle. \tag{2.21}$$

Since $\boldsymbol{u}$ is an equilibrium we have

$$0 = \nabla \cdot W_{\boldsymbol{H}}(\nabla \boldsymbol{u}) = \nabla \cdot QW_{\boldsymbol{H}}(\nabla \boldsymbol{u}) = \nabla \cdot N_{\boldsymbol{H}}(\nabla \boldsymbol{u}) + \nabla \cdot C_{\boldsymbol{H}}(\nabla \boldsymbol{u}) = \nabla \cdot C_{\boldsymbol{H}}(\nabla \boldsymbol{u}).$$

The equality of $W_{\boldsymbol{H}}(\boldsymbol{H}) = QW_{\boldsymbol{H}}(\boldsymbol{H})$ for any $\boldsymbol{H} \in \boldsymbol{\mathcal{A}}$ follows from the fact that the function $W(\boldsymbol{H}) - QW(\boldsymbol{H})$ is nonnegative, C^1 smooth and attains its minimum value of 0 at all $\boldsymbol{H} \in \boldsymbol{\mathcal{A}}$. Now, the integration by parts in the second term on the right-hand side in the inequality in (2.21) implies $\int_\Omega C(\nabla \boldsymbol{v})d\boldsymbol{x} \geq \int_\Omega C(\nabla \boldsymbol{u})d\boldsymbol{x}$. It follows that

$$\int_\Omega QW(\nabla \boldsymbol{v})d\boldsymbol{x} \geq \int_\Omega QW(\nabla \boldsymbol{u})d\boldsymbol{x}. \qquad \square$$

Remark 2.6 An important observation was made by Sivaloganathan and Spector in [58] that $C(\boldsymbol{H})$ does not need to be convex for inequality in (2.21) to hold. In fact, as the authors show, it is only required that $C^{**}(\nabla \boldsymbol{u}(\boldsymbol{x})) = C(\nabla \boldsymbol{u}(\boldsymbol{x}))$ for all $\boldsymbol{x} \in \Omega$, i.e. $C(\boldsymbol{H})$ agrees with its convex hull at all values of $\nabla \boldsymbol{u}$.

3 Global Minimizers

In this section we temporarily switch to a more general setting. For instance, we assume that the "deformation" $\boldsymbol{y}$ is a vector field $\boldsymbol{y} : \Omega \to \mathbb{R}^m$. In fact, the analysis presented below does not depend on either dimension of $\boldsymbol{x}$ space or the dimension of $\boldsymbol{y}$ space. It also does not depend on the specific form of $W(\boldsymbol{F})$.

Consider the problem of attainment in the definition of the quasiconvex envelope

$$QW(\boldsymbol{F}) = \inf_{\boldsymbol{y} \in \boldsymbol{F}\boldsymbol{x} + W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \fint_\Omega W(\nabla \boldsymbol{y})d\boldsymbol{x}, \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$ is a star-shaped Lipschitz domain. Below we show that Lipschitz equilibrium configurations for (3.1) must be global minimizers, provided they satisfy the necessary conditions for metastability and are sufficiently regular near $\partial\Omega$.

Recall first the well-known necessary conditions for Lipschitz strong local minima of variational functionals. The first classical necessary condition is the Euler-Lagrange equation

$$\nabla \cdot \boldsymbol{P}(\nabla \boldsymbol{y}) = \boldsymbol{0}, \qquad \boldsymbol{P}(\boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{F}), \tag{3.2}$$

which even weak local minimizers have to satisfy. The second, is the Noether equation

$$\nabla \cdot \boldsymbol{P}^*(\nabla \boldsymbol{y}) = \boldsymbol{0}, \qquad \boldsymbol{P}^*(\boldsymbol{F}) = W(\boldsymbol{F})\boldsymbol{I}_d - \boldsymbol{F}^t\boldsymbol{P}(\boldsymbol{F}). \tag{3.3}$$

We recall that if $\boldsymbol{y}(\boldsymbol{x})$ is of class C^2 then (3.3) is a consequence of (3.2), while there are Lipschitz configurations satisfying (3.2), but not (3.3). For instance, the Maxwell relation, which is the last equation in (2.17), is a consequence of (3.3), but not of (3.2).

Definition 3.1 A configuration $\boldsymbol{y} \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ is called stationary if it satisfies both (3.2) and (3.3).

The third classical necessary condition is quasiconvexity $\nabla \boldsymbol{y}(\boldsymbol{x}) \in \boldsymbol{\mathcal{A}}$ for a.e. $\boldsymbol{x} \in \Omega$, where $\boldsymbol{\mathcal{A}}$ is defined in (2.3). It was shown in [25] that for configurations, whose only singularities are smooth phase boundaries the combination of (3.2) and quasiconvexity implies stationarity. In general, however, an example in [40] shows that there could be weak local minimizers for strictly quasiconvex energies that are not strong local minimizers; we conjecture that the configuration in [40] fails stationarity.

It turns out that in the case of affine boundary conditions in a star-shaped domain, the three necessary conditions formulated above are sufficient for $\boldsymbol{y}(\boldsymbol{x})$ to be a global minimizer, provided we assume its regularity near the boundary of the domain.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^d$ be a star-shaped domain with a Lipschitz boundary. Assume that*

(i) $\boldsymbol{y} : \Omega \to \mathbb{R}^m$ *is Lipschitz continuous,*
(ii) $\boldsymbol{y}$ *solves* (3.2) *and* (3.3) *in the sense of distributions,*
(iii) $\nabla \boldsymbol{y}(\boldsymbol{x})$ *is locally stable for a.e.* $\boldsymbol{x} \in \Omega$,
(iv) $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{x}$ *for all* $\boldsymbol{x} \in \partial\Omega$,
(v) *There exists* $\epsilon > 0$, *such that* $\boldsymbol{y} \in C^1(\overline{\Omega_\epsilon}; \mathbb{R}^m)$, *where* $\Omega_\epsilon = \{\boldsymbol{x} \in \Omega : \mathrm{dist}(\boldsymbol{x}, \partial\Omega) < \epsilon\}$.

Then $\boldsymbol{y}(\boldsymbol{x})$ *is the global minimizer in* (3.1), *i.e.*

$$\int_\Omega W(\nabla \boldsymbol{y}) d\boldsymbol{x} = |\Omega| QW(\boldsymbol{F}). \tag{3.4}$$

Proof While each ingredient of the proof presented below was in one way or another already present in the uniqueness proof of Knops and Stuart [36], our theorem is new, since it does not assume that the energy density $W(\boldsymbol{F})$ is quasiconvex. The idea is to express the energy of an equilibrium configuration as a boundary integral using a far reaching generalization of a formula in [36] which, by itself, can be viewed as a nontrivial nonlinear generalization of the classical Clapeyron theorem[5]

Lemma 3.3 *Suppose* $\boldsymbol{y}(\boldsymbol{x})$ *is a Lipschitz stationary equilibrium in a Lipschitz domain* Ω. *Then*

$$E[\boldsymbol{y}] = \int_\Omega W(\nabla \boldsymbol{y}) = \frac{1}{d} \int_{\partial\Omega} \{\boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{y} + \boldsymbol{P}^*\boldsymbol{n} \cdot \boldsymbol{x}\} d\boldsymbol{x}, \tag{3.5}$$

where $\boldsymbol{P}\boldsymbol{n}$ *and* $\boldsymbol{P}^*\boldsymbol{n}$ *can be regarded as trace functionals, since they act on Lipschitz functions* $\boldsymbol{y}(\boldsymbol{x})$ *and* $\boldsymbol{x}$, *respectively.*

Proof

$$\int_{\partial\Omega} \boldsymbol{P}^*\boldsymbol{n} \cdot \boldsymbol{x} dS = \int_\Omega \langle \boldsymbol{P}^*, \nabla \boldsymbol{x} \rangle d\boldsymbol{x} = \int_\Omega \mathrm{Tr}(\boldsymbol{P}^*) d\boldsymbol{x} = d \int_\Omega W(\nabla \boldsymbol{y}) d\boldsymbol{x} - \int_\Omega \langle \boldsymbol{P}, \nabla \boldsymbol{y} \rangle d\boldsymbol{x}.$$

Equation (3.5) follows from

$$\int_\Omega \langle \boldsymbol{P}, \nabla \boldsymbol{y} \rangle d\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{P}\boldsymbol{n} \cdot \boldsymbol{y} dS.$$

□

We now apply formula (3.5) for the energy of $\boldsymbol{y}(\boldsymbol{x})$. Using the boundary condition (iv) and the assumption (v) we have at all $\boldsymbol{x} \in \partial\Omega$

$$\nabla \boldsymbol{y} = \boldsymbol{F} + \boldsymbol{a} \otimes \boldsymbol{n}, \qquad \boldsymbol{a} = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{n}} - \boldsymbol{F}\boldsymbol{n}.$$

[5] The classical Clapeyron theorem in linear elasticity states that the elastic energy $E[\boldsymbol{u}] = (1/2) \int_\Omega \langle \mathsf{C}(\boldsymbol{x}) e(\boldsymbol{u}), e(\boldsymbol{u}) \rangle d\boldsymbol{x}$ of an equilibrium configuration can be computed by the formula $E[\boldsymbol{u}] = (1/2) \int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{n} \cdot \boldsymbol{u} dS$, where $\boldsymbol{\sigma} = \mathsf{C}(\boldsymbol{x}) e(\boldsymbol{u})$ is the stress tensor, [19].

We therefore compute

$$dE[\boldsymbol{y}]=\int_{\partial\Omega}(W(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{n})-\boldsymbol{P}(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{n}\cdot\boldsymbol{x})dS$$

next, we use the local stability condition (iii) and appeal to [25, Lemma 4.2]. We quote the part of the lemma we need for the sake of completeness.

Lemma 3.4 *Let $V(\boldsymbol{F})$ be a rank-one convex function such that $V(\boldsymbol{F})\le W(\boldsymbol{F})$. Let*

$$\mathcal{A}_V=\{\boldsymbol{F}\in\mathcal{O}:W(\boldsymbol{F})=V(\boldsymbol{F})\},$$

where $\mathcal{O}$ is an open subset of $\mathbb{M}$ on which $W(\boldsymbol{F})$ is of class C^1. Then for every $\boldsymbol{F}\in\mathcal{A}_V$, $\boldsymbol{u}\in\mathbb{R}^m$, and $\boldsymbol{v}\in\mathbb{R}^d$

$$V(\boldsymbol{F}+\boldsymbol{u}\otimes\boldsymbol{v})\ge W(\boldsymbol{F})+\boldsymbol{P}(\boldsymbol{F})\boldsymbol{v}\cdot\boldsymbol{u}. \tag{3.6}$$

We now apply Lemma 3.4 by choosing $V(\boldsymbol{F})$ to be $QW(\boldsymbol{F})$. By assumption (iii) for each $\boldsymbol{x}\in\partial\Omega$ the field $\nabla\boldsymbol{y}(\boldsymbol{x})=\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{n}$ is in $\mathcal{A}_V$. Choosing $\boldsymbol{u}\otimes\boldsymbol{v}=-\boldsymbol{a}\otimes\boldsymbol{n}$, inequality (3.6) becomes

$$QW(\boldsymbol{F})\ge W(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{n})-\boldsymbol{P}(\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{n})\boldsymbol{n}\cdot\boldsymbol{a}. \tag{3.7}$$

Finally, we use the assumption of Ω being star-shaped. If we choose the origin at the star point, then the function $\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{x}$ is always non-negative at all points on $\partial\Omega$. Therefore, inequality (3.7) implies

$$dE[\boldsymbol{y}]\le QW(\boldsymbol{F})\int_{\partial\Omega}(\boldsymbol{n}\cdot\boldsymbol{x})dS=d|\Omega|QW(\boldsymbol{F}).$$

Since $|\Omega|QW(\boldsymbol{F})$ is the minimal value of $E[\boldsymbol{y}]$ we conclude that $E[\boldsymbol{y}]=QW(\boldsymbol{F})$, and $\boldsymbol{y}(\boldsymbol{x})$ is the global minimizer.

4 Multiplicity of Global Minimizers

The absence of strong local minimizers in hard device with affine boundary conditions demonstrated in the previous section leaves the question of the characterization of all global minimizers in

$$QW_0(\boldsymbol{H}_0)=\inf_{\boldsymbol{u}|_{\partial\Omega}=\boldsymbol{H}_0\boldsymbol{x}}\fint_{\Omega}W_0(\nabla\boldsymbol{u})d\boldsymbol{x}, \tag{4.1}$$

where $W_0(\boldsymbol{H})$ is defined in (1.3). We remark that the construction of the periodic laminate in Sect. 2 provides us with a minimizing sequence in (4.1) in *arbitrary domains*. In this section we raise the question of *attainability* of the minimum in (4.1). In this regard, laminates can no longer be used because the boundary conditions

$$\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{H}_0\boldsymbol{x},\quad \boldsymbol{x}\in\partial\Omega \tag{4.2}$$

are inconsistent with $\nabla \boldsymbol{u}$ taking exactly two specific values almost everywhere in Ω. We will therefore attempt to find energy minimizers using the sufficiency conditions from Theorem 3.2. In fact, we only need to find solutions $\boldsymbol{u}(\boldsymbol{x})$ of (3.2) that satisfies (4.2) while respecting the stability condition

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) \notin (\theta_1, \theta_2) \text{ for a.e. } \boldsymbol{x} \in \Omega. \tag{4.3}$$

Let $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{H}_0\boldsymbol{x}$. Then (3.2) can be written in terms of $\boldsymbol{\phi}$ as follows

$$\mu\Delta\boldsymbol{\phi} + \nabla\Phi'(\nabla \cdot \boldsymbol{u}) = \mu\nabla(\nabla \cdot \boldsymbol{\phi}), \quad \boldsymbol{x} \in \Omega. \tag{4.4}$$

Integrating the dot product of $\boldsymbol{\phi}(\boldsymbol{x})$ and (4.4) by parts we obtain

$$\int_\Omega \left\{\mu(|\nabla \cdot \boldsymbol{\phi}|^2 - |\nabla\boldsymbol{\phi}|^2) - \Phi'(\nabla \cdot \boldsymbol{u})\nabla \cdot \boldsymbol{\phi}\right\} d\boldsymbol{x} = 0.$$

Now we use the algebraic identity

$$|\nabla \cdot \boldsymbol{\phi}|^2 - |\nabla\boldsymbol{\phi}|^2 = 2J_2(\nabla\boldsymbol{\phi}) - \frac{1}{2}|\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\phi})^t|^2$$

and the fact that $J_2(\nabla\boldsymbol{\phi})$ given by (2.6) is a null-Lagrangian to obtain

$$\int_\Omega \left\{\frac{\mu}{2}|\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\phi})^t|^2 + \Phi'(\nabla \cdot \boldsymbol{u})\nabla \cdot \boldsymbol{\phi}\right\} d\boldsymbol{x} = 0. \tag{4.5}$$

Now let us show that (4.3) implies that

$$\int_\Omega \Phi'(\nabla \cdot \boldsymbol{u})\nabla \cdot \boldsymbol{\phi}(\boldsymbol{x})d\boldsymbol{x} \geq 0. \tag{4.6}$$

Indeed, from the geometric interpretation of θ_1 and θ_2 (as the points of common tangency) we conclude that in view of (4.3) we can write

$$\Phi(\nabla \cdot \boldsymbol{u}(\boldsymbol{x})) = \Phi^{**}(\nabla \cdot \boldsymbol{u}(\boldsymbol{x})) \tag{4.7}$$

for a.e. $\boldsymbol{x} \in \Omega$. For a convex function $\Phi^{**}(\theta)$ the tangent line to the graph of that function lies always below the graph which means that

$$\Phi^{**}(\eta) \geq \Phi'(\theta)(\eta - \theta) + \Phi(\theta)$$

for any $\eta \in \mathbb{R}$ and any $\theta \notin (\theta_1, \theta_2)$. Substituting $\eta = \operatorname{Tr} \boldsymbol{H}_0$ and $\theta = \nabla \cdot \boldsymbol{u}(\boldsymbol{x})$ we obtain

$$\Phi'(\nabla \cdot \boldsymbol{u}(\boldsymbol{x}))\nabla \cdot \boldsymbol{\phi}(\boldsymbol{x}) \geq \Phi(\nabla \cdot \boldsymbol{u}(\boldsymbol{x})) - \Phi^{**}(\operatorname{Tr} \boldsymbol{H}_0) = \Phi^{**}(\nabla \cdot \boldsymbol{u}(\boldsymbol{x})) - \Phi^{**}(\operatorname{Tr} \boldsymbol{H}_0)$$

for a.e. $\boldsymbol{x} \in \Omega$. Now (4.6) follows from Jensen's inequality for a convex function $\Phi^{**}(\theta)$ and the fact that

$$\frac{1}{|\Omega|}\int_\Omega \nabla \cdot \boldsymbol{u}(\boldsymbol{x})d\boldsymbol{x} = \operatorname{Tr} \boldsymbol{H}_0. \tag{4.8}$$

Now, inequality (4.6) together with (4.5) imply that $\nabla\boldsymbol{\phi} = (\nabla\boldsymbol{\phi})^t$. Therefore,

$$0 = \nabla \cdot (\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\phi})^t) = \Delta\boldsymbol{\phi} - \nabla(\nabla \cdot \boldsymbol{\phi}). \tag{4.9}$$

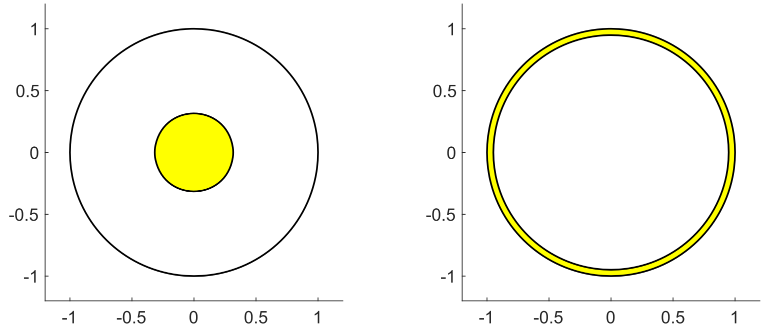

Fig. 3 Morphological hysteresis showing two different morphologies of phase arrangements corresponding to 10% volume fraction of phase 2 ($\omega = 0.1$), shown in yellow. In the left panel nucleation starts at the center. In the right panel nucleation starts at the boundary

Thus, (4.4) implies that $\nabla\Phi'(\nabla \cdot \boldsymbol{u}) = 0$, and there exists a constant P_0, such that

$$\Phi'(\nabla \cdot \boldsymbol{u}) = P_0. \tag{4.10}$$

The constraint (4.3) then implies that either $\nabla \cdot \boldsymbol{u} = \theta_0$ is a constant function in Ω, or that $P_0 = \Phi'(\theta_1) = \Phi'(\theta_2)$, in which case there exists a subset A of Ω such that

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = \theta_1 \chi_A(\boldsymbol{x}) + \theta_2 \chi_{\Omega\setminus A}(\boldsymbol{x}), \tag{4.11}$$

where $\theta_1 < \theta_2$ are the endpoints of the interval $\{\theta \in \mathbb{R} : \Phi^{**}(\theta) < \Phi(\theta)\}$ (see Fig. 2). Recalling (4.8) we conclude that we must have $\nabla \cdot \boldsymbol{u} = \operatorname{Tr} \boldsymbol{H}_0$, if $\operatorname{Tr} \boldsymbol{H}_0 \notin (\theta_1, \theta_2)$. But then $\nabla \cdot \boldsymbol{\phi} = 0$, and thus, according to (4.9), $\boldsymbol{\phi}$ solves $\Delta\boldsymbol{\phi} = 0$ in Ω, with $\boldsymbol{\phi} = 0$ on $\partial\Omega$. It follows that $\boldsymbol{\phi} = 0$ and we conclude that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{H}_0\boldsymbol{x}$ is the only equilibrium satisfying (4.3).

If $\operatorname{Tr} \boldsymbol{H}_0 \in (\theta_1, \theta_2)$, then in addition to $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{H}_0\boldsymbol{x}$ we may have other solutions satisfying (4.11), in which case it is convenient to introduce the scalar parameter

$$\omega = \frac{|A|}{|\Omega|} = \frac{\operatorname{Tr} \boldsymbol{H}_0 - \theta_2}{\theta_1 - \theta_2}, \tag{4.12}$$

which represents the volume fraction of the phase in which $\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = \theta_1$.

If, in addition, Ω is simply connected, then there exists a scalar potential $h \in W_0^{2,2}(\Omega)$, such that $\boldsymbol{\phi} = \nabla h$. In terms of the potential h we obtain a free boundary problem

$$\begin{cases} \Delta h = (\theta_1 - \theta_2)(\chi_A(\boldsymbol{x}) - \omega), & \boldsymbol{x} \in \Omega \\ h \in W_0^{2,2}(\Omega). \end{cases} \tag{4.13}$$

As formulas (4.12), (4.13) indicate, the volume fraction ω of the precipitate is uniquely determined by the hard device loading $\boldsymbol{H}_0$, whose shear component has no effect on the precipitate morphology. While (4.13) looks like a Cauchy problem for the Poisson equation, the right-hand side in (4.13) is not fixed and the problem would be solved if we can find the shape A of the inclusion for which the solution of the Dirichlet problem has the correct Neumann boundary conditions.

In fact, a solution of (4.13) can be easily found in explicit form for some particularly simple domain geometries. For instance, it is easy to construct a radially symmetric solution when Ω is the unit ball, taking the set A to be either the concentric ball of radius $r_0 = \omega^{1/d}$, or its complement, in which case $r_0 = (1-\omega)^{1/d}$. Then (4.13) is solved by $h(\boldsymbol{x}) = h_\omega(|\boldsymbol{x}|)$, given by

$$h_\omega(r) = \begin{cases} \dfrac{1}{2d}(\theta_1-\theta_2)(1-\omega)\left(r^2 - \dfrac{d(\omega^{2/d}-\omega)}{(d-2)(1-\omega)}\right), & 0 \le r \le \omega^{1/d} \\ -\dfrac{1}{2d}(\theta_1-\theta_2)\omega\left(r^2 + \dfrac{2}{(d-2)r^{d-2}} - \dfrac{d}{d-2}\right), & \omega^{1/d} \le r \le 1, \end{cases} \tag{4.14}$$

in the former case. In the latter case, $h(\boldsymbol{x})$ is given by the same formula (4.14) where θ_1 and θ_2 are interchanged and ω is replaced by $1-\omega$, i.e. $h(\boldsymbol{x}) = -h_{1-\omega}(|\boldsymbol{x}|)$. These two optimal configurations with the same energy are illustrated in Fig. 3. With the increase of the volume fraction ω the nucleus in the left panel (colored in yellow) will grow outward while the nucleus in the right panel (also colored in yellow) will grow inward. At $\omega = 1$ both transformation terminate as the second phase completely takes over the whole domain. Note that the associated morphological mechanisms of such phase transformations, which start and end in the same configurations and proceed through energetically equivalent states, are nevertheless different.

The “concentric sphere” solution (4.14) can be used to construct an infinite family of solutions in any other Lipschitz domain. This can be done using the so-called Hashin’s “concentric sphere” construction, [28]. In this construction the set A is a countable union of variously scaled copies of the radially symmetric solution (4.14) filling Ω up to a set of Lebesgue measure zero. Let $B(\boldsymbol{x}_i, a_i) \subset B(\boldsymbol{x}_i, R_i)$, $i = 1, 2, \ldots$ are the concentric balls used in Hashin’s construction, where

$$\frac{a_i^d}{R_i^d} = \omega.$$

The inner balls $B(\boldsymbol{x}_i, a_i)$ in the construction belong to the set A, while the spherical shells $B(\boldsymbol{x}_i, R_i) \setminus B(\boldsymbol{x}_i, a_i)$ belong to $\Omega \setminus A$. The function $h(\boldsymbol{x})$ restricted to the ball $B(\boldsymbol{x}_i, R_i)$ is given by

$$h(\boldsymbol{x}) = R_i^2\, h_\omega\left(\frac{|\boldsymbol{x}-\boldsymbol{x}_i|}{R_i}\right), \quad \boldsymbol{x} \in B(\boldsymbol{x}_i, R_i),$$

where h_ω is given by (4.14). The function $h(\boldsymbol{x})$ defined like this on each of the concentric balls does indeed solve (4.13), since on the boundary of each ball $B(\boldsymbol{x}_i, R_i)$ both $h(\boldsymbol{x})$ and $\nabla h(\boldsymbol{x})$ are zero.

It is intuitively clear that the “concentric sphere” construction in a non-circular domain will have an infinite surface area (for the formal proof see [44, 65]), which is unacceptable in the physically realistic contexts where phase boundaries (surfaces of gradient discontinuity) carry additional surface energy. For this reason one may be interested whether other optimal microstructures with finite surface area exist as well. In some cases, when the domain Ω is sufficiently simple, such low surface energy alternatives are indeed known to exist. For example, if Ω is an ellipsoid, then the low-surface-area minimizer A is a confocal ellipsoid (or its complement) [8, 22, 45, 62, 68]. However, if Ω is more complex, for instance, a square, the low-surface-area minimizer, if it exists, may be topologically nontrivial. To corroborate this claim, we show in Appendix B that in the case of a square domain an optimal inclusion with square symmetry and simple topology does not exist.

5 Conclusions

In this paper we considered a classical problem of nonlinear elasticity for a geometrically linear material undergoing purely volumetric phase transition. Mathematically it reduces to a non-convex vectorial problem of the calculus of variations. In this framework we presented a simple, yet non-trivial example of a material for which the absence of metastability in a hard device coexists with a nonuniqueness of global minimizers.

Metastability in elastostatics, understood as the existence of strong local minimizers that are not global, is ubiquitous as hysteresis is a typical phenomenon accompanying martensitic phase transitions. While for the Neumann boundary conditions the presence of such local minimizers is indeed common, as their existence can be linked to the generic incompatibility of the energy wells, here we showed that for the Dirichlet boundary conditions the incompatibility of the wells does not necessarily cause "metastability". More specifically, we have presented an explicit example of the energy density of a hyperelastic material with non-rank-one convex, double-well energy and non-rank one connected energy wells, for which we could prove the lack of strong local minimizers which are not global on any domain and for any Dirichlet boundary conditions. The analytical transparency of our arguments is due to the utmost simplicity of the chosen energy density which is both geometrically linear and isotropic.

For the same material model we could fully characterize the necessary and sufficient conditions defining global energy minima in hard device loading. The obtained conditions allowed us to reveal the multiplicity of global minimizers with the associated nonuniqueness unrelated to either objectivity or crystallographic symmetry. As an important element of this analysis we used a novel way of expressing the energy of equilibrium configurations as boundary integrals which can be viewed as a nontrivial nonlinear generalization of the classical Clapeyron theorem.

While we showed that the relaxation (quasi-convexification) of the energy in our model can be achieved by simple lamination or coated sphere construction, these optimal microstructures are hardly physical as the associated surface area is infinite. In real physical situations finite surface energy plays the role of the selection mechanism among otherwise energetically equivalent configurations [13, 35, 39], in particular, ruling out constructions with infinite surface area discussed in this paper. Therefore, of interest in physical applications are global minimizers with finite surface area, and, to demonstrate the existence of such minimizers, we explicitly computed a one-parameter family of non-affine energy minimizing configurations for the case of a finite domain with smooth boundary. An interesting aspect of the obtained solution is that, despite the absence of metastability and the associated constitutive hysteresis, it shows that the direct and the reverse transformation may follow different morphological paths while traversing energetically equivalent configurations. The possibility of such a morphological hysteresis may be found advantageous in applications if the ways of manipulating the microstructure, beyond the scale and range of classical continuum elasticity, are employed.

Appendix A: The Degeneracy of Acoustic Tensors of Rank-One Envelopes

Here we prove that the acoustic tensor of a rank-one convex envelope RW of the non rank-one convex energy must have a degenerate direction at all points $\boldsymbol{F}_0$ where $RW(\boldsymbol{F}_0) <$

$W(\boldsymbol{F}_0)$. We also show that in particularly simple situations, such as the one discussed in this paper, this property may be even sufficient to compute the whole rank-one convex envelope.

We recall that the acoustic tensor of the energy $W(\boldsymbol{F})$ at $\boldsymbol{F} = \boldsymbol{F}_0$ in the direction $\boldsymbol{n}$ is a quadratic form $\boldsymbol{A}(\boldsymbol{n})$ defined by

$$\boldsymbol{A}(\boldsymbol{n})\boldsymbol{a} \cdot \boldsymbol{a} = \langle W_{\boldsymbol{FF}}(\boldsymbol{F}_0)(\boldsymbol{a} \otimes \boldsymbol{n}), \boldsymbol{a} \otimes \boldsymbol{n} \rangle.$$

Theorem A.1 *Let $\boldsymbol{F}_0$ be fixed and suppose $RW(\boldsymbol{F}_0) < W(\boldsymbol{F}_0)$, where RW denotes the rank-1 convexification of W. Assume further that W and RW are C^2 near $\boldsymbol{F}_0$. Let $\boldsymbol{A}_0(\boldsymbol{n})$ be the acoustic tensor of RW at $\boldsymbol{F}_0$. Then there is a direction $\boldsymbol{n}$ such that* $\det \boldsymbol{A}_0(\boldsymbol{n}) = 0$.

Proof Assume that there is no such direction $\boldsymbol{n}$. Since RW is necessarily rank-1 convex we can conclude that for any direction $\boldsymbol{n}$ the matrix $\boldsymbol{A}_0(\boldsymbol{n})$ is positive semidefinite. Since the function $\boldsymbol{n} \mapsto \boldsymbol{A}_0(\boldsymbol{n})$ is continuous there exists a positive number α such that for every unit vector $\boldsymbol{n}$

$$\boldsymbol{A}_0(\boldsymbol{n}) \geq \alpha \boldsymbol{I}.$$

Since $RW \in C^2$ near $\boldsymbol{F}_0$ there exists a number $\delta > 0$ such that for every $\boldsymbol{F}$ satisfying $|\boldsymbol{F} - \boldsymbol{F}_0| < \delta$ the following inequalities hold:

1. $\boldsymbol{A}(\boldsymbol{n}) \geq \frac{1}{2}\alpha \boldsymbol{I}$, where $\boldsymbol{A}(\boldsymbol{n})$ is the acoustic tensor of RW at $\boldsymbol{F}$.
2. $W(\boldsymbol{F}) - RW(\boldsymbol{F}) \geq \frac{1}{2}(W(\boldsymbol{F}_0) - RW(\boldsymbol{F}_0)) = \beta > 0$.

Now consider a function

$$T_\epsilon(\boldsymbol{F}) = RW(\boldsymbol{F}) + \epsilon\phi\left(\frac{\boldsymbol{F} - \boldsymbol{F}_0}{\sqrt[4]{\epsilon}}\right),$$

where ϕ is a smooth nonnegative function supported on the unit ball in $\mathbb{M}$ and such that $\phi(\boldsymbol{0}) = 1$. We can choose ϵ so small that the following inequalities hold:

1. $\|\epsilon\phi\|_{L^\infty} < \frac{1}{2}\beta$,
2. $\|\sqrt{\epsilon}\nabla_{\boldsymbol{F}}\nabla_{\boldsymbol{F}}\phi\|_{L^\infty} \leq \frac{1}{4}\alpha$,
3. $\operatorname{supp} \phi\left(\frac{\boldsymbol{F}-\boldsymbol{F}_0}{\sqrt[4]{\epsilon}}\right) \subset B(\boldsymbol{F}_0, \delta)$, where $B(\boldsymbol{F}_0, \delta)$ is the ball of radius δ around $\boldsymbol{F}_0$ in $\mathbb{M}$.

Then $T_\epsilon(\boldsymbol{F}) \leq W(\boldsymbol{F})$ and the acoustic tensor $\boldsymbol{A}_\epsilon(\boldsymbol{n})$ of $T_\epsilon(\boldsymbol{F})$ satisfies

$$\boldsymbol{A}_\epsilon(\boldsymbol{n}) \geq \frac{1}{4}\alpha \boldsymbol{I}$$

in the sense of quadratic forms. Thus $T_\epsilon(\boldsymbol{F})$ is rank-1 convex and $T_\epsilon(\boldsymbol{F}) \leq W(\boldsymbol{F})$ but $T_\epsilon(\boldsymbol{F}_0) > RW(\boldsymbol{F}_0)$. Contradiction. Thus our assumption is false and at every point $\boldsymbol{F}_0$ where $RW(\boldsymbol{F}_0) < W(\boldsymbol{F}_0)$ there is a direction $\boldsymbol{n}$ such that the acoustic tensor $\boldsymbol{A}_0(\boldsymbol{n})$ is degenerate. □

Remark A.2 It follows that materials that transform by forming microstructures with sharp phase boundaries to accommodate deformations produced by the propagation of sound waves will have a direction with a zero sound speed.

We can apply Theorem A.1 to the energy (1.3). Since our material is isotropic, the degeneration of the acoustic tensor may be either through $\mu = 0$ or through $\lambda + 2\mu = 0$ with the latter also meaning that the bulk modulus $\kappa = -2\mu(d-1)/d$. The possibility that $\mu = 0$ is excluded because the tangential shear modulus is the same at every deformation.

Suppose that we have somehow guessed that if

$$W_0(\boldsymbol{H}) = f(\mathrm{Tr}\,\boldsymbol{\varepsilon}) + \mu|\mathrm{dev}(\boldsymbol{\varepsilon})|^2,$$

then

$$RW_0(\boldsymbol{H}) = F(\mathrm{Tr}\,\boldsymbol{\varepsilon}) + \mu|\mathrm{dev}(\boldsymbol{\varepsilon})|^2$$

for some function F, yet to be determined. In that case Theorem A.1 will let us determine the function $F(\theta)$. It is easy to compute that for any unit vector $\boldsymbol{n}$ and any vector $\boldsymbol{a} \in \mathbb{R}^d$

$$\langle RW_{0,\boldsymbol{HH}}(\boldsymbol{H})(\boldsymbol{n}\otimes\boldsymbol{a}), \boldsymbol{n}\otimes\boldsymbol{a}\rangle = F''(\mathrm{Tr}\,\boldsymbol{\varepsilon})(\boldsymbol{a},\boldsymbol{n})^2 + 2\mu\left|\frac{1}{2}(\boldsymbol{n}\otimes\boldsymbol{a}+\boldsymbol{a}\otimes\boldsymbol{n}) - \frac{1}{d}(\boldsymbol{a},\boldsymbol{n})\boldsymbol{I}\right|^2.$$

Therefore, we get a formula for the acoustic tensor $\boldsymbol{A}(\boldsymbol{n})$.

$$\boldsymbol{A}(\boldsymbol{n}) = \left(F''(\mathrm{Tr}\,\boldsymbol{H}) + \frac{d-2}{d}\mu\right)\boldsymbol{n}\otimes\boldsymbol{n} + \mu\boldsymbol{I}. \tag{A.1}$$

We see that $\det \boldsymbol{A}(\boldsymbol{n}) = \mu^2(F''(\mathrm{Tr}\,\boldsymbol{\varepsilon}) + 2(d-1)\mu/d)$ for all directions $\boldsymbol{n}$. Thus, for all $\boldsymbol{H}$ for which $RW_0(\boldsymbol{H}) < W_0(\boldsymbol{H})$ we get $F''(\mathrm{Tr}\,\boldsymbol{H}) = -2(d-1)\mu/d$. The continuity of $RW_{0,\boldsymbol{H}}$ implies that at the boundary points θ_1 and θ_2 of the binodal region we have $F'(\theta_1) = f'(\theta_1)$ and $F'(\theta_2) = f'(\theta_2)$. Therefore, in the binodal region, where $F'(\theta) = -2(d-1)\theta\mu/d + C$ for some constant C, we must have $C = \Phi'(\theta_1) = \Phi'(\theta_2)$. Further, the continuity of W_0 implies that the affine function[6] $y(\theta) = F(\theta) + (d-1)\mu\theta^2/d$ would be the equation of the common tangent to the graph of $\Phi(\theta)$. Thus, we obtain that $F(\theta) + (d-1)\mu\theta^2/d = \Phi^{**}(\theta)$, and we recover the rank-one convex envelope of $W_0(\boldsymbol{H})$, which in this case is seen to coincide with its quasiconvexification (2.4).

Appendix B: A Non-existence of a Topologically Simple Square-Symmetric Minimizer in a Square

We begin with a general observation that if $A \subset \Omega \subset \mathbb{R}^2$ is an open subset for which the problem (4.13) has a solution $h(x, y)$, then functions

$$\widetilde{h}_+(x,y) = h(x,y) + \frac{1}{4}(\theta_1 - \theta_2)\omega(x^2+y^2),$$

$$\widetilde{h}_-(x,y) = h(x,y) - \frac{1}{4}(\theta_1 - \theta_2)(1-\omega)(x^2+y^2)$$

are harmonic in $\Omega \setminus A$ and A respectively. We can then conclude that the functions $\partial\widetilde{h}_\pm/\partial x - i\partial\widetilde{h}_\pm/\partial y$ are analytic in the complex variable $z = x + iy$ on their respective domains. Thus,

[6]The function $y(\theta)$ is affine because the property of $F(\theta)$ can be written as $y''(\theta) = 0$.

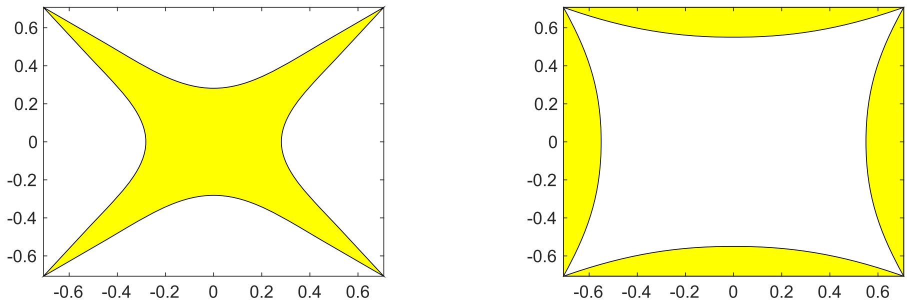

Fig. 4 Morphological hysteresis showing two different morphologies of phase arrangements in a square. The hypothetical coherent precipitate with simple topology and square symmetry corresponds to 30% volume fraction of phase 2 ($\omega = 0.3$) is shown in yellow. In the left panel nucleation starts at the center. In the right panel nucleation starts at the boundary

the functions

$$H_+(z) = \frac{2}{\theta_1 - \theta_2}\left(\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}\right) + \omega\bar{z}, \quad z \in \Omega \setminus A,$$

$$H_-(z) = \frac{2}{\theta_1 - \theta_2}\left(\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}\right) - (1-\omega)\bar{z}, \quad z \in A$$

are also analytic. The boundary conditions in (4.13) and the continuity of ∇h across ∂A, representing the kinematic compatibility of the displacement, imply that

$$\begin{cases} H_+(z) = \omega\bar{z}, & z \in \partial\Omega, \\ H_+(z) - H_-(z) = \bar{z}, & z \in \partial A. \end{cases} \tag{B.1}$$

Thus, the problem of finding global minimizers in Ω with affine boundary conditions reduces to the problem (B.1) in the theory of complex analytic functions of one complex variable.

We now attempt to solve this problem when Ω is a square centered at the origin with diagonal of length 2. Then along the bottom side of the square we have $H_+(x - i/\sqrt{2}) = \omega(x + i/\sqrt{2}) = \omega(z + i\sqrt{2})$ and therefore, $H_+(z) = \omega(z + i\sqrt{2})$. Along the right side of the square we have $H_+(1/\sqrt{2} + iy) = \omega(1/\sqrt{2} - iy) = \omega(\sqrt{2} - z)$, and therefore $H_+(z) = \omega(\sqrt{2} - z)$. These contradictory expressions for $H_+(z)$ can only be reconciled by a structure with topology indicated in Fig. 4, where the set A, in which $\nabla \cdot \boldsymbol{u} = \theta_1$, is shaded in yellow.

The square symmetry of the problem also suggests that we should look for a structure with square symmetry. Thus, if we assume that A has a simple topology, then we just need to find the curve Γ that joins the top and bottom ends of the right side of the square. Knowing $H_+(z)$ in all four regions adjacent to the sides of the square gives the boundary values for $H_-(z)$ on ∂A. The square symmetry of A together with (B.1) implies that

$$H_-(iz) = -iH_-(z), \quad z \in \partial A. \tag{B.2}$$

There is a holomorphic function H_- in A with given boundary values if and only if for every $n \geq 0$

$$\int_{\partial A} H_-(z) z^n dz = 0. \tag{B.3}$$

The integral in (B.3) can easily be written as an integral along Γ because of (B.2). If n is not a multiple of 4 then the integral in (B.3) will evaluate to zero by virtue of (B.2) alone. If $n = 4k$, then we require that

$$\int_{\Gamma} H_-(z) z^{4k} dz = 0, \quad k \geq 0. \tag{B.4}$$

On Γ we have $H_-(z) = \omega(\sqrt{2} - z) - \overline{z}$. Therefore, (B.4) is equivalent to

$$\int_{\Gamma} \overline{z} z^{4k} dz = \frac{(-1)^k i\omega}{(4k+1)(2k+1)}, \quad k \geq 0. \tag{B.5}$$

If we integrate by parts in (B.5):

$$\int_{\Gamma} \overline{z} z^{4k} dz = \int_{\Gamma} \overline{z}\, d\left(\frac{z^{4k+1}}{4k+1}\right) = -\int_{\Gamma} \frac{z^{4k+1}}{4k+1} d\overline{z}.$$

and parametrize the curve Γ by $z(t) = a(t) + it$, $t \in [-1/\sqrt{2}, 1/\sqrt{2}]$. Then $\overline{z'(t)} = z'(t) - 2i$, and we obtain

$$\int_{\Gamma} \overline{z} z^{4k} dz = \frac{(-1)^{k+1} i}{(4k+1)(2k+1)} + \frac{2i}{4k+1} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} z(t)^{4k+1} dt.$$

Thus, equation (B.5) becomes

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} z(t)^{4k+1} dt = \frac{(-1)^k(\omega+1)}{2(2k+1)}.$$

Due to the symmetry we have $z(-t) = \overline{z(t)}$. Therefore, we finally obtain

$$\mathfrak{Re}\left\{\int_0^{1/\sqrt{2}} (a(t) + it)^{4k+1} dt\right\} = \frac{(-1)^k(\omega+1)}{4(2k+1)}, \quad k \geq 0. \tag{B.6}$$

In Fig. 4 we show hypothetically existing exact solutions of (B.6) with square symmetry and simple topology which should be taken at this point just as an indication of a general structure of the purported solutions. As in Fig. 3, we illustrate that at a fixed value of the volume fraction ω there would always be two optimal configurations with the same energy. As we have already seen in Fig. 3, with the increase of ω the nucleus in the left panel in Fig. 4 (colored in yellow) will grow outward while the nucleus in the right panel (also colored in yellow) will grow inward. At $\omega = 1$ both transformations again terminate as the second phase completely takes over the whole domain.

Observe now that $|a(t) + it| < 1$ for all $t \in [0, 1/\sqrt{2})$ and $|a(t) + it| = 1$ for $t = 1/\sqrt{2}$. Thus, for large k the principal contribution to the integral (B.6) comes only from the neighborhood of $t = 1/\sqrt{2}$. Therefore, we approximate

$$a(t) = \frac{1}{\sqrt{2}} + m\left(t - \frac{1}{\sqrt{2}}\right) + O\left(\left(t - \frac{1}{\sqrt{2}}\right)^2\right), \text{ as } t \to \frac{1}{\sqrt{2}}. \tag{B.7}$$

Substituting the leading term in (B.6) and computing the integral explicitly we obtain

$$\frac{(-1)^k}{(4k+2)(m^2+1)} - \frac{(1-m)^{4k+2}m}{2^{2k+1}(4k+2)(m^2+1)} \approx \frac{(-1)^k(\omega+1)}{2(4k+2)}. \tag{B.8}$$

If we choose

$$m = \sqrt{\frac{1-\omega}{1+\omega}}, \tag{B.9}$$

then for large k we have equality in (B.6) up to an exponentially small error. Thus, we have determined the slope m with which the curve Γ enters the corner of the square. We can integrate by parts in (B.6), which we rewrite as

$$\frac{1}{4k+2}\mathfrak{Re}\left\{\int_0^{1/\sqrt{2}} \frac{\left[(a(t)+it)^{4k+2}\right]'}{i+a'(t)}dt\right\} = \frac{(-1)^k(\omega+1)}{4(2k+1)}.$$

We obtain

$$\frac{1}{4k+2}\mathfrak{Re}\left\{\frac{e^{i(4k+2)\pi/4}}{i+m}\right\} - \frac{1}{4k+2}\mathfrak{Re}\left\{\int_0^{1/\sqrt{2}} (a(t)+it)^{4k+2}\left[\frac{1}{i+a'(t)}\right]' dt\right\}$$
$$= \frac{(-1)^k(\omega+1)}{4(2k+1)}. \tag{B.10}$$

But then we conclude that (B.10) is equivalent to

$$\mathfrak{Re}\left\{\int_0^{1/\sqrt{2}} (a(t)+it)^{4k+2}\left[\frac{1}{i+a'(t)}\right]' dt\right\} = 0, \qquad k \geq 0. \tag{B.11}$$

We can now keep integrating by parts in (B.11)

$$\mathfrak{Re}\left\{\int_0^{1/\sqrt{2}} [(a(t)+it)^{4k+3}]' \frac{1}{i+a'(t)}\left[\frac{1}{i+a'(t)}\right]' dt\right\} = 0.$$

This implies that $a''(1/\sqrt{2}) = 0$. By induction we obtain $a^{(n)}(1/\sqrt{2}) = 0$ for all $n \geq 2$, showing the breaking of analyticity of $a(t)$ near $t = 1/\sqrt{2}$. This unexpected failure of analyticity is our first hint that the solution we seek might not exist.

To make a more persuasive argument that this is the case, multiply the kth equation in (B.6) by $x^{4k+2}/(4k+1)!$ and sum, obtain the "generating function equation"

$$\mathfrak{Re}\left\{\int_0^{1/\sqrt{2}} [\sinh(x(it+a(t))) + \sin(x(it+a(t)))]dt\right\} = \frac{\omega+1}{2ix}(\cosh(\sqrt{i}x) - \cos(\sqrt{i}x)).$$

Taking the real part we obtain

$$\int_0^{1/\sqrt{2}} [\cos(xt)\sinh(xa(t)) + \sin(xa(t))\cosh(xt)]dt = \frac{\omega+1}{x}\sinh\frac{x}{\sqrt{2}}\sin\frac{x}{\sqrt{2}}. \tag{B.12}$$

The two sides of (B.12) represent entire functions of x, and hence, the equality is valid for all $x \in \mathbb{C}$. It is convenient to rescale the equation: $u(s) = \sqrt{2}a(s/\sqrt{2})$, $z = x/\sqrt{2}$. Then

$$\int_0^1 [\cos(zs)\sinh(zu(s)) + \sin(zu(s))\cosh(zs)]ds = \frac{\omega+1}{z}\sinh z \sin z. \tag{B.13}$$

The function $u(s)$ is continuous and monotone increasing from $u(0) > 0$ to 1 on $[0, 1]$. It also satisfies $u'(0) = 0$. We know that $u(s) \to s$, when $\omega \to 0$ and we also know that $u(s) \sim 1 + m(s-1)$, when $s \approx 1$. Observe that $m \to 1$, when $\omega \to 0$, and $1 + m(s-1) \to s$. Let us then write $u(s) = u_0(s) + w(s)$, where $u_0(s) = 1 + m(s-1)$. Thus,

$$\sinh(zu(s)) = \sinh(zu_0(s)) + 2\sinh(zu_0(s))\sinh^2(zw(s)/2) + \cosh(zu_0(s))\sinh(zw(s)),$$

and similarly,

$$\sin(zu(s)) = \sin(zu_0(s)) - 2\sin(zu_0(s))\sin^2(zw(s)/2) + \cos(zu_0(s))\sin(zw(s)),$$

Observing that

$$\int_0^1 [\cos(zs)\sinh(zu_0(s)) + \sin(zu_0(s))\cosh(zs)]ds = \frac{\omega+1}{z}\sinh z \sin z + R(z),$$

where

$$R(z) = \sqrt{1-\omega^2}\frac{\cos((1-m)z) - \cosh((1-m)z)}{2z},$$

we obtain

$$\int_0^1 \Bigg[\cos(zs)\cosh(zu_0(s))\sinh(zw(s)) + \cosh(zs)\cos(zu_0(s))\sin(zw(s)) +$$
$$2\cos(zs)\sinh(zu_0(s))\sinh^2\left(\frac{zw(s)}{2}\right) - 2\cosh(zs)\sin(zu_0(s))\sin^2\left(\frac{zw(s)}{2}\right)\Bigg]ds =$$
$$\sqrt{1-\omega^2}\frac{\cos((1-m)z) - \cosh((1-m)z)}{2z}. \tag{B.14}$$

If we now examine equation (B.14) in the limit $z \to +\infty$ we can see that this is impossible.

Indeed, the oscillatory terms generate at most polynomial decay at infinity, while the exponential terms behave like $e^{zu_0(s)}$ attenuated by $\sin(w(s)z)$ or $\sinh(w(s)z)$. The main contribution to such integrals comes from the vicinity of the maximizer of $u_0(s)$, over the support $[0, \alpha]$ of $w(s)$. The integral will then have the exponential growth $e^{(1-m+m\alpha))z}$ possibly modulated by polynomially decaying factors. However, the right-hand side of (B.14) has the exponential growth $e^{z(1-m)}$. Observing that $1 - m = \min_{s\in[0,1]} u_0(s)$ we conclude that the integral on the left-hand side of (B.14) cannot possibly have that growth at infinity, unless the support of $w(s)$ is zero. To summarize, we have shown that the equality (B.14) cannot be satisfied for all $m \in (0, 1)$. We note, however that $w(s) = 0$ does satisfy (B.14) when $m = 0$ or $m = 1$.

Finally, we remark that even if we had not assumed square symmetry of the solution our conclusion would not have changed. The reason is that equation (B.6) corresponds to (B.3) for $n = 4k$. When n has a different remainder mod 4 square symmetry ensures that the corresponding equation is trivially satisfied. Breaking the symmetry simply adds additional infinite systems on the extra degrees of freedom, ultimately forcing square symmetry which, as we have seen, leads to non-existence.

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Author contributions Both authors conducted research and wrote the paper

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Declarations

Competing interests The authors declare no competing interests.

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