Supplemental Material for the paper "Printing non-Euclidean solids"

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I. PRELIMINARIES

Our subsequent derivations rely on the following elementary formula

$$\nabla\left(\int_{\tau(\boldsymbol{x})}^{t}\varphi(\boldsymbol{x},s)\,ds\right)= \int_{\tau(\boldsymbol{x})}^{t}\nabla\varphi(\boldsymbol{x},s)\,ds-\varphi(\boldsymbol{x},\tau(\boldsymbol{x}))\nabla\tau(\boldsymbol{x})$$

where $\varphi(\boldsymbol{x},t)$ is a smooth scalar function and ∇ is the gradient operator. Here $\nabla\tau(\boldsymbol{x}) = V^{-1}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})$ where $V(\boldsymbol{x})$ is the normal velocity of the accreting surface and $\boldsymbol{n}(\boldsymbol{x})$ the outer normal. By taking $\varphi(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{x},t)\cdot\boldsymbol{a}$ with $\boldsymbol{a}$ an arbitrary constant vector, we obtain

$$\nabla\left(\int_{\tau(\boldsymbol{x})}^{t}\boldsymbol{w}(\boldsymbol{x},s)\,ds\right)= \int_{\tau(\boldsymbol{x})}^{t}\nabla\boldsymbol{w}(\boldsymbol{x},s)\,ds-\boldsymbol{w}(\boldsymbol{x},\tau(\boldsymbol{x}))\otimes\nabla\tau(\boldsymbol{x})$$

where the tensor product defines through $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ for arbitrary vectors $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$. Next we recall the following identities

$$\text{div}\boldsymbol{w} = \text{tr}\nabla\boldsymbol{w},\,(\nabla\boldsymbol{w}-\nabla\boldsymbol{w}^{\mathsf{T}})\boldsymbol{a} = \text{curl}\boldsymbol{u}\times\boldsymbol{a},$$

and the properties of an arbitrary smooth second order tensor $\boldsymbol{A}$:

$$\text{div}\boldsymbol{A}\cdot\boldsymbol{a} = \text{div}(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{a})\qquad(\text{curl}\boldsymbol{A})\boldsymbol{a} = (\text{curl}\boldsymbol{A}^{\mathsf{T}})\boldsymbol{a}$$

where $(\text{curl}\mathbf{A})_{ij} = e_{ipq}A_{jq,p}$. We will also need the following identity

$$(\boldsymbol{a}\times\boldsymbol{w})\times\boldsymbol{b} = (\boldsymbol{w}\otimes\boldsymbol{a}-\boldsymbol{a}\otimes\boldsymbol{w})\boldsymbol{b}.$$

By taking $\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{A}(\boldsymbol{x},t)^{\mathsf{T}}\boldsymbol{a}$ we then get

$$\text{div}\left(\int_{\tau(\boldsymbol{x})}^{t}\boldsymbol{A}(\boldsymbol{x},s)\,ds\right)= \int_{\tau(\boldsymbol{x})}^{t}\text{div}\boldsymbol{A}(\boldsymbol{x},s)\,ds-\boldsymbol{A}(\boldsymbol{x},\tau(\boldsymbol{x}))\nabla\tau(\boldsymbol{x}).$$

Furthermore,

$$\text{curl}\left(\int_{\tau(\boldsymbol{x})}^{t}\boldsymbol{A}(\boldsymbol{x},s)\,ds\right)= \int_{\tau(\boldsymbol{x})}^{t}\text{curl}\boldsymbol{A}(\boldsymbol{x},s)\,ds-(\nabla\tau(\boldsymbol{x})\times)\boldsymbol{A}^{\mathsf{T}}(\boldsymbol{x},\tau(\boldsymbol{x})).$$

Note that we made use of the notation $\boldsymbol{w}\times$ for the skew tensor with axial vector $\boldsymbol{w}$, which implies that $(\boldsymbol{w}\times)\boldsymbol{a}\equiv\boldsymbol{w}\times\boldsymbol{a}$.

II. CURVATURE AND INCOMPATIBILITY

The strain incompatibility $\boldsymbol{\epsilon}$ is the linear counterpart of the Riemann-Christoffel curvature of the reference metric. For small strains, the metric of the reference state can be written as $\boldsymbol{g}^{\alpha} = \mathbf{I}+2\,\alpha\,\boldsymbol{\epsilon}$, where $\mathbf{I}$ is the identity matrix, $\boldsymbol{\epsilon}$ is the elastic strain and $\alpha>0$ a smal parameter. In the limit $\alpha\to 0$ the curvature tensor associated with $\boldsymbol{g}^{\alpha}$ can be written as

$$R^{\alpha}_{ijkl} = \alpha\,e_{ijs}e_{klr}\eta_{sr}+o(\alpha)$$

where e_{ijk} is the Ricci alternator, see for instance [1].

III. RESIDUAL STRESSES AND INCOMPATIBILITY

Strain incompatibility is the source of residual stresses. Denoting the latter by $\mathbf{T}$, we can write the system of equations :

$$\begin{cases}\text{div}\,\mathbf{T} = \mathbf{0} & \mathscr{B}\\ \text{curl}\,\text{curl}\,\mathbb{C}^{-1}\mathbf{T} = \boldsymbol{\eta} & \mathscr{B}\\ \mathbf{T}\mathbf{n} = \mathbf{n} & \partial\mathscr{B}.\end{cases}$$

The residual stresses depend not only on incompatibility $\boldsymbol{\eta}$ but also on elastic constants and the shape of the body $\mathscr{B}$. The strain incompatibility $\boldsymbol{\eta}$ may have diffuse and singular components. The global effects of the latter are characterized through the Burgers and Frank vectors, which are discussed below.

IV. BURGERS AND FRANK VECTORS

Here we provide additional details regarding the Burgers and Frank vectors and show how these vectors can be computed from a given elastic strain distribution $\boldsymbol{\epsilon}(\boldsymbol{x})$.

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First of all we recall that the regular part of the incompatibility can be extracted from the elastic strain by computing $\boldsymbol{\eta}^c = \mathrm{curlcurl}\boldsymbol{\epsilon}$. The symmetric tensor field $\boldsymbol{\eta}^c$ necessarily satisfies $\mathrm{div}\boldsymbol{\eta}^c = 0$.

To characterize the global (topological) component of incompatibility, which we associate with the singular term $\boldsymbol{\eta}^s$, we start with the special case $\boldsymbol{\eta}^c = \boldsymbol{0}$.

Consider a cut along a surface $\mathscr{S}$ terminating on a singular "defect line" and an arbitrary closed curve γ around this defect line passing through a point $\boldsymbol{p}$ on $\mathscr{S}$. Then a classical argument [3] shows that the relative displacement at $\boldsymbol{p}$ can be written in the form

$$[\![\boldsymbol{u}]\!](\boldsymbol{p}) = \mathbf{B} + \boldsymbol{\Omega} \times \boldsymbol{p},$$

where

$$\mathbf{B} = \oint_\gamma (\boldsymbol{\epsilon} + \boldsymbol{y} \times \mathrm{curl}\boldsymbol{\epsilon})\, d\boldsymbol{y}$$

and

$$\boldsymbol{\Omega} = \oint_\gamma \mathrm{curl}\boldsymbol{\epsilon}\, d\boldsymbol{y}$$

are, respectively, the Burgers and the Frank vectors. Using the singular part of the incompatibility $\boldsymbol{\eta}^s$, we may alternatively define the Burgers and Frank vectors as

$$\mathbf{B} = \int_D \boldsymbol{y} \times {\boldsymbol{\eta}_s}^{\mathsf{T}} \boldsymbol{n}\, da$$

and

$$\boldsymbol{\Omega} = \int_D {\boldsymbol{\eta}_s}^{\mathsf{T}} \boldsymbol{n}\, da,$$

where D is an arbitrarily oriented disk centeblue at the dislocation line. From these expressions one can recover the distributional representation for $\boldsymbol{\eta}^s(\boldsymbol{x})$ in terms of $\mathbf{B}$ and $\boldsymbol{\Omega}$, see ref.[35] of the main text for details.

Observe that when $\boldsymbol{\eta}^c = \boldsymbol{0}$, the relative displacements across $\mathscr{S}$ are rigid. For this reason, if $\boldsymbol{\Omega} \neq \boldsymbol{0}$, one can find a point $\boldsymbol{p}^*$ such that $[\![\boldsymbol{u}]\!](\boldsymbol{p}) = \boldsymbol{\Omega}\times(\boldsymbol{p}-\boldsymbol{p}^*)$. Instead, if $\boldsymbol{\Omega} = \boldsymbol{0}$, then $[\![\boldsymbol{u}]\!](\boldsymbol{p}) \equiv \mathbf{B}$. In both cases there are only three independent components that fully characterize $[\![\boldsymbol{u}]\!]$ and this is consistent with the fact that singular and regular parts of the incompatibility have only three independent components each.

When $\boldsymbol{\eta}^c \neq \boldsymbol{0}$ and the domain is simply connected we observe that

$$\lim_{h\to 0} \int_{D_h} \boldsymbol{y} \times {\boldsymbol{\eta}^c}^{\mathsf{T}} \boldsymbol{n}\, dA = \lim_{h\to 0} \int_{D_h} {\boldsymbol{\eta}^c}^{\mathsf{T}} \boldsymbol{n}\, dA = \boldsymbol{0}.$$

Denoting by $\boldsymbol{\eta} = \boldsymbol{\eta}^c + \boldsymbol{\eta}^s$ the total incompatibility, we get

$$\mathbf{B} = \lim_{h\to 0} \int_{D_h} \boldsymbol{y} \times \boldsymbol{\eta}^{\mathsf{T}} \boldsymbol{n}\, dA$$

and

$$\boldsymbol{\Omega} = \lim_{h\to 0} \int_{D_h} \boldsymbol{\eta}^{\mathsf{T}} \boldsymbol{n}\, dA,$$

where now D_h is an asymptotically shrinking disk with diameter h and orientation $\boldsymbol{n}$. These last two definitions can be equivalently rewritten as

$$\mathbf{B} = \lim_{h\to 0} \int_{\gamma_h} (\boldsymbol{\epsilon} + \boldsymbol{y} \times \mathrm{curl}\boldsymbol{\epsilon})\, d\boldsymbol{y}$$

$$\boldsymbol{\Omega} = \lim_{h\to 0} \int_{\gamma_h} (\mathrm{curl}\boldsymbol{\epsilon})\, d\boldsymbol{y}.$$

Observe that the only difference with the case $\boldsymbol{\eta}^c = \boldsymbol{0}$ is that here we use an asymptotically shrinking curve γ_h around the defect line, whereas when $\boldsymbol{\eta}^c = \boldsymbol{0}$ this curve was arbitrary.

Since for non-simply connected domains we can always assume that $\boldsymbol{\eta}^c = \mathbf{0}$ inside the "holes" carrying the singularities of incompatibility, we can also write

$$\mathbf{B} = \oint_{\gamma_i^+} (\boldsymbol{\epsilon} + \boldsymbol{y} \times \mathrm{curl}\boldsymbol{\epsilon})\, d\boldsymbol{y}$$

$$\boldsymbol{\Omega} = \oint_{\gamma_i^+} \mathrm{curl}\boldsymbol{\epsilon}\, d\boldsymbol{y}$$

where the curve γ_i^+ is a curve tracing the surface of the "hole" from the side of the body.

V. INCREMENTAL EQUILIBRIUM

Here we provide an alternative derivation of the condition on incremental stress (Eq. (3) in the main text), that in the main text was obtained by specialisation of the Hadamard compatibility condition. By making use of the identities derived above, together with the definition

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathring{\boldsymbol{\sigma}}_a(\boldsymbol{x}) + \boldsymbol{p}(\boldsymbol{x}) + \int_{\tau(\boldsymbol{x})}^t \dot{\boldsymbol{\sigma}}(\boldsymbol{x},s)\, ds,$$

we obtain that the equilibrium equation for $t \geq \tau(\boldsymbol{x})$ can be rewritten as

$$\begin{aligned}\boldsymbol{0} = \mathrm{div}\boldsymbol{\sigma}(\boldsymbol{x},t) &= \int_{\tau(\boldsymbol{x})}^t \mathrm{div}\dot{\boldsymbol{\sigma}}(\boldsymbol{x},s)\, ds \\ &+ \mathrm{div}(\mathring{\boldsymbol{\sigma}}_a(\boldsymbol{x}) + \boldsymbol{p}(\boldsymbol{x})) - \dot{\boldsymbol{\sigma}}(\boldsymbol{x},\tau(\boldsymbol{x}))\nabla\tau(\boldsymbol{x}).\end{aligned}$$

For $t \to \tau(\boldsymbol{x})$ this requirement gives a condition which is equivalent to Eq (3) of the main text (see also ref.[27] of the main text),

$$\dot{\boldsymbol{\sigma}}(\boldsymbol{x},\tau(\boldsymbol{x}))\boldsymbol{n}(\boldsymbol{x}) = |\nabla\tau(\boldsymbol{x})|^{-1}\mathrm{div}(\mathring{\boldsymbol{\sigma}}_a(\boldsymbol{x}) + \boldsymbol{p}(\boldsymbol{x})).$$

Since the remaining integral must vanish for all $t \geq \tau(\boldsymbol{x})$, we obtain the incremental bulk condition

$$\mathrm{div}\dot{\boldsymbol{\sigma}}(\boldsymbol{x},t) = \boldsymbol{0}$$

for all t.

VI. INCREMENTAL INCOMPATIBILITY

We start with the expression

$$\boldsymbol{\epsilon}(\boldsymbol{x},t) = \mathbb{C}^{-1}\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathring{\boldsymbol{\epsilon}}(\boldsymbol{x}) + \int_{\tau(\boldsymbol{x})}^{t} \dot{\boldsymbol{\epsilon}}(\boldsymbol{x},s)\,ds$$

where we have set $\mathring{\boldsymbol{\epsilon}}(\boldsymbol{x}) = \mathbb{C}^{-1}\mathring{\boldsymbol{\sigma}}(\boldsymbol{x})$ and where $\boldsymbol{\epsilon}(\boldsymbol{x},s) = \mathbb{C}^{-1}\dot{\boldsymbol{\sigma}}(\boldsymbol{x},t)$. Then,

$$\mathrm{curl}\left(\int_{\tau(\boldsymbol{x})}^{t} \dot{\boldsymbol{\epsilon}}(\boldsymbol{x},s)\,ds\right) = \int_{\tau(\boldsymbol{x})}^{t} \mathrm{curl}\dot{\boldsymbol{\epsilon}}(\boldsymbol{x},s)\,ds - \nabla\tau(\boldsymbol{x}) \times \dot{\boldsymbol{\epsilon}}^{\mathsf{T}}(\boldsymbol{x},\tau(\boldsymbol{x})).$$

and

$$\mathrm{curlcurl}\left(\int_{\tau(\boldsymbol{x})}^{t} \dot{\boldsymbol{\epsilon}}(\boldsymbol{x},s)\,ds\right) = \int_{\tau(\boldsymbol{x})}^{t} \mathrm{curlcurl}\dot{\boldsymbol{\epsilon}}(\boldsymbol{x},s)\,ds - \mathrm{curl}\left[\nabla\tau(\boldsymbol{x}) \times \dot{\boldsymbol{\epsilon}}(\boldsymbol{x},\tau(\boldsymbol{x}))\right] - \nabla\tau(\boldsymbol{x}) \times \left[\mathrm{curl}\dot{\boldsymbol{\epsilon}}(\boldsymbol{x},t)\right]^{\mathsf{T}}_{t=\tau(\boldsymbol{x})}.$$

Since incremental elastic strains are assumed compatible, the term $\mathrm{curlcurl}\dot{\boldsymbol{\epsilon}}(\boldsymbol{x},s)$ vanishes and by making use of the additive structure of $\boldsymbol{\epsilon}(\boldsymbol{x},t)$ we obtain Eq. (4) in the main text. The derivation above can be also given in components if we observe that

$$(\mathrm{curlcurl}\boldsymbol{\epsilon})_{ij} = e_{ikl}e_{jmn}\varepsilon_{ln,km}$$

where e_{ijk} is the Levi-Civita symbol. Then by using

$$\varepsilon_{ln}(\boldsymbol{x},t) = \mathring{\varepsilon}_{ln}(\boldsymbol{x}) + \int_{\tau(\boldsymbol{x})}^{t} \dot{\varepsilon}_{ln}(\boldsymbol{x},s)\,ds$$

and

$$e_{ikl}e_{jmn}\dot{\varepsilon}_{ln,km} = 0,$$

we obtain the component-version of the surface compatibility Eq. (4)

$$e_{ikl}e_{jmn}\left[\mathring{\varepsilon}_{ln,km}(\boldsymbol{x}) - \dot{\varepsilon}_{ln,k}(\boldsymbol{x},\tau(\boldsymbol{x})) - \left(\dot{\varepsilon}_{ln}(\boldsymbol{x},\tau(\boldsymbol{x}))\tau_{,k}(\boldsymbol{x})\right)_{,m}\right] = \eta^{p}_{ij}. \quad (1)$$

VII. POLAR SYMMETRY

In polar symmetry and plane strain, the requirement that two solids have the same regular and singular contributions to incompatibility is equivalent to a single scalar condition. Suppose that polar coordinates $\boldsymbol{x} = (r,\theta,z)$ are associated with the local basis $\{\boldsymbol{e}(\theta),\boldsymbol{\nu}(\theta),\boldsymbol{k}\}$ which includes radial, hoop and axial vectors. We obtain that

$$\mathrm{curl}\boldsymbol{\epsilon}(\boldsymbol{x}) = \varphi(r)\boldsymbol{k}\otimes\boldsymbol{\nu}(\theta),$$

where we have set $\varphi = {\varepsilon_\theta}' + (\varepsilon_\theta - \varepsilon_r)/r$. The regular part of the incompatibility reads

$$\boldsymbol{\eta}^{c} = \eta^{c}\boldsymbol{k}\otimes\boldsymbol{k}$$

with $\eta^{c}(r) = (\varphi(r)\,r)'/r$. Therefore the two states $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ have the same continuous incompatibility if $(\varphi_1(r)\,r)' = (\varphi_2(r)\,r)'$ for all r. Now observe that by taking γ_z, an arbitrary circle of radius z, we can write

$$\oint_{\gamma_z}(\mathrm{curl}\boldsymbol{\epsilon})d\boldsymbol{y} = 2\pi z\varphi(z)\boldsymbol{k}.$$

So for a hollow disk with internal radius r_i, the condition $\Omega_1 = \Omega_2$ implies $\varphi_1(r_i) \equiv \varphi_2(r_i)$. Hence , to ensure that $\eta^c_1(r) = \eta^c_2(r)$ we must require that $\varphi_1(r) \equiv \varphi_2(r)$ for all r .

VIII. FEM SIMULATION

We have used the open source FreeFem++ [2]. Residual stresses and displacements in sliced bodies were obtained by using the weak formulation of the equilibrium equations:

$$\int_R \mathbb{C}\boldsymbol{\epsilon}\cdot\mathrm{sym}\nabla\boldsymbol{v} = 0 \quad (2)$$

where $\boldsymbol{\epsilon} = \mathrm{sym}\nabla\boldsymbol{u} - \boldsymbol{\epsilon}^{\mathrm{pl}}$, $\boldsymbol{u}$ is the displacement and $\boldsymbol{\epsilon}^{\mathrm{pl}}$ is the inelastic strain. We denoted by $\boldsymbol{v}$ a test displacement field.

Denote the radial and hoop components of stress $\boldsymbol{\epsilon}^{\mathrm{pl}}$ by (γ_r,γ_θ) and the corresponding components of strain $\boldsymbol{\epsilon}$ by $(\varepsilon_r,\varepsilon_\theta)$. Then $u' = \varepsilon_r + \gamma_r$ and $u/r = \varepsilon_\theta + \gamma_\theta$. Since in the case of polar symmetry the incompatibility is fully controlled by

$$\varphi = {\varepsilon_\theta}' + (\varepsilon_\theta - \varepsilon_r)/r = -({\gamma_\theta}' + (\gamma_\theta - \gamma_r)/r),$$

we can set $\gamma_\theta = 0$ so that $\gamma_r = r\varphi$. Any distribution of inelastic strains with the same φ produces identical residual stresses in the body. For the target condition of uniform hoop stress, we use

$$\varphi(r) = \frac{2\mu+\lambda}{4\mu(\mu+\lambda)}\frac{p\,r_i r_f}{r^2(r_f - r_i)}.$$

whereas for the target $\eta^c_p = 0$ we use

$$\varphi = c/r$$

with c an arbitrary constant. To emphasize the effect we amplified our infinitesimal displacements by assuming that the Young modulus is $E = 0.2p$ where p is the pressure in physiological conditions, and assumed that $\nu = 0.33$.

IX. EXPERIMENTS ON NATURAL AND MANUFACTURED ARTERIES

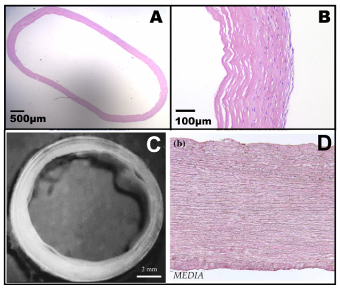

FIG. 1.

In figure Fig.1 we report a comparison between real human arteries and manufactured ones. Insets A,B, taken from Ref.[30] in the manuscript, refer to arteries that are manufactured through the wind-rolling of sheets of mesenchymal cells. Insets C,D, taken from Ref.[51] in the manuscript, refer to real human arteries. The comparison clearly shows that wind-rolling reproduces the inherently layered structure of real arteries.

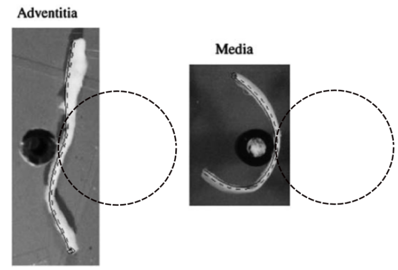

FIG. 2.

In Fig.2 we show the opening angles of the media and of the adventitia layers, taken from Ref.[51] of the manuscript. The former has an higher opening angle relative to the latter, which is consistent with the FEM results shown in Fig.3 of the manuscript.

[1] Maggiani G.B., Scala R., Van Goethem N., Math.Meth.Appl.Sci. doi: 10.1002/mma.3450 (2015)

[2] Hecht, F. J. Numer. Math. 20, 251–265 (2012).

[3] Boley A.B., Weiner J.H., Theory of thermal stresses, Dover Publications Inc., (1997)