Supplemental Material for the paper "Rigidity-controlled crossover: from spinodal to critical failure"

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To obtain the avalanche distribution in our generalized FBM problem with controlled *length*, we follow the general methodology largely developed by Hansen and collaborators in their studies of the classical FBM problem which implies control of the *force* [1–5].

Metastable states. First, we use the condition $\partial_X \mathcal{H} = 0$ to obtain $X(\boldsymbol{x}, \varepsilon) = \frac{1}{\lambda+\Lambda}\Big(\Lambda\varepsilon + \lambda\frac{1}{N}\sum_{i=1}^{N} x_i\Big)$, and the condition $\partial_{x_i}\mathcal{H} = 0$ to obtain $u'(x_i) = \lambda(X - x_i)$. In view of permutational invariance, we can characterize the microscopic state by the number of broken bonds, k, which gives

$$\hat{X}(k,\varepsilon) = \frac{(1+\lambda)\Lambda\varepsilon}{\lambda(1-k/N)+\lambda\Lambda+\Lambda}. \tag{1}$$

For the attached links we have

$$\hat{x}_0(k,\varepsilon) = \frac{\lambda\Lambda\varepsilon}{\lambda(1-k/N)+\lambda\Lambda+\Lambda}, \tag{2}$$

and for the broken links

$$\hat{x}_1(k,\varepsilon) = \frac{(1+\lambda)\Lambda\varepsilon}{\lambda(1-k/N)+\lambda\Lambda+\Lambda}. \tag{3}$$

The energy of the equilibrium configurations can be written as

$$\mathcal{H}(k,\varepsilon) = a_k\varepsilon^2 + S_k, \tag{4}$$

where $a_k = \dfrac{1}{2}\dfrac{\lambda\Lambda(N-k)}{\lambda(N-k)+N(\lambda\Lambda+\Lambda)}$, and S_k is the energy of the broken bonds. If $\bar{x}_i, i = 1, ..., N$ is the ordered sequence of failure thresholds, $\bar{x}_1 \le \bar{x}_2 \le \cdots \le \bar{x}_N$, we can write $S_k = \dfrac{1}{N}\displaystyle\sum_{i=1}^{k}\frac{\bar{x}_i^2}{2}$, and $S_0 = 0$. We observe that a_k is a (strictly) monotonically decreasing sequence while S_k is a (strictly) monotonically increasing sequence.

The stress-strain relation for a microscopic state characterized by the parameter k is

$$\sigma(k,\varepsilon) = \frac{\partial\mathcal{H}(k,\varepsilon)}{\partial\varepsilon} = \frac{\lambda\Lambda(N-k)\varepsilon}{\lambda(N-k)+N(\lambda\Lambda+\Lambda)}. \tag{5}$$

Each value of k defines an equilibrium branch extending between the two limits induced by the inequalities $\hat{x}_0(k,\varepsilon) < \bar{x}_k$ and $\hat{x}_1(k,\varepsilon) > \bar{x}_k$. For the failure thresholds we can then write

$$\varepsilon_k^f = \frac{\lambda+1}{\lambda}\left[\left(1-\frac{k}{N}\right)\nu + 1\right]\bar{x}_k, \tag{6}$$

where $0 \le k < N$. Similar expressions can be obtained for the rebuilding thresholds

$$\varepsilon_k^r = \left[\left(1-\frac{k}{N}\right)\nu + 1\right]\bar{x}_k, \tag{7}$$

where $0 < k \le N$. The ensuing equilibrium branches are represented by the gray lines in Fig. 1 (b, c) in the main text.

To analyze their (local) stability, we need to study the positive definiteness of the Hessian matrix for the energy $\mathcal{H}(\boldsymbol{x}, X)$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} M_1 & 0 & \dots & 0 & -\lambda \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & M_N & -\lambda \\ -\lambda & \dots & \dots & -\lambda & N(\lambda+\Lambda) \end{pmatrix}, \tag{8}$$

where M_i is either $\lambda + 1$, for $1 \le i < N - k$, or λ, for $N - k \le i \le N$. The sufficient condition for stability is that all the principal minors of $\mathcal{M}$ are positive. The first N minors are just the product of diagonal therms are therefore always positive. The last principal minor, the determinant

$$\det(\boldsymbol{\mathcal{M}}) = \prod_{i=1}^{N} M_i \sum_{i=1}^{N}\left(\lambda + \Lambda - \frac{\lambda^2}{M_i}\right). \tag{9}$$

is also positive implying stability of the obtained equilibrium configurations; the unstable configurations must contain at least one element in the spinodal state represented in our model by a single point.

Equilibrium (global minimum) path. For large N we can write

$$S_k = \frac{1}{N}\sum_{i=0}^{k}\frac{\bar{x}_i^2}{2} \approx \int_{\bar{x}_1}^{\bar{x}_k}\frac{x^2}{2}f(x)dx, \tag{10}$$

where we used the fact that for ordered distributions we can use the approximation $k/N \sim F(\bar{x}_k)$ [6]. We can

then write the continuous approximation of the discrete energy in the form

$$\mathcal{H}(x,\varepsilon) = \frac{\lambda\Lambda(1-F(x))}{\lambda(1-F(x))+\Lambda(\lambda+1)}\frac{\varepsilon^2}{2} + \int_0^x f(x')\frac{x'^2}{2}dx'. \tag{11}$$

Using the equilibrium condition $\partial\mathcal{H}(x,\varepsilon)/\partial x = 0$, and applying it for the discrete values $\bar{x}_k$, we obtain

$$\varepsilon_k^g = \frac{1}{\sqrt{\Lambda\nu}}\left[\left(1-\frac{k}{N}\right)\nu+1\right]\bar{x}_k. \tag{12}$$

Note that the three formulas (6), (7) and (12) are different only by constant multipliers.

Out-of-equilibrium (zero viscosity limit) path. Each microscopic configuration characterized by parameter k exists in an extended domain of the loading parameter ε between the failure strain ε_k^f and the rebinding strain ε_k^r. At large N, we can use the approximation

$$\bar{\varepsilon}_f(x) = \frac{\lambda+1}{\lambda}\left[\big(1-F(x)\big)\nu+1\right]x. \tag{13}$$

and $\bar{\sigma}_f(x) = [1-F(x)]x$. Similarly, along the reverse path,

$$\bar{\varepsilon}_r(x) = \left[\big(1-F(x)\big)\nu+1\right]x. \tag{14}$$

and $\bar{\sigma}_r(x) = \frac{\lambda}{\lambda+1}[1-F(x)]x$. Both, equilibrium and out of equilibrium (averaged) stress-strain relations are illustrated in Fig. 1.

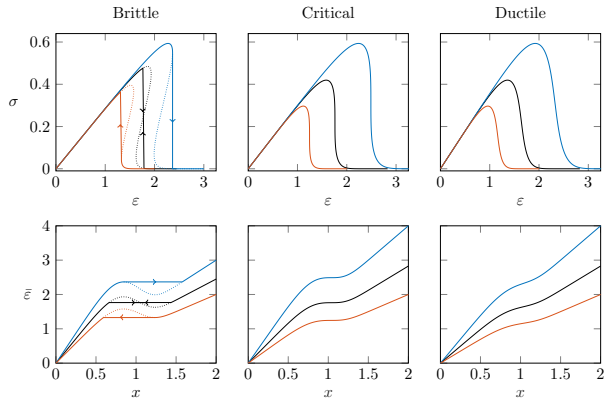

Figure 1. First row: avergaed stress-strain relations , second row: averaged strain dependence on the internal variable x; blue (red) curves correspond to the loading (unloading) out-of-equilibrium paths; black curves correspond to the equilibrium (global minimum) path.

Brittle to ductile transition. It is easier to see if the system is brittle if we consider the out-of-equilibrium (marginal stability) path, even though the actual ductility threshold would be the same if we consider the global minimum path. All we need to check is the condition that the curve $\bar{\varepsilon}(x)$ has a local maximum, which reads $[1-F(x_c)] - f(x_c)x_c + \nu^{-1} = 0$. To locate the brittle to ductile transition we need to find the inflection point on the curve $\bar{\varepsilon}(x)$ characterized by the condition $-2f(x_c) - f'(x_c)x_c = 0$.

In the case of Weibull distribution, we obtain from the first of these two conditions

$$x_c = \left[\frac{1}{\rho} - W\left(-\frac{\exp(1/\rho)}{\rho\nu}\right)\right]^{1/\rho}, \tag{15}$$

where $W(x)$ is the Lambert function, defined through the equation $x = W(x)e^{W(x)}$. Then the second condition gives $\nu = e^{\frac{1}{\rho}+1}/\rho$, which delineates the boundary between brittle and ductile regimes.

Statistics of avalanches. We first compute the avalanche distribution for the case of the *out-of-equilibrium* loading path; it will be clear that the same procedure can be adapted for the *out-of-equilibrium* unloading path and for the reversible *equilibrium* paths.

For an avalanche of size s to take place along the *out-of-equilibrium* loading path and be associated with the failure of kth (in strength) spring, we must have $\varepsilon_{k+j} \leq \varepsilon_k$, for $j = 1, 2, \ldots, s-1$, and $\varepsilon_{k+s} > \varepsilon_k$, which, following [3], we called the *forward condition*; to secure that ε_k is larger than previous thresholds, we must also require that $\varepsilon_j \leq \varepsilon_k$, for all $j < k$, which we call the *backwards condition*. Given that the rebinding sequence for the unloading out-of-equilibrium path is $\varepsilon_k^r = \frac{\lambda}{\lambda+1}\varepsilon_k$ and for the *equilibrium* path is $\varepsilon_k^g = \sqrt{\frac{\lambda}{\lambda+1}}\varepsilon_k$, the avalanche condition in those two cases and the ensuing avalanche statistics will be the same as in the case of out-of-equilibrium loading path, so it is sufficient to deal with this case only.

Since we are interested in the asymptotics for the avalanche distribution at large N, we assume that $s \ll N$. Using the ordered thresholds $\bar{x}_i$, and Eq. (6) for the sequence ε_k, we can then rewrite $\varepsilon_{k+j} \gtrless \varepsilon_k$ in the form

$$\bar{x}_{k+j} \gtrless \bar{x}_k\left(1+\frac{j}{N-k-j+N\nu^{-1}}\right). \tag{16}$$

Defining $\delta_k = \dfrac{\bar{x}_k}{N-k+N\nu^{-1}}$, and using the assumption that $j \ll N-k$, we can simplify these relation further

$$\bar{x}_{k+j} \gtrless \bar{x}_k + j\delta_k. \tag{17}$$

Note next that breaking of one spring at the elongation ε_k, corresponding to a threshold $\bar{x}_k = x$, raises the load on the remaining fiber by δ_k. The average number of fibers that breaks as a result of this load increase is equal to the number of thresholds in the interval $(x, x+\delta_k)$, which is $Nf(x)\delta_k$. Thus, the average number of fibers

breaking as a result of the failure of the kth fiber is,

$$g(x) = \frac{f(x)x}{1 - F(x) + \nu^{-1}}, \tag{18}$$

where we again used the approximation $k/N \sim F(x)$ [6]. For an avalanche of size s, the increase in load will be approximately $s\delta_k$, which leads to $g(x)s$ broken springs. The (forward) probability that the additional $s-1$ springs break is then given by a Poisson distribution with the rate $g(x)s$,

$$\tilde{p}_f(s,x) = \frac{(g(x)s)^{s-1}}{(s-1)!} e^{-g(x)s}. \tag{19}$$

To complete this expression, we still need to secure the condition stating that all the $s-1$ inequalities $\bar{x}_{k+1} < x + \delta_k$, $\bar{x}_{k+2} < x + 2\delta_k$, $\cdots$, $\bar{x}_{k+s-1} < x + (s-1)\delta_k$ are satisfied. To this end, we divide the interval $(x, x + s\delta_k)$ into s sub-intervals of size δ_k. For our condition to be satisfied, we must have at least one threshold value in the first interval $(x, x + \delta_k)$, at least two in the first two intervals, and at least $s-1$ in the first $s-1$ intervals. To ensure that $\varepsilon_{k+s} > \varepsilon_k$, there should be no threshold values in the last interval $(x + (s-1)\delta_k, x + s\delta_k)$. It can be shown that such combinatorial problem can be solved giving $p[s-1, s] \sim \frac{1}{s}$, [3]. We can then write the probability that the forward condition is satisfied in the form

$$p_f(s,x) = \tilde{p}_f(s,x)p[s-1,s] = \frac{(g(x)s)^{s-1}}{s!} e^{-g(x)s}. \tag{20}$$

We still need to satisfy the backward condition that the threshold ε_k is necessarily bigger than its predecessors. To find the corresponding (backward) probability, we consider a finite number n of such elements, $k-1, k-2, \ldots, k-n$ and search for the condition that $\bar{x}_{k-1} < x - \delta_k$, $\bar{x}_{k-2} < x - 2\delta_k$, $\cdots$, $\bar{x}_{k-n} < x - n\delta_k$. If there are no thresholds in $(x - \delta_k, x)$, at most one in $(x - 2\delta_k, x)$, at most two in $(x - 3\delta_k, x), \ldots$, and at most $n-1$ in $(x - n\delta_k, x)$, then all our inequalities are fulfilled. This implies that the number m, not exceeding $n-1$, must be in the interval $(x - n\delta_k, x - \delta_k)$, while all the remaining $k-1-m$ thresholds must be smaller than $x - n\delta_k$. The corresponding probability is given again by a Poisson distribution,

$$\tilde{p}_b(s) = \frac{(g(x)n)^m}{m!} e^{-g(x)n}. \tag{21}$$

We can now compute the probability that m thresholds are randomly distributed among these n intervals such that no threshold value lies in the interval $(x - \delta_k, x)$, at most one in the interval $(x - 2\delta_k, x - \delta_k)$, at most two in the interval $(x - 3\delta_k, x - 2\delta_k)$, and so on. This is again a combinatorial problem whose solution is $p[m, n] \sim 1 - \frac{m}{n}$ [3]. The probability for the backwards condition to be fulfilled is

$$\begin{aligned} p_b(s,x) &= \tilde{p}_b(s,x)p[m,n] \\ &= e^{-g(x)n} \sum_{m=0}^{n-1} \frac{\big(g(x)n\big)^m}{m!n} (n-m). \end{aligned} \tag{22}$$

Rearranging the summation in Eq. (22), we can re-write it as,

$$\begin{aligned} p_b(s,x) = (1-g)\, e^{-g(x)n} \sum_{m=0}^{n-1} \frac{\big(g(x)n\big)^m}{m!} \\ + e^{-g(x)n} \frac{\big(g(x)n\big)^n}{n!}. \end{aligned} \tag{23}$$

In the limit $n \to \infty$, we have $\sum_{m=0}^{n-1} \frac{(g(x)n)^m}{m!} \to e^{g(x)n}$; and with the use of the Stirling approximation, $n! \approx n^n e^{-n}\sqrt{2\pi n}$, we can show that the last term in Eq. (23) vanishes for $g \leq 1$. The (backwards) probability is then,

$$p_b(s,x) = 1 - g(x). \tag{24}$$

The probability of the avalanche of size s starting at the element k with the threshold value $x_k = x$ can be now written as the product of the forward (20) and the backward (24) probabilities,

$$p(s,x) = \frac{s^{s-1}}{s!} g(x)^{s-1} e^{-sg(x)} (1 - g(x)). \tag{25}$$

The final expression for the integrated avalanche distribution takes the form

$$p(s) = \frac{s^{s-1}}{s!} \int_0^{x_c} \phi(x) e^{[-g(x) + \ln g(x)]s} dx, \tag{26}$$

where $\phi(x) = \big[1 - g(x)\big] \dfrac{f(x)}{g(x)}$, and x_c is the maximum of the averaged curve $\varepsilon(x)$.

Asymptotic analysis. We now focus on the tail of the distribution $p(s)$ assuming that $N \to \infty$. We use the saddle-point approximation, which implies that the main contribution to the integral will come from the vicinity of $x = x_0$, where the function $h(x) = g(x) - \ln g(x)$ reaches its global minimum. To find x_0, we need to solve the equation $h'(x) = \frac{g'(x)}{g(x)}(g(x) - 1) = 0$. There are three possibilities,

1. $g(x_0) \neq 1$ and $g'(x_0) = 0$ (ductile regime),
2. $g(x_0) = 1$ and $g'(x_0) = 0$ (critical regime)
3. $g(x_0) = 1$ and $g'(x_0) \neq 0$ (brittle regime).

If $g(x_0) \neq 1$, and $g'(x_0) = 0$, we can write, $h(x) \approx g(x_0) - \ln g(x_0) + \frac{g''(x_0)}{2g(x_0)}(g(x_0) - 1)(x - x_0)^2$. Then using the saddle-point approximation in (26), and applying the

Stirling approximations $s! \approx s^s e^{-s}\sqrt{2\pi s}$, we obtain

$$p(s) = \frac{s^{s-1}}{s!} e^{-sh(x_0)} \phi(x_0) \sqrt{\frac{2\pi}{s|h''(x_0)|}} \sim s^{-2} e^{-s(h(x_0)-1)}. \tag{27}$$

When simultaneously $g(x_0) = 1$ and $g'(x_0) = 0$ we have $h''(x_0) = 0$, and $h'''(x_0) = 0$; therefore the Taylor expansion is $h(x) \approx 1 + \frac{3g''(x_0)^2}{4!}(x - x_0)^4$. We can also write $\phi(x) \approx -\frac{f(x_0)g''(x_0)}{2}(x - x_0)^2$, which allows us to re-write the integral (26) in the form,

$$p(s) = \frac{s^{s-1}e^{-s}}{s!} \int_0^{x_0} -f(x_0)g''(x_0)(x - x_0)^2 \times e^{-s\frac{3g''(x_0)^2}{4!}(x-x_0)^4} dx. \tag{28}$$

Computing the integral explicitly and using Stirling's approximation we obtain $p(s) \sim s^{-9/4}$.

In the brittle regime we need to consider separately equilibrium and out of equilibrium paths.

Consider first the out-of-equilibrium path. We need to expand the function $h(x) = g(x) - \ln g(x)$ up to second order to obtain $h(x) \approx 1 + \frac{g'^2(x_0)}{2}(x - x_0)^2$. We can also expand $\phi(x)$ to obtain $\phi(x) \approx -g'(x_0)f(x_0)(x - x_0)$. These expansions allow us to approximate the integral (26) by

$$p(s) = \frac{s^{s-1}}{s!} e^{-s} \int_0^{x_0} g'(x_0)f(x_0)(x_0 - x)e^{-s\frac{g'(x)^2}{2}(x-x_0)^2} dx. \tag{29}$$

Along the out-of -equilibrium path, the avalanches are counted up to $x = x_0$; and if we compute the integral explicitly, and use the Stirling approximations, we obtain $p(s) \sim s^{-5/2}$.

Consider now the equilibrium path. The actual equilibrium SNAP event takes place at some $x_* < x_0$, given by the Maxwell construction. The counting of avalanches should be then performed only up to the point x_*, and in the integral (26), we must put $x_c = x_*$. The function $h(x)$ will attain its minimum in the boundary point x_*, which is the upper limit of integration. In such case, the following asymptotic representation holds at $N \to \infty$ [7]

$$\int_{x_{inf}}^{x_{sup}} e^{-Nh(x)} dx \to \frac{e^{Nh(x_*)}}{Nh'(x_*)} \tag{30}$$

This allows to write, $p(s) \sim s^{-5/2} e^{-s(1-h(x_*))}$.

Mapping on RFIM. Using the condition $\partial_X \mathcal{H} = 0$, we obtain $X(\boldsymbol{x}, \varepsilon) = \frac{1}{\lambda+\Lambda}\left(\Lambda\varepsilon + \lambda\frac{1}{N}\sum_{i=1}^N x_i\right)$. If we substitute this expression back into $\mathcal{H}$ we obtain

$$\mathcal{H} = -\frac{1}{N^2}\sum_{i,j} J x_i x_j - \frac{1}{N}\sum_i [H x_i - v_i(x_i)], \tag{31}$$

where $J = \frac{\lambda^2}{2(\lambda+\Lambda)}$, $H = \frac{\lambda\Lambda\varepsilon}{\lambda+\Lambda}$, and

$$v_i(x_i) = u_i(x_i) + x_i^2 + \frac{\lambda\Lambda\varepsilon}{2(\lambda+\Lambda)}.$$

Initial condition for the Burgers equation. In the case of finite N, the equilibrium condition $\partial_{x_i}\mathcal{H} = 0$ allows us to write

$$\mathcal{H}(X, \varepsilon) = \frac{1}{N}\sum_{i=1}^N e_i(X) + \frac{\Lambda}{2}(\varepsilon - X)^2.$$

Here, two metastable branches $e_i = \frac{\lambda}{\lambda+1}\frac{X^2}{2}\Theta(l_i - \frac{\lambda}{\lambda+1}X) + \frac{l_i^2}{2}\Theta(X - l_i)$ are defined in each interval $X \in [l_i, \frac{\lambda+1}{\lambda}l_i]$. If we choose the branch with the minimal energy, the remaining problem reduces to finding

$$\tilde{\mathcal{H}}(\varepsilon, \nu) = \min_{X\in\mathbb{R}}\left\{\frac{1}{2\nu}(\varepsilon - X)^2 + q(X)\right\},$$

where $q(X) = \frac{1}{N}\sum_{i=1}^N \frac{X^2}{2}\Theta(l_i - \sqrt{\frac{\lambda}{\lambda+1}}X) + \frac{\lambda+1}{\lambda}\frac{l_i^2}{2}\Theta(X - \sqrt{\frac{\lambda}{\lambda+1}}l_i)$. The initial data for the associated Burgers equation are

$$\sigma_0 = \partial_\varepsilon q = \frac{1}{N}\sum_{i=1}^N \varepsilon\Theta\left(l_i - \sqrt{\frac{\lambda}{\lambda+1}}\varepsilon\right).$$

In the limit $N \to \infty$ we have $\frac{1}{N}\sum_{i=1}^N \Theta(l_i - \sqrt{\frac{\lambda}{\lambda+1}}X) \sim \int_{\sqrt{\frac{\lambda}{\lambda+1}}X}^\infty f(l)dl$ and $\frac{1}{N}\sum_{i=1}^N \frac{l_i^2}{2}\Theta(X - \sqrt{\frac{\lambda}{\lambda+1}}l_i) \sim \int_0^{\sqrt{\frac{\lambda}{\lambda+1}}X} f(l)\frac{l^2}{2}dl$. Then, in this limit,

$$\tilde{\mathcal{H}}(\varepsilon, \nu) = \min_{X\in\mathbb{R}}\left\{\frac{1}{2\nu}(\varepsilon - X)^2 + q^\infty(X)\right\},$$

where $q^\infty(X) = \sqrt{\frac{\lambda}{\lambda+1}}\int_0^X f(\sqrt{\frac{\lambda}{\lambda+1}}X')(X'^2/2)dX' + [1 - F(\sqrt{\frac{\lambda}{\lambda+1}}X)](X^2/2)$. The initial condition for the associated Burgers equation is $\sigma_0(\varepsilon) = \varepsilon[1 - F(\sqrt{\frac{\lambda}{\lambda+1}}\varepsilon)]$.

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