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From Discrete Visco-Elasticity to Continuum Rate-Independent Plasticity: Rigorous Results

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Abstract

We show that continuum models for ideal plasticity can be obtained as a rigorous mathematical limit starting from a discrete microscopic model describing a visco-elastic crystal lattice with quenched disorder. The constitutive structure changes as a result of two concurrent limiting procedures: the vanishing-viscosity limit and the discrete-to-continuum limit. In the course of these limits a non-convex elastic problem transforms into a convex elastic problem while the quadratic rate-dependent dissipation of visco-elastic lattice transforms into a singular rate-independent dissipation of an ideally plastic solid. In order to emphasize our ideas we employ in our proofs the simplest prototypical system mimicking the phenomenology of transformational plasticity in shape-memory alloys. The approach, however, is sufficiently general that it can be used for similar reductions in the cases of more general plasticity and damage models.

1. Introduction

Continuum models involving rate-independent hysteresis appear in various solid mechanics problems ranging from friction and earthquakes to plasticity and damage. Typically, the associated systems of phenomenological equations contain empirical functions characterizing failure thresholds and the hardening rates. In sharp contrast to elastic moduli, these characteristics of out-of-equilibrium behavior can rarely be rigorously linked to the structure of the underlining small-scale system. The main difficulty originates from the fact that at finite temperature the mesoscopic dissipation is necessarily rate dependent (see for example [44]) while the observed macroscopic dissipation may be rate independent. This means that the correct coarse graining, implying an averaging of the mesoscopic time and space scales, must necessarily involve the basic change of the model structure. We use the notion “mesoscopic” here to indicate that we are interested in inherently discrete phenomena at a scale that is larger than the microscopic level (atomistic length scale) but still smaller that the macroscopic level of a sample.

To be more precise, we need to understand the limiting transition from models with quadratic dissipative potentials of the Onsager type (employed in modeling kinetics of lattice defects) to models with singular dissipative potentials used in the description of rate-independent dissipative processes (continuum plasticity). The main physical ingredients of such a limit were identified in [53], and here we present the first mathematical limit analysis of the corresponding discrete system leading to a system of partial differential equations in space and time.

The foundations of the general phenomenological theory of rate-independent systems have been set in [23,36] (see also [14,18,21,43,45,47]). For our models we need to consider the nonconvex version of the theory which can be treated in the framework of the general concept of *energetic rate-independent systems (ERIS)* introduced in [35,38]. This approach has been already used in the description of fracture [11,12], plasticity [8,9,29], delamination [25,55], damage [4,17,19] and phase transformations [24,38,54,62].

The microscopic models in all these areas rely on the existence of characteristic defects carrying inelastic deformation (for example dislocations, phase boundaries, fracture fronts, etc.) The microscopic dynamics of the individual defects is understood only in simplified settings and their interaction is so complex that a detailed bridge in the sense of rigorous coarse graining of quantum mechanical or atomistic models does not seem feasible. In this situation turning to simple prototypical mesoscopic models, like the extremely schematic one discussed in this paper, still offers mathematical and physical insight. The fundamental value lies in the identification of spatial and temporal scales whose averaging ensures the transition between the two different dissipative mechanisms.

In the framework of plasticity, the microscopic origin of rate-independent dissipation was first studied by using models describing a single particle on a periodic landscape (for example [10,48]). Later, such models were applied to a wide range of rate-independent dissipative phenomena from charge density waves and friction to phase transitions [7,16,20,22,46,58]. The study of discrete chain of particles coupled through bi-stable springs (sometimes also called soft spins) represents the next level of schematization allowing one to model realistic hysteretic behavior without explicitly introducing a periodic landscape [15,39,51,53,63]. Higher-dimensional models with bi-stable or multi-stable springs allow one to go much further and study pinning–depinning transitions, criticality and power-law structure of fluctuations, for example [49,56].

Despite the considerable literature on the subject, only few attempts have been made for bridging the gap between viscous and rate-independent systems by rigorous mathematical analysis. In the simplest *zero-dimensional* case (a single "mass point" in a multi-dimensional configurational space) hysteretic models have been derived in [1,30,57,59] starting from macroscopic potentials with superimposed microscopic periodic perturbations. In the present paper we establish exact convergence results for the simplest *one-dimensional* system (a chain of "mass points" having both configurational and spatial dimensions). Already, here we are confronted with several major mathematical difficulties associated with coarse graining in both space and time, and we expect that some of our techniques are also useful for more general systems. In this relation we mention that somewhat related models for crack propagation, which also rely on the passage from a viscous system to a rate-independent one, have been recently considered in [26,28,40].

In mechanical terms our meso-scale model can be viewed as a quasi-statically driven discrete chain of bi-stable, visco-elastic springs and our goal is to obtain, in the slow-loading limit, a coarse-grained model that is equivalent to continuum rate-independent plasticity. The main ingredient of the discrete model making this reduction possible is the rugged energy landscape incorporating some quenched irregularity. Due to quasistatic external driving, our system remains in a local equilibrium (metastable state) until it is forced to undergo a fast transition from an unstable state to a new metastable state. An important conclusion of our work is that the energy dissipated by viscosity during such fast transitions can be described in the continuum limit by a dissipation potential that is a homogeneous function of degree one. Some formal computations justifying this limit have been presented in [53]. In particular, it was realized that the transition from a viscous to a plastic model must involve simultaneous averaging over the fast relaxational time scale and homogenization over spatial discreteness. In this paper we provide a first rigorous analysis of the discrete dynamical system and show that in order to obtain in the limit a spatially nontrivial rate-independent plasticity it is necessary to introduce disorder. Previously, disorder has been used to obtain hardening and generate realistic inner hysteresis loops for a bi-stable chain viewed as a mass-point system in the high dimensional phase space, see [52].

In mathematical terms, our starting point is a constrained gradient system system of N ordinary differential equations, where the constraint makes the system non-autonomous through time-dependent applied displacement on the boundary (that is, a hard loading device). We first identify two main small parameters which control the dynamics. The parameter δ is the rate of viscous relaxation on the time scale of the loading. This parameter goes to 0 either when driving is quasi-static or when the internal relaxation is fast. The second small parameter $\varepsilon = 1/N$ is the normalized reference length of the bi-stable springs, which prescribes the scale of the inhomogeneity.

By performing a series of numerical experiments (see Section 2.2) we motivate the need of a third parameter that breaks the permutational symmetry arising from the original spatial homogeneity. Indeed, given that the coupling between different elastic units is only through a mean-field, the system with identical springs is highly degenerate and is predisposed to the Neĭshtadt type phenomena [42] of delayed bifurcation. Hence, we introduce random inhomogeneity characterized by its variance r. We fix a finite value $r > 0$ and focus on the double limit: first $\delta \to 0$, then $\varepsilon \to 0$. Our main result is that in this limit the original finite dimensional visco-elastic system converges in some well defined sense to an infinite-dimensional continuum model describing elastoplasticity and exhibiting rate-independent hysteretic behavior.

The constitutive structure changes as a result of two concurrent limiting procedures: the vanishing-viscosity limit and the discrete-to-continuum limit. In the course of these limits a non-convex elastic energy functional, formulated in terms of mesoscopic strains, transforms into a convex elastic energy in terms of two macroscopic variables, namely the macroscopic elastic strain and the averaged phase indicators, which we interpret as plastic strain. Simultaneously, the quadratic rate-dependent dissipation of the visco-elastic lattice, which is given in terms of the rate of mesoscopic strains, transforms into a singular rate-independent dissipation of an ideally plastic solid (given in terms of the rate of the plastic strain). As intermediate constructions we encounter jump discontinuities in time and parametric measure-valued solutions in space. The proof of convergence involves three main steps. The first step is the reduction of a finite-dimensional ODE gradient system to a discrete automaton, which prescribes a truly quasi-static evolution on the time-dependent set of local energy minima. In the second step this automaton is reformulated as an ERIS characterized by an energy functional and a dissipation distance. The third step is the discrete-to-continuum passage in the spirit of Γ-convergence for ERIS, see [37]. Here we exploit the Young measures generated through the disorder and thus are able to pass to the limit in both the energy functional and the dissipation potential.

The paper is organized as follows. In Sections 2 and 3 we set the general dynamic problem for the overdamped ODE system and present the results of direct numerical simulations justifying the subsequent regularization through quenched disorder. We then define the macroscopic variables by embedding the discrete system into $\mathrm{L}^2(\Omega)$ where $\Omega =]0, 1[$ is the reference configuration of a continuum bar. Most of the rigorous analysis is done under the assumption that the spring potential Φ is bi-quadratic, that the total length ℓ of the chain (Dirichlet loading) is piecewise monotone with $|\dot{\ell}(t)| \geqq \lambda > 0$ almost everywhere, and that the body forces g_{ext} are time independent, see (3.7). These assumptions are not essential and are used only to make calculations simpler and the proofs more transparent.

In Section 4 we deal with the vanishing-viscosity limit $\delta \to 0$ for fixed ε. We present careful estimates comparing the viscous solutions to the limiting discrete automaton, which is rate independent. The control of the phase state in the individual springs is possible because both the quenched (that is independent of time) disorder and the dynamics are consistent with a suitable ordering property, see (4.2); this property is analogous to the no-passing property in the depinning dynamics, see [31]. We show that the evolution of ordered states splits into periods of slow conservative evolution along a branch of the family of equilibria and rapid dissipative transitions between different branches. In the zero viscosity limit the dissipative stages can be replaced by jump discontinuities in isolated moments of time. We show that the limiting system can be reformulated as ERIS, and we can take advantage of the fact that the energetic solutions (see Definition (4.3)) are controlled only by the energy functional and the dissipation potential, rather than by their derivatives.

In Section 5 the limit $\varepsilon = 1/N \to 0$ is obtained through embedding the system into $\mathcal{Q} = \mathrm{L}^2(\Omega)^2$ and controlling the joint Young measures for elastic and plastic strains. The convergence to the limiting plasticity model is interpreted in terms of Γ-convergence of energetic rate-independent systems by using the methods developed in [37]. In Section 5.3 we show that in the case of a bi-quadratic potential a more general double limit $(\varepsilon, \delta) \to (0, 0)$ with $\delta \leqq \kappa_+\varepsilon$ for some $\kappa_* > 0$ produces the same plasticity model and we do not expect the restriction $\delta \leqq \kappa_+\varepsilon$ to be sharp.

In Section 6 we return to the case of general (not necessarily bi-quadratic) potentials Φ and general time-dependent body forces. We first study the ordered double limit "$\lim_{\varepsilon\to 0} \lim_{\delta\to 0}$" and present a formal calculation showing how the effective dissipation potential and the effective stored-energy density can be obtained from the microscopic elastic potential and the probability distribution of the quenched disorder. We then sketch the proof of the convergence, heavily relying on the corresponding proofs in the case of bi-quadratic potential. Finally, in Section 6.5 we briefly discuss convergence along more general sequences in the (ε, δ) plane.

2. Setup and Modeling

2.1. Description of the Discrete Mesoscopic Model

Consider a macroscopic interval $[0, 1]$ containing $N-1$ massless particles at the reference positions $x_j^N = j/N$, $j = 1, \dots, N-1$. The boundary points $j = 0$ and $j = N$ are assumed to be controlled and undergoing prescribed displacements. The remaining points are linked in series by N identical bi-stable springs, which we call simply *snap-springs* from now on. The chain is springs is depicted in Fig. 1.

The most important ingredient of the model is the bi-stability of the individual elastic springs. To be more precise we write the normalized elastic energy of the chain in the form

$$\widetilde{\mathscr{E}}(\boldsymbol{e}) = \frac{1}{N}\sum_{j=1}^{N} \boldsymbol{\Phi}(e_j) \quad \text{with } \boldsymbol{e} = (e_1, \dots, e_N) \in \mathbb{R}^N,$$

where e_j is the strain in the jth snap-spring. We assume that the elastic energy of a snap-spring $\Phi : \mathbb{R} \to \mathbb{R}$ is a non-convex two-well potential. This means that the function $\phi = \Phi'$ is decreasing on the interval $\left]e_-, e_+\right[$, which is called *spinodal region*. Moreover, we assume that ϕ is strictly increasing on the two intervals $\left]-\infty, e_-\right[$ and $\left]e_+, \infty\right[$, representing phase "+" and phase "−", respectively (see Fig. 2). We can formally define the corresponding energy wells by setting

$$\sigma_+ := \phi(e_-) > \sigma_- := \phi(e_+).$$

For future convenience we denote by $\psi_{+1} : \left[\sigma_-, \infty\right[\to \left[e_+, \infty\right[$ and $\psi_{-1} : \left]-\infty, \sigma_+\right] \to \left]-\infty, e_-\right]$ the inverse functions of $\phi : \left[e_+, \infty\right[\to \left[\sigma_-, \infty\right[$ and

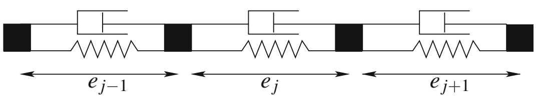

Fig. 1. Viscoelastic chain with "massless points" (*black*), bi-stable springs, and viscous dashpots

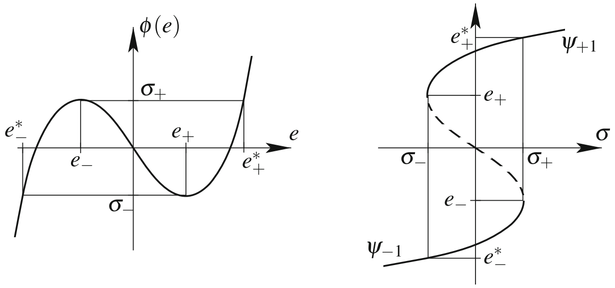

Fig. 2. *Left* non-monotone stress–strain relation $\sigma = \phi(e)$. *Right* two branches ψ_{+1} and ψ_{-1} of the inverse strain–stress relation

$\phi : \left]-\infty, e_-\right] \to \left]-\infty, \sigma_+\right]$, respectively. We also define $e_+^* = \psi_{+1}(\sigma_+) > e_+$ and $e_-^* = \psi_{-1}(\sigma_-) < e_-$ (see Fig. 2).

In what follows a prominent role will be played by a particular bi-quadratic potential

$$\Phi_{\text{biq}}(e) := \frac{k}{2} \min\{(e+a)^2, (e-a)^2\}, \tag{2.1}$$

giving

$$\phi_{\text{biq}}(e) = \begin{cases} k(e+a) & \text{for } e < 0, \\ k(e-a) & \text{for } e > 0. \end{cases}$$

Note that in this case ϕ is not continuous at $e = 0$, where ϕ can take the value either ka or $-ka$. For the bi-quadratic energy Φ_{biq} we find

$$e_\pm = 0, \quad e_\pm^* = \pm 2a, \quad \sigma_\pm = \pm ka, \quad \psi_{\pm 1}(\sigma) = \frac{1}{k}\sigma \pm a.$$

We impose a time-dependent Dirichlet boundary condition (hard device) representing external control of the total average strain $\widetilde{\ell}$, namely

$$\mathscr{C}_\varepsilon(\boldsymbol{e}) := \frac{1}{N}\sum_{j=1}^{N} e_j(\tau) = \widetilde{\ell}(\tau) \quad \text{with } \varepsilon = 1/N \quad \text{and} \quad \boldsymbol{e} = (e_1, \ldots, e_N) \in \mathbb{R}^N. \tag{2.2}$$

In addition, the chain is loaded by a time-dependent macroscopic body force with potential

$$\widetilde{G}(\tau, j/N) = \int_0^{j/N} \widetilde{g}_{\text{ext}}(\tau, y)\,\mathrm{d}y, \tag{2.3}$$

and the total energy of the chain can be written as

$$\widetilde{\mathscr{E}}(\tau, \boldsymbol{e}) = \frac{1}{N}\sum_{j=1}^{N}\Big(\Phi(e_j) + \widetilde{G}(\tau, j/N)\widetilde{e}_j\Big). \tag{2.4}$$

The normalization is such that in the continuum limit (see below) this expression converges to

$$\int_0^1 \Phi(u_x(x)) + \widetilde{G}(\tau, x)u_x(x)\,\mathrm{d}x = \int_0^1 \Phi(u_x(x)) - \widetilde{g}_{\text{ext}}(\tau, x)u(x)\,\mathrm{d}x,$$

if $u(1) = 0$. Note that our Dirichlet conditions impose only $u(\tau, 1) - u(\tau, 0) = \widetilde{\ell}(\tau)$ such that translation invariance allows us to fix any one of the ends.

In the framework of quasi-static elasticity the mechanical problem for the driven chain reduces to the minimization of the energy $\widetilde{\mathscr{E}}(\tau, \boldsymbol{e})$ for each τ. Due to bi-stability of the individual springs the number of critical points is exponentially large with respect to N, which shows that $\widetilde{\mathscr{E}}(\tau, \cdot)$ has a wiggly energy landscape. One can also expect that the corresponding metastable (local minimum) branches $\boldsymbol{e}(\tau)$ are not continuous with respect to the parameter τ. In this situation, knowledge of the dynamics is necessary to uniquely define the evolution of the system.

In the overdamped, viscous case (see [13] for the inertial limit) the dissipative forces are obtained as the differential of a dissipation potential $\mathscr{R}(\dot{\boldsymbol{e}})$. Taking into account the constraint $\mathscr{C}_\varepsilon(\boldsymbol{e}) = \widetilde{\ell}(\tau)$, see (2.2), we find the following abstract force balance:

$$0 = \mathrm{D}_{\dot{\boldsymbol{e}}}\mathscr{R}(\dot{\boldsymbol{e}}) + \mathrm{D}_{\boldsymbol{e}}\widetilde{\mathscr{E}}(\tau, \boldsymbol{e}) - \widetilde{\sigma}(t)\mathrm{D}\mathscr{C}_\varepsilon(\boldsymbol{e}), \qquad \mathscr{C}_\varepsilon(\boldsymbol{e}) = \widetilde{\ell}(\tau). \tag{2.5}$$

Here $\widetilde{\sigma}$ acts as a Lagrange multiplier that determines the strength of the constraint. Note that (2.5) has the form of a constraint gradient flow. From now on, we will restrict our attention to the standard viscosity model with the dissipation potential

$$\mathscr{R}(\dot{\boldsymbol{e}}) = \frac{\nu}{2N}\sum_{j=1}^{N} \dot{e}_j^2,$$

where ν is the viscosity parameter. In this case, the Lagrange parameter $\widetilde{\sigma}$ can be calculated explicitly, viz. We have

$$\widetilde{\sigma}(t) = \frac{1}{N}\sum_{j=1}^{N}\Big(\phi\big(e_j(t)\big) + \widetilde{G}(\tau, j/N)\Big) + \nu\dot{\widetilde{\ell}}(\tau). \tag{2.6}$$

We further assume that the loading rate is *small*, that is,

$$\widetilde{\ell}(\tau) = \ell(\widetilde{\delta}\tau),$$

where $\ell(\cdot)$ is a given smooth function and $\widetilde{\delta}$ is a measure of the loading rate. By introducing the slow time $t = \widetilde{\delta}\tau$ and defining $G(t, j/N) = \widetilde{G}(t/\widetilde{\delta}, j/N)$ and $\mathscr{E}(t, \boldsymbol{e}) = \widetilde{\mathscr{E}}(t/\widetilde{\delta}, \boldsymbol{e})$, system (2.5) yields our final ODE system, namely

$$\left.\begin{aligned} &\delta\dot{e}_j = -\phi(e_j) + \mu_j^N - G(t, j/N) + \sigma(t) \quad \text{for } j = 1, \ldots, N, \\ &\mathscr{C}_\varepsilon(\boldsymbol{e}) = \tfrac{1}{N}\sum_{j=1}^{N} e_j(t) = \ell(t). \end{aligned}\right\} \tag{2.7}$$

The new non-dimensional parameter $\delta = \widetilde{\delta}\nu$ is the ratio of the rate of loading and the rate of viscous relaxation (see also [53]).

In (2.7) we already introduced the components of the external point force vector $(\mu_j^N)_{j=1,\dots,N}$, which will be chosen randomly to destroy the permutational symmetry of the system for the case $G \equiv 0$, see below. Thus, the component μ_j^N represents the stress bias of the jth spring, as now the springs are no longer identical.

2.2. Numerical Experiments

To gain some insight into the behavior of system (2.7) when subjected to quasi-static loading, we perform several numerical experiments. In all these experiments we set $G \equiv 0$ and assume $\phi(e) = e^3 - e$. We also assume that viscosity is small and fix it to $\delta = 0.015$. The initial data are chosen to be randomly distributed around the value $e_j(0) \approx -1.3$, which is well in phase "−". For the loading we choose the simple function $\ell(t) = \min\{\alpha+\beta t, \gamma-\beta t\}$ and study the behavior of the average stress $\widehat{\sigma} = \frac{1}{N}\sum_1^N \phi(\varepsilon_j) = \sigma(t) - \delta\dot{\ell}(t)$ as well as the individual strains e_j as functions of ℓ and of t, respectively.

The first experiment is conducted with a homogeneous chain where all snap-springs are identical, that is $\mu_j^N \equiv 0$. The resulting stress–strain diagram ($\widehat{\sigma}$ versus ℓ) and the strains e_j inside individual snap-springs are shown in Fig. 3. Instead of plasticity-like hysteretic behavior we observe a "snap phenomenon", where a large number of springs transform simultaneously from phase -1 to $+1$ while the rest of the springs relaxes within the phase -1. To understand the unusual stress–strain diagram on the left plot of Fig. 3, one must recall that for this model the spinodal region is between $-1/\sqrt{3} \approx -0.57$ and $1/\sqrt{3} \approx 0.58$. Looking at the right plot of Fig. 3, we start at $t = 0$ with $e_j(0) \approx -1.3$. For $t < 57$ all strains remain approximately identical, even though they have reached the spinodal domain at $t \approx 35$. Only at time $t \approx 57$ is the instability of the homogeneous state realized, when five springs jump to phase $+1$ while the other four relax back. At time $t \approx 80$ the four formerly relaxed springs reach the spinodal region again and then jump to phase $+1$ and join the others. Again the springs synchronize quickly and upon unloading

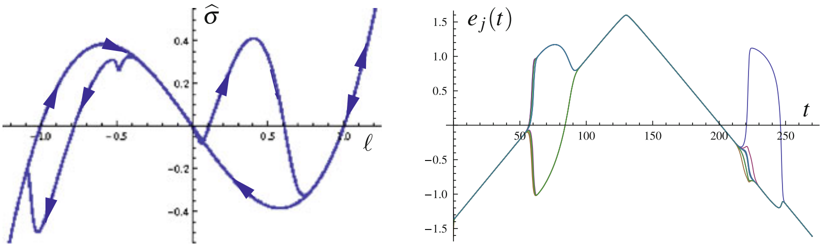

Fig. 3. Simulation of system (2.7) for $N = 9$. *Left* $\widehat{\sigma} = \frac{1}{N}\sum_1^N \phi(e_j)$ versus ℓ. *Right* $e_1, \dots, e_9$ versus t

they jointly move into the spinodal region at $t \approx 177$ but remain on the unstable branch until $t \approx 213$.

We interpret the "snap behavior" as synchronization, which leads to a delayed bifurcation, known as the Neĭshtadt phenomenon [41,42]. Recall that in order to leave the state of unstable equilibrium the system must be perturbed. In the stable regime $\ell(t) < e_-$ the strains $e_j^N(t)$ are always close to the quasistatic equilibrium value $\ell(t)$ and the deviations from the homogeneous state decay exponentially. More precisely, the decay rate is $-\lambda_{\min}/\delta$, where $\lambda_{\min} > 0$ is the smallest eigenvalue of the Hessian of the energy at $\boldsymbol{e} = (\ell, \ldots, \ell)$. Hence, if a solution starts in the stable regime at $t = t_0$ with perturbations of order 1 and reaches the spinodal region at $t = t_1$, the deviations will be of order $e^{-\lambda_{\min}(t_1 - t_0)/\delta}$. Now, the instability of the steady state $\boldsymbol{e}(t) = (\ell(t), \ldots, \ell(t))$ in the spinodal region needs some time to establish itself; the unstable eigenvalue will be of the form $\widehat{\lambda}/\delta$, and to obtain perturbations of order 1 we need to wait until t_2 satisfies $\widehat{\lambda}(t_2 - t_1)/\delta = \lambda_{\min}(t_1 - t_0)/\delta$. The point is that $(t_2 - t_1)/(t_1 - t_0) = \lambda_{\min}/\widehat{\lambda}$ is independent of δ. Recalling $e_\pm = \pm 1/\sqrt{3} \approx \pm 0.58$, Fig. 3 shows that the triple (t_0, t_1, t_2) is given approximately by $(0, 30, 50)$ for the loading phase and by $(100, 170, 210)$ in the unloading phase. Hence, in both cases we have $(t_2 - t_1)/(t_1 - t_0) \approx 0.67$.

To obtain more realistic hysteresis loops, one has to allow for a better separation of the trajectories. One possibility is to replace our visco-elasticity by an environmental viscosity represented by a dashpot connection of each "massless point" to a rigid foundation. Another possibility, which we pursue here, is to break the permutational symmetry of the problem. To this end, in our second numerical experiment we choose nontrivial biases $\mu_j^N = 0.05(j-5)$ in (2.7) while keeping $N = 9$. Such inhomogeneity allows us to generate an unsynchronized response, where each spring transforms at its own critical stress starting from the weakest one, see also [52]. The results are shown in Fig. 4. Now, instead of one big "snap", we observe a series of small "popping events" so that the inhomogeneous system produces realistic plasticity-type behavior (with hardening).

However the plastic deformation in this model (phase transition in our case) propagates through the system in the form of a single front. This is not realistic because we know that (outside very special "easy glide" regimes) plasticity usually develops simultaneously all over the sample. To achieve the stochastic separation

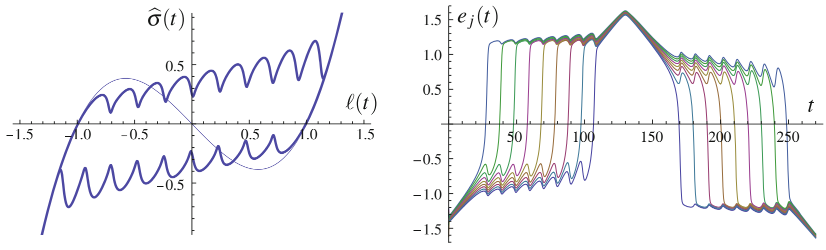

Fig. 4. Stresses and strains for a model with $N = 9$ and linear bias $\mu_j^N = 0.05(j-5)$. *Left* $\widehat{\sigma}$ versus ℓ. *Right* $e_1, \ldots, e_9$ versus t

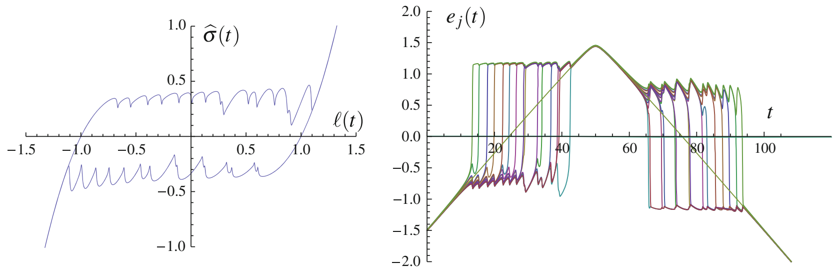

Fig. 5. Simulation of ODE with $N = 15$ and random bias. *Left* $\widehat{\sigma}$ versus ℓ. *Right* e_j versus t

of the trajectories we need to assume that parameters μ_j are stochastically independent.

The results of a numerical loading-unloading test for the case when μ_j are equi-distributed in the segment $[-0.1, 0.1]$ is presented in Fig. 5. We see that the overall behavior of the system is basically the same as in the previous case modulo the dispersion of the "popping events". The important difference, however, is that now the strain distribution inside the sample is no longer monotone and instead becomes strongly oscillatory, making the system macroscopically homogeneous. The ensuing homogeneity at the coarse-grained scale is exactly the property which is necessary to obtain a nontrivial continuum limit.

3. General Description of the Main Results

To formulate the main results we first specify the random nature of the biases μ_j^N representing quenched disorder. We assume that all μ_j^N are random variables that are identically and independently distributed according to the probability density $f \in \mathrm{L}^1(\mathbb{R})$, which satisfies the following natural constraints

$$f \geqq 0, \int_{\mathbb{R}} f(\mu)\,\mathrm{d}\mu = 1, \int_{\mathbb{R}} \mu f(\mu)\,\mathrm{d}\mu = 0, \text{ and } \quad r^2 = \int_{\mathbb{R}} \mu^2 f(\mu)\,\mathrm{d}\mu > 0. \tag{3.1}$$

Thus, our dynamical system (2.7) now depends on the three non-dimensional parameters

$$\varepsilon = 1/N > 0, \quad \delta > 0, \quad \text{and} \quad r > 0,$$

where ε is the discreteness level, $\delta > 0$ is the normalized viscosity, and $r > 0$ measures the strength of the quenched disorder. As our numerical experiments suggest, one can expect to obtain a macroscopic continuum rate-independent plasticity model only in a certain triple limit of the form $(\varepsilon, \delta, r) \to (0, 0, 0)$.

We have seen that the limit $r \to 0$ at fixed ε, δ may lead to "snap behavior", and the subsequent driving ε and δ to 0 does not save the situation. To obtain "pop behavior" we need first to assume that $r > 0$ and consider the limit $(\varepsilon, \delta) \to (0, 0)$,

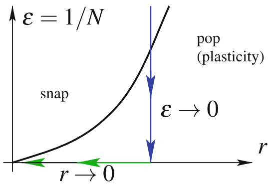

Fig. 6. Schematic phase diagram in the space of small parameters indicating location of the 'popping' domain which we associate with rate-independent plasticity response. The '*pop*' domain is separated from the '*snap*' domain by a crossover line

see Fig. 6. We can then continue along the parametric path $r \to 0$ leading to ideal plasticity limit.

At fixed $r > 0$, as is assumed for all our rigorous results, one can find for each ε and δ a set of solutions of the microscopic problem $\boldsymbol{e}^{\varepsilon,\delta} : [0, T] \to \mathbb{R}^N$, where $\boldsymbol{e}^{\varepsilon,\delta}(t) = (e_j^{\varepsilon,\delta}(t))_{j=1,\ldots,N}$. In the vanishing viscosity limit $\delta \to 0$ the solutions $\boldsymbol{e}^{\varepsilon,\delta}(t)$ can be expected to stay close to elastic equilibrium most of the time. The corresponding elastic problem reduces to solving the equations

$$0 = -\phi(e_j) + \mu_j^N - G(t, j/N) + \sigma^N(t) \quad \text{for } j = 1, \ldots, N, \quad \mathscr{C}_\varepsilon(\boldsymbol{e}) = \ell(t). \tag{3.2}$$

Since Φ has a double-well structure, that is $\phi = \Phi'$ is non-monotone, there may be many equilibria. In fact, for $\ell(t) \in \left]e_-, e_+\right[$ the number of equilibria is of the order 2^N, giving rise to a complex energy landscape.

The challenge now is to determine the limiting trajectory $\boldsymbol{e}^{\varepsilon,0} : [0, T] \to \mathbb{R}^n$ through the set of all local minima (metastable equilibria). To facilitate the bookkeeping we introduce the phase indicators

$$z_j = \operatorname{sign}(e_j) \in \{-1, 0, 1\},$$

specifying three individual sheets of the inverse function $\psi_{z_j}(\cdot)$ (two stable phases and the spinodal region, see Fig. 2). In terms of these new spin-type variables we can rewrite (3.2) explicitly as

$$e_j = \psi_{z_j}\big(\sigma^N(t) + \mu_j^N - G(t, j/N)\big). \tag{3.3}$$

The phase indicators identify individual branches of the equilibrium stress–strain relation and, if the solution remains close to a particular branch, the phase indicators remain unchanged. The spin variables z_j are the discrete precursors of continuum plastic strain variables, which we introduce in the next section. One can see that if the phase configuration $(z_j)_{j=1,\ldots,N}$ is given, the elastic strains e_j can be easily recovered from the solution of the monotone system (3.3).

In Section 4 we first establish the convergence $\boldsymbol{e}^{\varepsilon,\delta}(t) \to \boldsymbol{e}^{\varepsilon,0}(t)$ for $\delta \to 0$ and characterize the path $\boldsymbol{e}^{\varepsilon,0} : [0, T] \to \mathbb{R}^N$ as the solutions of an integer valued discrete automaton (see also [49,50,56]). Here we use the crucial observation that all the relevant equilibria are given by special phase indicators in $\{-1, 1\}^N$, see the "ordered states" discussed in Section 4.1.

To obtain the limit $\varepsilon = 1/N \to 0$ (see Section 5) we use the fact that due to the quenched disorder the phase indicators $z^{\varepsilon,\delta}(t) \in \{-1, 0, 1\}^N$ and, consequently, the strains $e^{\varepsilon,\delta}(t) \in \mathbb{R}^N$ fluctuate spatially in a random fashion. The independence of the random choices at different spatial points leads (due to central limit theorem) to controllable properties of the mean values and thus allows one to construct a coarse-grained theory and explicate the macroscopic properties.

More precisely, we assume that the quantities varying at the scale ε are mesoscopic, while those varying at the scale 1 are macroscopic. To define the macroscopic averages we first need to introduce a spatial averaging operator. We begin by embedding the solutions $e \in \mathbb{R}^N$ into $\mathrm{L}^2(\Omega)$ via the characteristic functions

$$\chi_j^N : \Omega \to \mathbb{R}; \quad x \mapsto \begin{cases} 1 & \text{for } x \in](j-1)/N, j/N[\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4a}$$

This allows us to define the elastic strain field $\overline{e}^{\varepsilon,\delta} \in \mathrm{L}^2(\Omega)$ as follows

$$\overline{e}^{\varepsilon,\delta}(t, x) = \sum_{j=1}^{N} e_j^{\varepsilon,\delta}(t)\chi_j^N(x). \tag{3.4b}$$

Similarly, we introduce a continuum phase indicator (plastic strain) $\overline{z}^{\varepsilon,\delta} \in \mathrm{L}^2(\Omega)$ via

$$\overline{z}^{\varepsilon,\delta}(t, x) = \sum_{j=1}^{N} \widehat{s}(e_j^{\varepsilon,\delta}(t))\chi_j^N(x), \quad \text{where } \widehat{s}(e) = \begin{cases} -1 & \text{for } e \leqq e_-, \\ 0 & \text{for } e_- < e < e_+, \\ +1 & \text{for } e \geqq e_+. \end{cases} \tag{3.4c}$$

In the discrete-to-continuum limit $\varepsilon = 1/N \to 0$ the strong limits of the pairs $(\overline{e}^{\varepsilon,0}(t), \overline{z}^{\varepsilon,0}(t)) \in \mathrm{L}^2(\Omega)^2$ do not exist due to the fluctuations induced by the quenched disorder. Using explicit error bounds established in Theorem 4.5 we obtain $\|\overline{e}^{\varepsilon,\delta}(t)-\overline{e}^{\varepsilon,0}(t)\|_{\mathrm{L}^2} + \|\overline{z}^{\varepsilon,\delta}(t)-\overline{z}^{\varepsilon,0}(t)\|_{\mathrm{L}^2} \to 0$, and the remaining task is to characterize the weak limits

$$(\overline{e}^{\varepsilon,0}(t, \cdot), \overline{z}^{\varepsilon,0}(t, \cdot)) \rightharpoonup (\overline{e}(t, \cdot), \overline{z}(t, \cdot)) \quad \text{in } \mathscr{Q} = \mathrm{L}^2(\Omega)^2,$$

that is $\int_\Omega \overline{e}^{\varepsilon,0}(t, x)v_1(x)+\overline{z}^{\varepsilon,0}(t, x)v_2(x)\,\mathrm{d}x \to \int_\Omega \overline{e}(t, x)v_1(x)+\overline{z}(t, x)v_2(x)\,\mathrm{d}x$ for $\varepsilon \to 0$ for all test functions $v_1, v_2 \in \mathrm{L}^2(\Omega)$.

Quite expectedly, the limiting mixtures of phases cannot be fully characterized by the value of the average elastic strain $\overline{e}$. The missing information, allowing one to close the coarse-grained description at the macro-scale, is exactly the limit of the indicator function $\overline{z}$. To control the nonlinear terms we use the Young measure of the pair $(\overline{e}^{\varepsilon,0}(t), \overline{z}^{\varepsilon,0}(t))$, which can be characterized using the ordering property and the strong law of large numbers associated to the quenched disorder.

The final result states that there exists $\kappa_* > 0$ such that in the limit $(\varepsilon, \delta) \to (0, 0)$ under the constraint $\delta \leqq \kappa_*\varepsilon$, the functions $(\overline{e}^{\varepsilon,\delta}(t), \overline{z}^{\varepsilon,\delta}(t))$ converge to the unique solution $(\overline{e}, \overline{z})$ of a one-dimensional elasto-plasticity problem in the form

$$0 = \mathrm{D}_{\overline{e}}\overline{\mathscr{E}}(t, \overline{e}, \overline{z}(t)) + \sigma(t) \quad \text{for } x \in \Omega, \quad \int_\Omega e(t, x)\,\mathrm{d}x = \ell(t); \tag{3.5a}$$

$$0 \in \partial\overline{\mathscr{R}}(\dot{\overline{z}}(t)) + \mathrm{D}_{\overline{z}}\overline{\mathscr{E}}(\overline{e}(t), \overline{z}(t)). \tag{3.5b}$$

Here the macroscopic elastic energy $\overline{\mathscr{E}}$ is given by

$$\overline{\mathscr{E}}(t, \overline{e}, \overline{z}) = \int_\Omega \overline{\Phi}(\overline{e}(x), \overline{z}(x)) + G(t, x)e(x)\,\mathrm{d}x,$$

where G is defined in (2.3) and the macroscopic energy density $\overline{\Phi}$ depends on Φ and the probability density f from (3.1). In the bi-quadratic case $\Phi = \Phi_{\text{biq}}$ (see (2.1)) we obtain the explicit formula

$$\overline{\Phi}(\overline{e}, \overline{z}) = \frac{k}{2}\big(\overline{e} - a\,\overline{z}\big)^2 + \overline{H}(\overline{z}), \tag{3.6}$$

where the kinematic hardening potential $\overline{H}$ is determined by f, see (5.2). In the general case $\overline{\Phi}$ is given in (6.5), and the macroscopic rate-independent dissipation potential $\overline{\mathscr{R}}$ takes the form

$$\overline{\mathscr{R}}(\dot{\overline{z}}) = \int_\Omega \overline{R}(\dot{\overline{z}}(x))\,\mathrm{d}x \quad \text{with } \overline{R}(v) = \begin{cases} \rho_+ v & \text{for } v \geqq 0, \\ \rho_- |v| & \text{for } v \leqq 0, \end{cases}$$

where ρ_+ and ρ_- can be expressed in terms of Φ, see (6.7). In the bi-quadratic case $\Phi = \Phi_{\text{biq}}$ we obtain $\rho_\pm = 2ka^2$.

While we described above convergence result for general double-well potentials Φ and general loadings G and ℓ, we will give the detailed proofs only under more restrictive conditions. Our main assumptions are the following:

$$\text{all } \mu_j^N \text{ are identically independently distributed with law } f; \tag{3.7a}$$

$$\Phi = \Phi_{\text{biq}} \quad (\text{cf. } (2.1)); \tag{3.7b}$$

$$G(t, x) = G(x) \text{ with } G \in \mathrm{H}^1(\Omega); \tag{3.7c}$$

$$\ell : [0, T] \to \mathbb{R} \text{ is piecewise } \mathrm{C}^1 \text{ with } |\ell(t)| \geqq \lambda_0 > 0 \text{ a.e. on } [0, T]. \tag{3.7d}$$

Despite those simplifying assumptions, we believe that our methods can be extended to handle the more general situation as well.

The most unexpected feature of our result is the fundamental change in the nature of the dynamical system in the limit. Indeed, while (2.7) is an N-dimensional ODE derived from the constraint gradient flow (2.5) with quadratic dissipation potential $\mathscr{R}_{\varepsilon,\delta}$, the limit is a rate-independent system, where the dissipation-related forces $\partial\overline{\mathscr{R}}(\dot{\overline{z}})$ are homogeneous of degree 0 in $\dot{\overline{z}}$ (as the dissipation potential $R(\cdot)$ is homogeneous function of degree 1). The origin of the change in the structure of the dissipation is the “constructive interference” of micro-elasticity and micro-viscosity in the continuum limit (see also [1]). Notice also that the memory of the specific nature of the microscopic dissipation has been lost in the macroscopic double limit, suggesting that linear viscosity is not the only microscopic dissipative mechanism leading to our rate-independent macro-model.

If introduction of quenched disorder is perceived as an auxiliary technical step, it may be of interest to perform yet another limit $r \to 0$ by following a well-established path known in classical elasto-plasticity, see for example [5]. From the definition (5.2) of the hardening potential $\overline{H}_f$ in (3.6), it follows that it depends on f in such a way that $r^2 = \int_{\mathbb{R}} \mu^2 f(\mu)\,\mathrm{d}\mu \to 0$ implies $\overline{H}_f(z) \to 0$ for all

$z \in]-1, 1[$ (while $\overline{H}_f(z) = \infty$ if $|z| > 1$), see for example (5.3). Therefore the limiting model, given again by (3.5) with $\overline{\Phi}$ from (3.6), has the property that $\overline{H}(z) = 0$ for $|z| \leqq 1$. One can see that the resulting $\overline{\Phi}$, and hence $\overline{\mathscr{E}}$, are degenerate (while convex), which means that the model is not well-posed. Indeed, as is well known in ideal plasticity, several solutions may exist for given initial data.

4. The Vanishing-Viscosity Limit

In this section we fix $\varepsilon = 1/N > 0$ and $r > 0$ and consider the limit $\delta \to 0$. In fact, the assumption $r > 0$ is not crucial in this section; the only required property of the parameters $\mu_1^N, \ldots, \mu_N^N$ is that the effective biases $h_j = \mu_j^N - G(t, j/N)$ are pairwise different.

4.1. Energy Landscape and Ordered States

We begin by reviewing the structure of the elastic energy landscape at the given loads (see [51]). Here the elastic potential Φ is a general double-well potential as specified in Section 2.1.

Let us fix the time $t = t_0$ and consider the problem of minimizing the energy

$$\mathscr{E}_\varepsilon(t_0, \boldsymbol{e}) = \frac{1}{N}\sum_1^N \big(\phi(e_j) - h_j e_j\big) \text{ subject to } \mathscr{C}_\varepsilon(\boldsymbol{e}) = \tfrac{1}{N}\sum_1^N e_j = \ell = \ell(t_0).$$

The critical points of (2.7) can be obtained as solutions of the algebraic equations

$$0 = -\phi(e_j) + h_j + \sigma \quad \text{for } j = 1, \ldots, N, \quad \mathscr{C}_\varepsilon(\boldsymbol{e}) = \tfrac{1}{N}\sum_1^N e_j = \ell. \tag{4.1}$$

Metastable equilibria (local minima of the energy) are selected by the condition of the positive definiteness of the Hessian matrix $\mathrm{D}^2\mathscr{E}_\varepsilon(t_0, \boldsymbol{e}) - \sigma \mathrm{D}^2\mathscr{C}_\varepsilon(\boldsymbol{e}) = \mathrm{D}^2\mathscr{E}_\varepsilon(t_0, \boldsymbol{e})$. For sufficiently large N none of the metastable strains e_j can lie in the spinodal region $\big]e_-, e_+\big[$, see [51]. To identify the remaining two phases, -1 and $+1$, we define the phase indicator $\boldsymbol{z} = (z_j)_j \in \{-1, +1\}^N$, such that (3.3) holds. Hence, a metastable equilibrium exists when the equations

$$\frac{1}{N}\sum_{j=1}^N \psi_{z_j}(h_j + \sigma) = \ell \quad \text{and} \quad \begin{cases} h_j + \sigma \geqq \sigma_- & \text{if } z_j = 1, \\ h_j + \sigma \leqq \sigma_+ & \text{if } z_j = -1 \end{cases}$$

can be satisfied simultaneously. For each metastable branch parameterized by $\boldsymbol{z}$ we can define the equilibrium response functions $\sigma = \widehat{\sigma}(\ell, \boldsymbol{z})$.

A crucial observation allowing us to advance is that, due to imposed inhomogeneity, not all metastable equilibria are accessible by our dynamics. Indeed, we work under the assumption that the bias coefficients h_j are pairwise different, which allows us to define the subclass of *ordered states* via the following condition:

$$\textit{ordering condition:} \quad \forall\, j, k \in \{1, \ldots, N\} : \quad h_j < h_k \implies e_j < e_k. \tag{4.2}$$

It will turn out that the knowledge of the set of ordered states is sufficient for the study of the limiting macroscopic problem because the set of ordered states is invariant under the evolution for both the viscous and for the limiting inviscid systems (see (4.8) and (DA1)–(DA3) in Definition (4.2)). Moreover, one can see that a system that starts non-ordered will have the tendency to return to an ordered state. For instance, the chain will acquire ordering if it is ever stretched beyond the transformation thresholds and will then maintain its ordering at all future times. Nevertheless, the system may have an initial nontrivial virgin curve involving some non-ordered states, which our limiting theory would not capture.

Remark 4.1. The disorder entering through the random microscopic body forces is special in the sense that it leads to a particular simple structure of the inner hysteresis loops. A different way of bringing disorder into the model would be through a randomization of the thresholds σ_- and σ_+ as in [52]. Such choice, however, brings additional technical complications, which we would like to avoid here.

In this section it will be convenient to simplify the ordering condition by using the permutational symmetry of the system. Indeed, without loss of generality we can assume that the biases h_j are ordered as $h_1 < h_2 < \cdots < h_N$, such that (4.2) reduces to the condition

$$e_1 < e_2 < \cdots < e_N. \tag{4.3}$$

In Section 5, however, we would need to return to the original ordering condition (4.2) because the strains $(e_j)_{j=1,\ldots,N}$ of the springs in a one-dimensional bar $\Omega =]0, 1[$ are naturally ordered according to the material points $x_j = j/N$, see (3.4).

The class of *ordered equilibria* in the sense of (4.3) has a simple characterization: for each ordered equilibrium state there exists a threshold $\widehat{h}$ such that all j with $h_j \geqq \widehat{h}$ are in phase $z_j = +1$ while those with $h_j < \widehat{h}$ are in phase $z_j = -1$. We can then associate with each threshold a particular distribution of snap-springs between the two energy wells

$$z_j = \mathrm{sign}(h_j - \widehat{h}), \tag{4.4}$$

where $\mathrm{sign}(h_j - \widehat{h}) = 1$ for $h_j \geqq \widehat{h}$ and $\mathrm{sign}(h_j - \widehat{h}) = -1$ for $h_j < \widehat{h}$. It will also be convenient to introduce the following two functions

$$\overline{h}_+(\widehat{h}) = \min\{\, h_j \mid h_j \geqq \widehat{h}\,\}, \quad \overline{h}_-(\widehat{h}) = \max\{\, h_j \mid h_j < \widehat{h}\,\}. \tag{4.5}$$

Notice that $\overline{h}_\pm : \mathbb{R} \to \mathbb{R}$ are nondecreasing piecewise constant functions such that $\overline{h}_-(\widehat{h}) < \widehat{h} \leqq \overline{h}_+(\widehat{h})$. We shall also define $\overline{h}_+(\widehat{h}) = \infty$ if all $h_j < \widehat{h}$ and $\overline{h}_-(\widehat{h}) = -\infty$ if all $h_j \geqq \widehat{h}$.

The main advantage of the restriction to ordered states is that there are only $N + 1$ different branches of equilibria, which are parametrized by the number $m \in \{0, \ldots, N\}$ of phase indicators with $z_j = 1$. Our analysis below will show that the dynamics of the ODE (2.7) is such that none of the other possible 2^N equilibria is relevant.

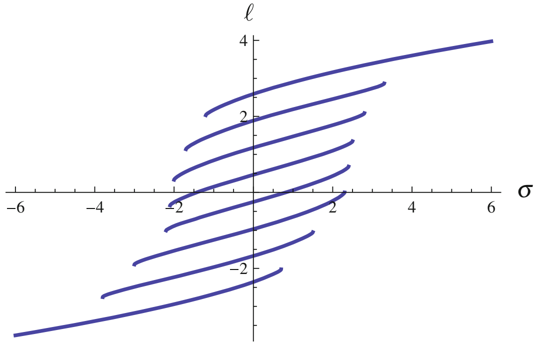

Fig. 7. Eight monotone stress–strain equilibrium branches $\ell = M(\widehat{h}, \sigma)$ representing ordered choices of the phases

For a given threshold $\widehat{h} \in \mathbb{R}$ the average strain $\ell = \frac{1}{N}\sum_j e_j$ of an ordered equilibrium $\boldsymbol{e}$ depends only on the stress σ. It is given by the function $M(\widehat{h}, \cdot) : [\sigma_- - \overline{h}_+(\widehat{h}), \sigma_+ - \overline{h}_-(\widehat{h})] \to \mathbb{R}$ such that

$$M(\widehat{h}, \sigma) = \frac{1}{N}\sum_{j=1}^{N} \psi_{\mathrm{sign}(h_j - \widehat{h})}(h_j + \sigma).$$

Of course, there are only $N+1$ different functions $M(\widehat{h}, \cdot)$. Each of these functions is strictly increasing, which means that the equation $M(\widehat{h}, \sigma) = \ell$ has at most one solution (see Fig. 7), giving rise to the stress–strain relation

$$\sigma = \widetilde{\sigma}(\ell, \xi), \quad \text{where } \xi = m/N \in [0, 1]. \tag{4.6}$$

Hence, for ordered states the metastable branch is defined by the single parameter $\xi \in [0, 1]$, which characterizes the fraction of transformed springs and serves as a precursor of the plastic strain appearing later in the limiting continuum problem.

For the case of a bi-quadratic potential Φ_{biq} in (2.1) the functions $M(\widehat{h}, \cdot)$ take the form

$$M(\widehat{h}, \sigma) = \frac{1}{k}\sigma + \frac{1}{kN}\sum_{j=1}^{N} h_j + \frac{a}{N}\sum_{j=1}^{N} \mathrm{sign}(h_j - \widehat{h}).$$

These are $N+1$ parallel lines shifted by the same constant $2a/N$. Under the simplifying assumption that $\sum_1^N h_j = 0$, we can then find the explicit representation of the equilibrium branches

$$e_j = \ell + a\,\mathrm{sign}(e_j) + h_j/k + a(1-2\xi). \tag{4.7}$$

4.2. Jump Discontinuities

This subsection motivates formal quasistatic dynamics $\boldsymbol{e}^0 : [0, T] \to \mathbb{R}^N$ on the set of equilibria, denoted by $\boldsymbol{e}^{\varepsilon,0}$ in Section 3. Here we dropped the superscript $\varepsilon = 1/N$, as N is fixed throughout Section 4. The reduced dynamics via motion along equilibrium branches is discussed in the following two subsections, while its status as a true limit for $\delta \to 0$ will be established in Section 4.5.

Suppose again that $(h_j)_j$ is strictly ordered and time-independent, while $\ell \in \mathrm{C}^{\mathrm{Lip}}([0, T])$ is a function of time through $\ell(t)$. The main system of ODEs can be written in the form

$$\delta \dot{e}_j = -\phi(e_j) + h_j + \sigma(t), \quad \mathscr{C}_\varepsilon(\boldsymbol{e}) = \frac{1}{N} \sum_{j=1}^{N} e_j(t) = \ell(t), \tag{4.8}$$

and we restrict our attention to the evolution of ordered configurations.

Moreover, for simplicity of describing the limiting system, we restrict ourselves to the easily computable case of the bi-quadratic potential $\Phi = \Phi_{\mathrm{biq}}$. However, the mapping $\phi = \Phi'_{\mathrm{biq}} : e \mapsto k(e - a \operatorname{sign} e)$ is not continuous and therefore, even for the viscous problem with $\delta > 0$, we may not have unique solutions. The existence of solutions is easily obtained by using set-valued analysis for ODEs (see [2]) and setting $\phi(0) = [-ka, ka]$. This leads to a set-valued inclusion with compact and convex right-hand side, which is upper semicontinuous.

The encountered non-uniqueness is not a major problem, as in the limit $\delta = 0$ uniqueness is lost anyway. Nevertheless, for fixed $\delta > 0$ we select for each ordered initial state a unique solution as follows. Observe that if all the $e_j(t)$ are ordered and are different from 0, then the solution of the ODE (4.8) can be extended uniquely as a differentiable function. Such a differentiable extension will work up to the time t_* when $e_{j_*}(t_*^-) = 0$ for some j_* (here $e_j(s^-) = \lim_{t \nearrow s} e_j(t)$ means the limit from the left), and until that time the solution is unique. If the solution is smoothly extendable, then we choose this as the unique extension, that is, e_{j_*} does not change sign at t_* (and we ignore the other solution where e_{j_*} would change sign and $\dot{\boldsymbol{e}}$ has a jump at t_*). If there is no extension where $\dot{\boldsymbol{e}}$ is continuous, we can construct a unique differentiable solution on $[t_*, t_*{+}\tau]$ with initial condition $\boldsymbol{e}(t_*)$, which is uniquely determined by choosing $e_{j_*}(\cdot)$ such that its signs differ for $t < t_*$ and $t > t_*$. Concatenating this to the solution on $[0, t_*]$ defines the unique global solution, which is still Lipschitz continuous in time. Observe that the system always remains in the set of ordered states.

We shall now describe the limit solution $\boldsymbol{e}^0(t)$ for $\boldsymbol{e}^\delta(t)$ while referring to Section 4.5 for the proof. The limiting trajectory stays in the time-dependent family of ordered equilibria (described in the previous subsection), which consists of finitely many separated branches. As we show later the solution corresponding to $\boldsymbol{e}^0(t)$ stays on such a branch as long as it remains metastable. If $\boldsymbol{e}^0(t)$ reaches the end of a branch, it jumps to the neighboring branch, meaning that only one of the springs changes phase. Of course, the jump is such that it dissipates energy, which is the only memory of the viscous dissipative mechanism (see also [53]).

The parameter defining plastic dissipation in the coarse-grained model is the release of energy in a jump of a single spring. The energy is defined as follows

$$\boldsymbol{E}(t,\boldsymbol{e}) = \begin{cases} \frac{1}{N}\sum_{j=1}^{N}\big(\Phi_{\text{biq}}(e_j) - h_j e_j\big) & \text{if } \frac{1}{N}\sum_{j=1}^{N} e_j = \ell(t), \\ \infty & \text{else.} \end{cases} \tag{4.9}$$

Using the form Φ_{biq} the energy release can be calculated explicitly

$$\boldsymbol{E}(t_*,\boldsymbol{e}(t_*^-)) - \boldsymbol{E}(t_*,\boldsymbol{e}(t_*^+)) = \rho_N/N > 0 \quad \text{where } \rho_N = 2ka^2 - 2ka^2/N. \tag{4.10}$$

Here the first term in ρ_N corresponds to the integral $\int_{e_-^*}^{e_+^*} \sigma_+ - \phi(e)\,\mathrm{d}e$, see Lemma 6.2. The second term is due to the relaxation of the stress from $\sigma(t_*^-) = \sigma_\pm$ to $\sigma(t_*^+) = \sigma_\pm \mp 2ak/N$. Because of our special choice of the type of disorder the critical values e_- and e_+ are not affected by the disorder. For Φ_{biq} both thresholds are equal to 0 and the strains satisfy the following explicit jump relations

$$e_j(t_*^+) = e_j(t_*^-) - a\hat{\Delta}/N \text{ for } j \neq j_*, \quad e_{j_*}(t_*^-) = 0, \quad e_{j_*}(t_*^+) = a(1-1/N)\hat{\Delta}, \tag{4.11}$$

where $\hat{\Delta} = z(t_*^+) - z(t_*^-) \in \{-2, 2\}$.

4.3. The Discrete Automaton

The resulting dynamics consists of two different regimes, namely (i) when the system remains on one of the metastable branches with parameter ξ fixed and (ii) the jumps, when ξ changes and the system switches to another metastable branch. The limiting dynamical system then takes the form of a discrete threshold-type automaton, see [49,50,56]).

Definition 4.2. Assume that $\Phi = \Phi_{\text{biq}}$, $(h_j)_j$ is a strictly ordered bias vector, and that $\ell \in \mathrm{C}^{\text{Lip}}([0,T])$. Then, a function $\boldsymbol{e} : [0,T] \to \mathbb{R}^N$ is called a *solution of the discrete automaton* if the following conditions hold:

(DA1) For all $t \in [0,T]$ the state $\boldsymbol{e}(t)$ is an ordered equilibrium (that is (4.1) and (4.2) hold) with $\mathscr{C}_\varepsilon(\boldsymbol{e}(t)) = \ell(t)$.

(DA2) There are at most finitely many times $0 = t_0 < t_1 < t_2 < \cdots t_L = T$ such that for $l = 1,\ldots,L$ the function $\boldsymbol{e}|_{]t_{l-1},t_l[}$ has a C^1 extension to $[t_{l-1},t_l]$.

(DA3) At each jump time t_l, $l = 1,\ldots,L-1$ the following holds:
- (i) one of strains is critical, that is, there $\exists\, j_* :\ e_{j_*}(t_l^-) = 0$,
- (ii) the jump conditions (4.11) hold for $t_* = t_l$, and
- (iii) the energy release $\boldsymbol{E}(t_l,\boldsymbol{e}(t_l^-)) - \boldsymbol{E}(t_l,\boldsymbol{e}(t_l^+))$ is exactly ρ_N/N.

Recall that all ordered equilibria are locally stable, which is crucial for the subsequent convergence analysis. Moreover, we will show in the next subsection that solutions of the discrete automaton are globally stable in the sense of rate-independent systems, see (S) in (4.12). The jump conditions in (DA3) are redundant and it would be sufficient to state only (iii), since the special form of $\phi = \Phi'_{\text{biq}}$ implies that (i) and (ii) must hold. This will be implicitly shown in the proof of Proposition 4.4. We stated the redundant conditions here to highlight all the special features of the jumps.

Another technical issue is that, as in the case of the viscous ODE system (4.8), the solution of the discrete automaton is not unique. A non-uniqueness can occur if a steady state reaches $e_{j_*}(t_*) = 0$ exactly at a moment when ℓ has a local extremum. Then, the phase jump may occur or may not occur. We define a unique extension by asking the solution to stay continuous as long as possible, that is, we assume that jumps occur only if they are necessary. This additional "rule" for the bi-quadratic problem can be obtained rigorously if one considers an additional limit when a finite spinodal region is asymptotically shrinking to 0.

4.4. An Energetic Rate-Independent System

Before giving the convergence proof for $\delta \to 0$, we show that the automaton (DA1)–(DA3) can be reformulated in terms of an *energetic rate-independent system* (ERIS) in the sense of [33]. This reformulation will serve as a basis of the subsequent continualization of our discrete dynamical system in Section 5.

A general ERIS is given in terms of the state space $\boldsymbol{Q}$, time-dependent energy functional $\boldsymbol{E} : [0, T] \times \boldsymbol{Q} \to \mathbb{R} \cup \{\infty\}$, and a dissipation distance $\boldsymbol{D} : \boldsymbol{Q} \times \boldsymbol{Q} \to [0, \infty]$. Our state space is $\boldsymbol{Q} = \mathbb{R}^N$ and the energy functional $\boldsymbol{E}$ is defined in (4.9). The new quantity is the dissipation distance $\boldsymbol{D}$, which measures the energy that is dissipated due to fast viscous motion. If the strains vary quasistatically in one of the two wells, there will be no dissipative contribution in the inviscid limit $\delta \to 0$. However, if a strain jumps into the other well (that is by changing sign), then the viscous motion is fast, namely of order $1/\delta$, and the energy $\int_{t_1(\delta)}^{t_2(\delta)} \frac{1}{N} \sum_{j=1}^{N} \delta \dot{e}_j^2(t)\,\mathrm{d}t$ has a finite limit (see also [53]).

We define the dissipation distance by counting the number of phase jumps:

$$\mathscr{D}(\boldsymbol{e}^0, \boldsymbol{e}^1) = \frac{1}{N} \sum_{j=1}^{N} D_N(e_j^0, e_j^1) \text{ with } D_N(\widetilde{e}, \hat{e}) = \begin{cases} \rho_N & \text{if } \widetilde{e}\,\hat{e} < 0 \text{ (phase jump)}, \\ 0 & \text{if } \widetilde{e}\,\hat{e} \geqq 0 \text{ (no phase jump)}, \end{cases}$$

where ρ_N is defined in (4.10). Using the triple $(\boldsymbol{Q}, \boldsymbol{E}, \boldsymbol{D})$ we can further define the notion of *energetic solutions* as follows (for example [33,34]).

Definition 4.3. Assume that $\Phi = \Phi_{\text{biq}}$, $(h_j)_j$ is a strictly ordered bias vector, and that $\ell \in \mathrm{C}^{\text{Lip}}([0, T])$. Then, a function $\boldsymbol{e} : [0, T] \to \boldsymbol{Q}$ is called an *energetic solution* of the ERIS $(\boldsymbol{Q}, \boldsymbol{E}, \boldsymbol{D})$, if for all $t \in [0, T]$ we have the stability **(S)** and the energy balance **(E)**:

$$\begin{aligned} &\textbf{(S)} \qquad \infty > \boldsymbol{E}(t, \boldsymbol{e}(t)) \leqq \boldsymbol{E}(t, \widetilde{\boldsymbol{e}}) + \boldsymbol{D}(\boldsymbol{e}(t), \widetilde{\boldsymbol{e}}) \quad \text{for all } \widetilde{\boldsymbol{e}} \in \boldsymbol{Q}, \\ &\textbf{(E)} \qquad \boldsymbol{E}(t, \boldsymbol{e}(t)) + \mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e}, [0, t]) = \boldsymbol{E}(0, \boldsymbol{e}(0)) - \int_0^t \Sigma(\boldsymbol{e}(s))\dot{\ell}(s)\,\mathrm{d}s, \end{aligned} \tag{4.12}$$

where $\mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[0,t])$ is the supremum of $\sum_{k=1}^{M} \boldsymbol{D}(\boldsymbol{e}(\tau_{k-1}),\boldsymbol{e}(\tau_k))$ over all $M \in \mathbb{N}$ and all partitions $0 \leqq \tau_0 < \tau_1 < \cdots < \tau_M \leqq t$ of $[0,t]$ and $\Sigma(\boldsymbol{e}) = \frac{1}{N}\sum_{j=1}^{N}\big(\phi(e_j)-h_j\big)$.

Note that the dissipation functional $\mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[r,t])$ gives a counting measure, since it is equal to ρ_N/N times the number of all the phase jumps of $\boldsymbol{e}$ in the time interval $[r,t]$.

The global stability condition (S) immediately implies a local stability condition, namely with respect to variations $\widetilde{\boldsymbol{e}}$ that have the same phase vector $\boldsymbol{z}$ as $\boldsymbol{e}(t)$, which implies $\boldsymbol{D}(\boldsymbol{e}(t),\widetilde{\boldsymbol{e}}) = 0$. In general, for ERIS with nonconvex energies the global stability condition is much stronger than local stability. In the present case, the strength of $\boldsymbol{D}$ is tuned via D_N or ρ_N in such a way that energetic solutions cannot jump too early; the energy balance (E) enforces that every jump is accompanied by an energy release compensating for the dissipation.

The following result states that the evolution given in terms of the discrete automaton is exactly the same as the energetic solution of $(\boldsymbol{Q},\boldsymbol{E},\boldsymbol{D})$. To obtain this result the ordering property of the solutions is, in fact, not necessary and the proposition below also applies to non-ordered solutions. The subsequent assumption that $(h_j)_j$ is fixed, means that the external loading through G is time-independent, see (3.7c).

Proposition 4.4. *Assume that $\Phi = \Phi_{biq}$, $(h_j)_j$ is a strictly ordered bias vector, and that $\ell \in \mathrm{C}^{Lip}([0,T])$. Then, an ordered function $\boldsymbol{e} : [0,T] \to \boldsymbol{Q} = \mathbb{R}^N$ is an energetic solution of $(\boldsymbol{Q},\boldsymbol{E},\boldsymbol{D})$ given via* (4.12) *if and only if it satisfies* (DA1)–(DA3) *in Definition* 4.2.

Proof. (S)&(E) ⇒ (DA1)–(DA3). From (S) we conclude that for each $t \in [0,T]$ the solution satisfies the length constraint and is in equilibrium. For the latter, simply consider variations $\widetilde{\boldsymbol{e}}$ such that $\boldsymbol{D}(\boldsymbol{e}(t),\widetilde{\boldsymbol{e}}) = 0$, that is, with no additional phase jumps. Then, $\boldsymbol{e}(t)$ is a local minimizer of $\boldsymbol{E}(t,\cdot)$ and thus a stable equilibrium. Thus, (DA1) is established. In particular, we know that $\boldsymbol{e}(t)$ lies in the finite set of stable equilibria. Along these branches the dependence of $\boldsymbol{e}(t)$ on $\ell(t)$ is smooth, see (4.7).

From (E) we conclude that $\mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[0,T])$ is finite. Since $\boldsymbol{D}$ takes only the discrete values $\{\,k\rho_N/N \mid k = 0,1,\ldots,N\,\}$, we conclude that the monotone function $\hat{\delta} : [0,T] \to [0,\infty[\,;\, t \mapsto \mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[0,t])$ is piecewise constant with finitely many jump points $t_1 < \cdots < t_{L-1}$, where each jump is an integer multiple of ρ_N/N. Since jumping between the solution branches generates a jump in $\hat{\delta}$, we conclude that on the intervals $\big]t_{l-1},t_l\big[$ the solution remains on one branch and hence can be extended smoothly to $[t_{l-1},t_l]$. Hence (DA2) is established.

Using (E) we obtain energy balance on all subintervals $[r,t]$, namely $\boldsymbol{E}(t,\boldsymbol{e}(t)) + \mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[r,t]) = \boldsymbol{E}(r,\boldsymbol{e}(r)) - \int_r^t \Sigma(\boldsymbol{e}(s))\dot{\ell}(s)\,\mathrm{d}s$. Taking the limits $t \to t_l^+$ and $r \to t_L^-$ we find the jump relation

$$\boldsymbol{E}(t_l,\boldsymbol{e}(t_l^+)) + \boldsymbol{D}(\boldsymbol{e}(t_l^-),\boldsymbol{e}(t_l^+)) = \boldsymbol{E}(t_l,\boldsymbol{e}(t_l^-)). \tag{4.13}$$

However, the choice of ρ_N was exactly such that it corresponds to the energy loss for a jump arising from critical strains $e_{j_*}(t_l^-) \in \{e_-, e_+\}$, which establishes (i). Properties (ii) and (iii) follow from the assumption that all h_j are pairwise disjoint. Then, at most one e_j can have a phase jump.

(DA1)–(DA3) $\Rightarrow$ **(S)&(E)**. From (DA1) we easily obtain (S): Every stable equilibrium is globally stable in the sense of (S), since stability with respect to $\widetilde{\boldsymbol{e}}$ satisfying $\boldsymbol{D}(\boldsymbol{e}(t), \widetilde{\boldsymbol{e}}) = 0$ follows from the equilibrium conditions and convexity of Φ in the two wells. Moreover, ρ_N was chosen as the maximal energy loss when jumping from one branch to a neighboring one. Thus, the energy release $\boldsymbol{E}(t, \boldsymbol{e}(t)) - \boldsymbol{E}(t, \widetilde{\boldsymbol{e}})$ will be always less than $\boldsymbol{D}(\boldsymbol{e}(t), \widetilde{\boldsymbol{e}})$.

Using (DA2) and (DA3), the energy balance (E) is obtained by joining the smooth parts in $\big]t_{l-1}, \min\{t, t_l\}\big[$ and the jumps. In the first case set $t_* = \min\{t, t_l\}$; the smoothness gives $\boldsymbol{E}(t_*, \boldsymbol{e}(t_*^-)) = \boldsymbol{E}(t_{l-1}, \boldsymbol{e}(t_{l-1}^+)) - \int_{t_{l-1}}^{t_*} \Sigma(\boldsymbol{e}(s))\dot{\ell}(s)\,\mathrm{d}s$. At the jumps we have (4.13) and (E) follows by addition. □

4.5. Convergence Proof for $\delta \to 0$

We finally prove the convergence for $\delta \to 0$ of the viscous ODE system (4.8) to the automaton (DA1)–(DA3), and consequently to the ERIS system $(\boldsymbol{Q}, \boldsymbol{E}, \boldsymbol{D})$. The proof is constructive and provides explicit error estimates in terms of the small parameters δ and $\varepsilon = 1/N$.

Notice that different sources of error would need to be estimated in different norms. During the equilibrium phase, when the system slides close to a particular metastable equilibrium branch, the non-zero viscosity prevents the solution from relaxing to the exact equilibrium state, and this gives rise to an error (i) of order δ in all of the components. Two other errors occur during jumps: (ii) one of the strains, namely e_{j_*}, is far away from a stable steady state, while (iii) all the other strains have an error of order ε. The first and the third types of error are most efficiently measured in the maximum norm $|R|_\infty = \max\{\,|R_j| \mid j = 1, \ldots, N\,\}$ whereas the second type of error is better evaluated in the 1-norm $|R|_1 = \sum_1^N |R_j|$.

For the following result, which is the first of the two main steps in the theory, we have to strengthen our assumptions a little bit further by assuming that the time-derivative $\dot{\ell}$ is bounded away from 0. This assumption is very helpful (but probably not essential) in controlling the jumps in the limit $\delta \to 0$. We would simply like to avoid the technicalities in the situation that ℓ has a local extremum at $t = t_*$ with $\dot{\ell}(t_*)$ and $\ddot{\ell}(t_*) \neq 0$. In such a case it would be delicate to control the number of springs with a phase jump in the case of $\varepsilon = 1/N \to 0$. Since this limit is our main goal, we decided to incorporate assumption (3.7d), which allows us to decompose the time interval in finitely many pieces where ℓ is smooth and $\dot{\ell}$ is bounded away from 0.

Note also that the assumption of time-independence of $(h_j)_j$ (see (3.7c)) is essential to guarantee that the ordering property (4.2) is independent of time. Dropping this assumption would destroy the strict validity of the ordering property; however, it would still be true in large regions as the stability requirement would pull the system always back into ordered states.

Theorem 4.5. *Assume that $\Phi = \Phi_{biq}$, $(h_j)_j$ is a strictly ordered bias vector, and that ℓ satisfies (3.7d). Take any ordered steady state $\overline{\boldsymbol{e}}^0 \in \mathbb{R}^N$ associated with $\ell = \ell(0)$. Then, the solution $\boldsymbol{e}^\delta \in \mathrm{C}^{\mathrm{Lip}}([0,T];\mathbb{R}^N)$ of (4.8) with $\boldsymbol{e}^\delta(0) = \boldsymbol{e}^0$ constructed above converges to the unique solution $\boldsymbol{e}^0 : [0,T] \to \mathbb{R}^N$ with $\boldsymbol{e}^0(0) = \overline{\boldsymbol{e}}^0$ of the discrete automaton (DA1)–(DA3) constructed above, that is, for almost every $t \in [0,T]$ we have $\boldsymbol{e}^\delta(t) \to \boldsymbol{e}^0(t)$ as $\delta \to 0$.*

Moreover, there are positive constants C and κ_* such that for all $\delta \in\,]0,1]$ and $N \in \mathbb{N}$ with $\delta N \leqq \kappa_*$, we have $\boldsymbol{e}^\delta(t) = \boldsymbol{e}^0(t) + R^1(t) + R^2(t)$ with

$$|R^1(t)|_\infty \leqq C(\delta+1/N) \quad \text{and} \quad |R^2(t)|_1 \leqq C. \tag{4.14}$$

Proof. To simplify the notations we drop the superscript δ for the viscous solutions but keep the superscript 0 for the limit. Throughout the proof the constant C may vary, but it is always independent of δ, N and the given solutions. We sometimes use constants $C_1, C_2, \ldots$ to indicate how certain estimates follow from others.

Using (3.7d) we decompose $[0,T]$ into finitely many subintervals, on each of which ℓ is monotone. If we allow for a suitable error for the initial condition it is then sufficient to consider only one of these intervals. Indeed, without loss of generality we can assume that ℓ is monotonically increasing on $[0,T]$, however, to be able to concatenate several pieces we allow for a nontrivial shift $\boldsymbol{e}(0) - \boldsymbol{e}^0(0)$.

From the monotonicity of ℓ and the ordering of the solutions $\boldsymbol{e}$ we obtain jump times $0 < t_1 < \cdots < t_L < T$. For the following it is more convenient to reorder these numbers and to use as the switching times parameters s_j, $j = 1, \ldots, N$ defined such that $\operatorname{sign} e_j(t) = \operatorname{sign}(t - s_j)$. Then, $0 \leqq s_N \leqq s_{n-1} \leqq \cdots s_1 \leqq T$, where strict inequality holds as soon as the times are different from 0 or T. With $m(t)$ we count the number of $e_j(t)$ and $e_j^0(t)$ bigger than 0, namely $m(t) = N - j$ for $t \in \big]s_{j-1}, s_j\big[$. Similarly, for the solution $\boldsymbol{e}^0$, where $\delta = 0$, we define s_j^0 and $m^0(t)$ having exactly the same properties.

For sufficiently small $\delta + 1/N$ we conclude that $m(0) = m^0(0)$. Using m^0 and m the average stresses σ^0 and σ can be calculated as

$$\sigma(t) = \frac{1}{N}\sum_{j=1}^{N}\big(\phi(e_j(t)) + h_j + \delta \dot{e}_j(t)\big) = k\ell(t) + \delta\dot{\ell}(t) + \frac{ak}{N}(2m(t) - N),$$

$$\sigma^0(t) = k\ell(t) + \frac{ak}{N}(2m^0(t) - N).$$

With these stress histories known, the strains solving (4.8) have the explicit representation

$$e_j(t) = \mathrm{e}^{-kt/\delta} e_j(0) + \int_0^t \mathrm{e}^{-k(t-s)/\delta}\frac{1}{\delta}\big(ak\operatorname{sign}(s - s_j) + h_j - \sigma(s)\big)\,\mathrm{d}s, \tag{4.15a}$$

$$e_j^0(t) = a\operatorname{sign}(t - s_j^0) + \frac{1}{k}(h_j + \sigma^0(t)). \tag{4.15b}$$

We write the difference $\rho_j(t) = e_j(t) - e_j^0(t)$ in the form

$$\begin{aligned}
&\rho(t) = \rho_j^1(t) + \rho_j^2(t) + \rho_j^3(t) + \rho_j^4(t) \text{ with}\\
&\rho_j^1(t) = \mathrm{e}^{-kt/\delta}\rho_j(0), \qquad \rho_j^2(t) = \int_0^t \mathrm{e}^{-k(t-s)/\delta} k\dot{\ell}(s)\,\mathrm{d}s,\\
&\rho_j^3(t) = \int_0^t \mathrm{e}^{-k(t-s)/\delta}\frac{2ak}{\delta N}\big(m^0(t) - m(s)\big)\,\mathrm{d}s,\\
&\rho_j^4(t) = \int_0^t \mathrm{e}^{-k(t-s)/\delta}\frac{ak}{\delta}\big(\operatorname{sign}(t-s_j^0) - \operatorname{sign}(s-s_j)\big)\,\mathrm{d}s.
\end{aligned}$$

We immediately find $|\rho_j^1(t)| + |\rho_j^2(t)| \leqq C(\delta+1/N)$ as desired.

To estimate the other terms we have to control $s_j - s_j^0$. The nontrivial s_j^0 are defined via

$$0 = -a + h_j/k + \ell(s_j^0) + a(2j-N)/N, \tag{4.16}$$

which implies $\ell(s_j^0) - \ell(s_{j+1}^0) = (h_{j+1}-h_j)/k + 2a/N > 2a/N$. Hence, we find

$$|s_j^0 - s_l^0| \geqq \frac{|j-l|}{CN} \quad \text{for } j,l = 1,\ldots,N, \quad \text{where } C = a\|\dot{\ell}\|_\infty/2. \tag{4.17}$$

For the moment we assume a similar estimate

$$|s_j - s_l| \geqq \tfrac{|j-l|}{C_m N} \quad \text{for } j,l = 1,\ldots,N, \tag{4.18}$$

where the constant C_m is still to be determined by choosing $\delta N \leqq \kappa_*$ sufficiently small. Using this assumption we can estimate $\dot{e}_j(s_j^-)$ (limit from the left) via the explicit form of e_j in (4.15a). Note that σ is piecewise smooth with jumps of size $O(1/N)$ at each s_l. The contributions of the initial condition and the smooth parts are bounded by a constant C_1 independently of δ, N and C_m. Including the terms from the jumps gives the estimate

$$|\dot{e}_j(s_j^-)| \leqq C_1 + CC_m\gamma\big(1/(C_m\delta N)\big), \quad \text{where } \gamma(r) = \sum_{l=j+1}^{N} r\mathrm{e}^{-(l-j)r} \leqq 1 + r.$$

As the nontrivial s_j satisfy

$$0 = e_j(s_j) = -ah_j/k + \ell(s_j) + a(2j-N)/N + \delta\big(\dot{\ell}(s_j) - \dot{e}_j(s_j^-)\big),$$

we can compare with (4.16). Using $\lambda \leqq |\dot{\ell}(t)| \leqq C$ and $|\dot{e}_j(s_j^-)| \leqq C(1+C_m)$ we find a constant C such that

$$|s_j - s_j^0| \leqq \tfrac{\delta}{\lambda}\big(C(1+C_m) + \|\dot{\ell}\|_\infty\big) =: \delta C_2(1+C_m). \tag{4.19}$$

From this we derive (4.18) as follows. For nontrivial j and l with $j \neq l$ we have

$$\begin{aligned}
|s_j - s_l| &\geqq |s_j^0 - s_l^0| - |s_j^0 - s_j| - |s_l^0 - s_l| \geqq \frac{|j-l|}{CN} - 2\delta C_2(1+C_m)\\
&\geqq \frac{|j-l|}{CN}\big(1 - 2\delta NCC_2(1+C_m)\big) \geqq_{(*)} \frac{|j-l|}{C_m N}.
\end{aligned}$$

To justify "$\geqq_{(*)}$" we use $\delta N \leqq \kappa_*$ with $\kappa_* := 1/(4C_2 \max\{C, 2C^2\})$ and set $C_m = (2\kappa_* C_2)^{-1/2}$. Thus, (4.18) is finally established.

Using the above estimates between the jump times s_j and s_l^0 we are able to control the difference between $m^0(t)$ and $m(s)$. First assume $m(t) = N-j \geqq m^0(t) = N-l$, then by the definition of m and m^0 we have $s_j \geqq s_{l-1}^0$. Thus, we find

$$s_j^0 + \delta C \geqq s_j \geqq s_{l-1}^0 \geqq s_j^0 + \frac{l-1-j}{CN},$$

which yields $l-j \leqq 1+\delta N C^2$. Hence, $l-j \leqq N_* := \lfloor 1+\kappa_* C^2 \rfloor \in \mathbb{N}$. With a similar argument for $m(t) = N-j \leqq m^0(t) = N-l$ and using (4.17) we obtain

$$|m(s) - m^0(t)| \leqq N_* + CN(t-s) \quad \text{for } 0 \leqq s \leqq t \leqq T.$$

Hence, ρ_j^3 can be estimated via $|\rho_j^3(t)| \leqq C(\delta + 1/N)$ for $j = 1, \ldots, N$ and $t \in [0, T]$. Let $s_j^{\min}$ and $s_j^{\max}$ be the minimum and maximum of $\{s_j, s_j^0\}$. Using (4.19) yields

$$|\rho_j^4(t)| \leqq \begin{cases} 0 & \text{for } s \leqq s_j^{\min}, \\ 2 & \text{for } s_j^{\min} < s \leqq s_j^{\max}, \\ 2\mathrm{e}^{-k(t-s_j^{\max})/\delta} & \text{for } s \geqq s_j^{\max}. \end{cases}$$

To conclude the theorem we define R^1 via $R_j^1(t) = \rho_j^1(t) + \rho_j^2(t) + \rho_j^3(t)$ and obtain immediately $|R^1(t)|_\infty \leqq C(\delta + 1/N)$. For $R_j^2(t) = \rho^4(t)$ we use the fact that, in a given time t, only for a few js has there been a recent jump, namely

$$|\rho^4(t)|_1 = \sum_{j=1}^{N} |\rho_j^4(t)| \leqq 2\big(N_* + \sum_1^N \mathrm{e}^{-k/(C\delta)}\big) \leqq C_4.$$

Thus, estimate (4.14) is established.

We still have to show the convergence $R^{\delta,1}(t) + R^{\delta,2}(t) \to 0$ for $\delta \to 0$ but N fixed. We now display the dependence on δ again by adding the superscript δ where convenient. We show that this convergence holds for all t in $\mathscr{T} := [0, T] \setminus \{s_1^0, \ldots, s_N^0\}$, which is a set of full measure.

It is now easy to see that $\rho_j^{\delta,1}(t) + \rho_j^{\delta,2}(t) \to 0$ for all t. To estimate $\rho_j^{\delta,3}$ and $\rho_j^{\delta,4}$ we fix $t \in \mathscr{T}$ and let $\tau = \frac{1}{2}\mathrm{dist}(t, \{s_1^0, \ldots, s_N^0\})$. Then, for all sufficiently small δ the interval $]t-\tau, t[$ does not contain any s_l^0 or s_l^δ. Whence $m^0(t) = m^\delta(s)$ and $\mathrm{sign}(t-s_j^0) = \mathrm{sign}(s-s_j^\delta)$ for $s \in [t-\tau, t]$, because $s_l^\delta \to s_l^0$, and $\rho_j^{\delta,3}(t) + \rho_j^{\delta,4}(t) \to 0$ follows easily.

Thus, the proof of Theorem 4.5 is complete. □

5. The Continuum Limit

We now focus on the limit $\varepsilon \to 0$, by assuming that the number N of springs goes to infinity. This means that we apply the second limiting procedure to the discrete automaton, or equivalently to the energetic rate-independent system (ERIS)

$(\boldsymbol{Q}_N, \boldsymbol{E}_N, \boldsymbol{D}_N)$, representing the primary inviscid limit of the original ODE system. The main challenge is to replace the automaton-type evolution of the plastic variable formulated in terms of *discrete space* and *discrete time* by a dynamical system employing a *continuous time* variable t and *continuous space* variable x. This is feasible because in the limit $\varepsilon \to 0$ the *elastic stages* become progressively shorter while the *plastic jumps* becomes weaker and more frequent (see also [53]). As a result, the limiting evolution involves *simultaneous elastic and plastic stages* and the corresponding continuum variables change all the time.

To justify this picture it will be convenient to use the formulation based on the energetic system $(\boldsymbol{Q}_N, \boldsymbol{E}_N, \boldsymbol{D}_N)$. The strategy is to embed our system into a system defined on $\mathscr{Q} = \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$, which contains the strains and a plastic variable. For the embedded ERIS $(\mathscr{Q}, \mathscr{E}_N, \mathscr{D}_N)$ the limit passage $\varepsilon \to 0$ can be performed in the pure rate-independent setting, following the philosophy developed in [37].

Recall that now we are treating a sequence of problems with N as a parameter. Hence, for each N there is a bias vector with components $h_j^N = \mu_j^N - G(j/N)$, $j = 1, \ldots, N$. All considered solutions $\boldsymbol{e}(t) \in \mathbb{R}^N$ satisfy the original ordering condition (4.2), which is independent of t by the assumed time-independence of G, see (3.7c).

The embedding in $\mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$ is done as explained in (3.4), but now with $e_- = e_+ = 0$, the piecewise constant interpolants $\overline{e}^N$ and $\overline{p}^N = a\overline{z}^N$ for the elastic and plastic strains are given by

$$\begin{aligned} &\mathscr{P}_N : \mathbb{R}^N \to \mathscr{Q} := \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega), \quad \mathscr{P}_N(\boldsymbol{e}) := (\overline{e}^N, \overline{z}^N) \quad \text{with} \\ &\overline{e}^N(t,x) = \sum_{j=1}^N e_j(t)\chi_j^N(x) \quad \text{and} \quad \overline{p}^N(t,x) = a \sum_{j=1}^N \operatorname{sign}(e_j(t))\chi_j^N(x). \end{aligned} \tag{5.1}$$

For notational convenience and implying similarity with classical notations in elastoplasticity, we denote the plastic strain by $p = az \in [-a, a]$, where $z \in [-1, 1]$ is our phase indicator function introduced earlier.

5.1. Macroscopic System

To specify the structure of the limiting energy, which incorporates kinematic hardening, we associate with each probability density function f satisfying (3.1) an auxiliary function $\mathscr{F}^*$. We first define

$$F : \mu \mapsto \int_{-\infty}^{\mu} f(y)\,\mathrm{d}y \quad \text{and} \quad \mathscr{F} : \mu \mapsto \int_{-\infty}^{\mu} F(y)\,\mathrm{d}y,$$

which gives $\mathscr{F}''(\mu) = f(\mu) \geqq 0$. Now, $\mathscr{F}^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is the Legendre transform $\mathscr{F}^*(\eta) := \sup\{\mu\eta - \mathscr{F}(\mu) \mid \mu \in \mathbb{R}\}$. Thus, $\mathscr{F}^*$ is convex and with $\mathscr{F}^*(\eta) = \infty$ for $\mu \notin [0, 1]$. The (kinematic) hardening potential $H : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ associated with f is defined as

$$H(p) = 2a\mathscr{F}^*\Big((a-p)/(2a)\Big), \tag{5.2}$$

which is convex and satisfies $H(p) = \infty$ for $|p| > a$, by definition. In the special case when $f(\mu) = \frac{1}{2\mu_*}\chi_{[-\mu_*,\mu_*]}$ we obtain $H(p) = \mu_*(p^2 - a^2)/(2a)$.

Consider now a family of densities f_r satisfying $f_r(\mu) = \frac{1}{r} f_1(\frac{\mu}{r})$. Then, we obtain $F_r(\mu) = F_1(\mu/r)$ and $\mathscr{F}_r(\mu) = r\mathscr{F}_1(\mu/r)$. For the Legendre transform this leads to $\mathscr{F}_r^*(\eta) = r\mathscr{F}_1(\eta)$. Thus, we obtain that

$$H_r(p) = rH_1(p) \to 0 \quad \text{for } r \to 0 \text{ and } |p| < a \text{ fixed.} \tag{5.3}$$

By using the definitions above we can now describe the limiting continuum problem. We define an effective macroscopic energy functional $\mathscr{E} : [0, T] \times \mathscr{Q} \to \mathbb{R} \cup \{\infty\}$ and the macroscopic dissipation functional $\mathscr{D}$ as follows:

$$\mathscr{E}(t, \overline{e}, \overline{p}) = \begin{cases} \mathscr{E}_0(\overline{e}, \overline{p}) & \text{for } \int_\Omega e(x)\,\mathrm{d}x = \ell(t), \\ \infty & \text{otherwise,} \end{cases} \quad \text{and} \tag{5.4a}$$

$$\mathscr{D}(\overline{p}_0, \overline{p}_1) = \int_\Omega 2ka|\overline{p}_1(x) - \overline{p}_0(x)|\,\mathrm{d}x, \tag{5.4b}$$

$$\text{where } \mathscr{E}_0(\overline{e}, \overline{p}) = \int_\Omega \overline{\Phi}(\overline{e}(x), \overline{p}(x)) + G(x)e(x)\,\mathrm{d}x - \Gamma_f \tag{5.4c}$$

$$\text{with } \overline{\Phi}(\overline{e}, \overline{p}) = \frac{k}{2}(\overline{e} - \overline{p})^2 + H(\overline{p}) \text{ and } \Gamma_f = \frac{1}{2k}\int_{\mathbb{R}} \mu^2 f(\mu)\,\mathrm{d}\mu. \tag{5.4d}$$

Here $\overline{\Phi}$ is the continuum energy density depending on the macroscopic elastic and the plastic strain variables.

Using the uniform convexity of H one can show that the macroscopic ERIS $(\mathscr{Q}, \mathscr{E}, \mathscr{D})$ has a unique energetic solution for each stable initial condition $(\overline{e}^0, \overline{p}^0)$. This solution $(\overline{e}, \overline{p})$ is Lipschitz continuous in time and satisfies the following plasticity problem (see [6,27,33,64]):

$$k\big(\overline{e}(t, x) - \overline{p}(t, x)\big) + G(x) = \sigma(t), \quad \int_\Omega \overline{e}(t, y)\,\mathrm{d}y = \ell(t), \tag{5.5a}$$

$$0 \in ka \ \mathrm{Sign}\,\big(\dot{\overline{p}}(t, x)\big) + k\big(\overline{p}(t, x) - \overline{e}(t, x)\big) + \partial H(\overline{p}(t, x)), \tag{5.5b}$$

where "Sign" denotes the set-valued function with $\mathrm{Sign}(0) = [-1, 1]$ and $\mathrm{Sign}(v) = \{\mathrm{sign}(v)\}$ for $v \neq 0$. Introducing the displacement $u(t, x) = \int_0^x \overline{e}(t, y)\,\mathrm{d}y$ we can rewrite the system in the more classical form

$$-\partial_x\Big(k\big(\partial_x u(t, x) - \overline{p}(t, x)\big)\Big) = g_{\mathrm{ext}}(x), \quad u(t, 0) = 0, \quad u(t, 1) = \ell(t),$$
$$0 \in ka\,\mathrm{Sign}\big(\dot{\overline{p}}(t, x)\big) + k\big(\overline{p}(t, x) - \partial_x u(t, x)\big) + \partial H(\overline{p}(t, x)).$$

Note that H_r and Γ_{f_r} are the only terms in $\mathscr{E}$ and $\mathscr{D}$ depending on the probability distribution density f_r. Obviously, Γ_{f_r} is irrelevant for the elasto-plastic evolution, whereas the hardening potential H_r is essential. When $r \to 0$ one can show that $\mathscr{F}_r(\mu) \to \max\{0, \mu\}$ and $H_r(p) \to H_0(p) = 0$ for $|p| < a$, see (5.3) for a special case. As we have already mentioned, there is no hardening in the case $H = H_0$, therefore existence of solutions can still be established but uniqueness fails.

5.2. Convergence Proof for $\varepsilon = 1/N \to 0$

In this subsection we prove our second main theorem, which justifies the discrete-to-continuum limit $\varepsilon = 1/N \to 0$ from the discrete automaton (DA1)–(DA3)) to the elastoplastic system (5.5). More precisely we consider the sequence of discrete ERIS $(\mathbb{R}^N, \boldsymbol{E}_N, \boldsymbol{D}_N)$ described in Section 4.4 with solutions $\boldsymbol{e}_N : [0, T] \to \mathbb{R}^N$ and show that the embedded functions $(\overline{e}^N, \overline{p}^N) = \mathscr{P}_N(\boldsymbol{e}_N) : [0, T] \to \mathscr{Q}$ weakly converge to the unique solution of the macroscopic ERIS $(\mathscr{Q}, \mathscr{E}, \mathscr{D})$, where $\mathscr{P}_N$ is defined in (5.1). We show that the associated energy and dissipation converge and ensure that the limit is an energetic solution for $(\mathscr{Q}, \mathscr{E}, \mathscr{D})$. While we use the methods from the general Γ-convergence theory for ERIS developed in [37], the limit passage relies on the law of large numbers applied to the quenched disorder, which allows us to control the full Young measure of the weakly convergent sequences $\mathscr{P}_N(\boldsymbol{e}^N(t))$ (see Step 3 in the proof).

Theorem 5.1. *Assume that $(\mu_j^N)_j$, Φ, G and ℓ satisfy (3.7). For $N \in \mathbb{N}$ choose initial conditions $\boldsymbol{e}_0^N \in \mathbb{R}^N$ that are ordered and satisfy*

$$\mathscr{P}_N(\boldsymbol{e}_0^N) \rightharpoonup (\overline{e}_0, \overline{p}_0) \text{ in } \mathscr{Q} = \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega) \text{ and}$$
$$\boldsymbol{E}_N(0, \boldsymbol{e}_0^N) \to \mathscr{E}(0, \overline{e}_0, \overline{p}_0) < \infty.$$

Then, with probability 1 the embeddings of the ordered solutions of $\boldsymbol{e}^N : [0, T] \to \mathbb{R}^N$ of $(\mathbb{R}^N, \mathscr{E}_N, \mathscr{D}_N)$ with $\boldsymbol{e}^N(0) = \boldsymbol{e}_0^N$ constructed in Section 4.2 converge to the unique solution $(\overline{e}, \overline{p}) : [0, T] \to \mathscr{Q}$ of $(\mathscr{Q}, \mathscr{E}, \mathscr{D})$ with $(\overline{e}(0), \overline{p}(0)) = (\overline{e}_0, \overline{p}_0)$, viz.

$$\mathscr{P}_N(\boldsymbol{e}^N(t)) \rightharpoonup (\overline{e}(t), \overline{p}(t)) \text{ in } \mathscr{Q} \quad \text{for all } t \in [0, T].$$

Moreover, $\boldsymbol{E}_N(t, \boldsymbol{e}^N(t)) \to \mathscr{E}(t, \overline{e}(t), \overline{p}(t))$ and $\mathrm{Diss}_{\boldsymbol{D}_N}(\boldsymbol{e}^N, [0, t]) \to \mathrm{Diss}_{\mathscr{D}}(\overline{p}, [0, t])$.

Proof. Since with probability 1 the biases $h_j^N = \mu_j^N - G(j/N)$ satisfy $h_i^N \neq h_j^N$ for $i \neq j$, the theory of Section 4 is applicable.

Step 1: For the proof we use our precise knowledge of the solutions $\boldsymbol{e}^N$. Note that the ordered states are uniquely determined by the function $m^N(t) : [0, T] \to \{0, \ldots, N\}$ counting the number of j such that $e_j^N(t)$ is bigger than 0. Moreover, we have

$$\sigma^N(t) = k\ell(t) - ak(2m^N(t) - N)/N. \tag{5.6}$$

Thus, $\sigma^N(t)$ also allows us to recover the solution $\boldsymbol{e}^N(t)$ completely as follows. For given t we define $h_+^N(t) > h_-^N(t)$ such that $m^N(t) = \#\{\, j \mid h_j \geqq h_+^N(t)\,\}$,

$$h_+^N(t) = \min\{\, h_j^N \mid h_j^N \geqq h_+^N(t)\,\}, \text{ and } \quad h_-^N(t) = \max\{\, h_l^N \mid h_l^N < h_+^N(t)\,\}.$$

Along solutions, the values of $h_\pm$ are equal to those of $\overline{h}_\pm$ (see (4.5)), but now they depend on $t \in [0, T]$. We have

$$e_j^N(t) = \mathrm{sign}(e_j^N(t))a + \tfrac{1}{k}(\sigma^N(t) + h_j^N)$$
$$\text{and} \quad \mathrm{sign}(e_j^N(t)) = \begin{cases} 1 & \text{for } h_j^N \geqq h_+^N(t), \\ -1 & \text{for } h_j^N \leqq h_-^N(t). \end{cases} \tag{5.7}$$

Step 2: We prove that convergence holds along a subsequence. However, since the limit problem has a unique solution, we know a priori that the whole sequence must converge. To find a convergent subsequence we consider the functions σ^N. By (3.7d) the interval $[0, T]$ can be decomposed into finitely many, let us say P, subintervals where ℓ is monotone. However, each m^N is also monotone in these subintervals. Since the variation of m^N in a monotone part is bounded by N, the variation of each m^N is at most PN. By (5.6) the variation of σ^N is bounded by $k\|\dot\ell\|_{L^1}+2akP$, and Helly's selection principle yields a subsequence (not relabeled) such that $\sigma^N(t) \to \sigma^\infty(t)$ for all $t \in [0, T]$, giving

$$m^N(t)/N \to \xi^\infty(t) = \big(k(\ell(t)-a) - \sigma^\infty(t)\big)/(2ak). \tag{5.8}$$

Step 3: Next we show that this convergence implies convergence of $(\overline{e}^N, \overline{p}^N) = \mathscr{P}_N(\boldsymbol{e}^N)$ as well as that of the energy and the dissipation. In fact, we show that for each $t \in [0, T]$ the sequence $(\overline{e}^N(t), \overline{p}^N(t))_{N\in\mathbb{N}}$ generates a well-defined Young measure $\nu(t) : \Omega \to \mathrm{Prob}(\mathbb{R}^2)$ (Radon measures on $\mathbb{R}$ with total measure 1). This follows from the independent random choices of μ_j^N using the law of large numbers. It is here, where we exploit the disorder in an essential fashion. Because the biases μ_j^N are chosen independently and identically distributed (see (3.7a)), the law of large numbers can be applied to any continuous function $\Xi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ to obtain

$$\frac{1}{N}\sum_{j=1}^{N} \Xi(j/N, \mu_j^N) \quad \to \quad \int_\Omega \int_{\mathbb{R}} \Xi(x, \mu) f(\mu)\,\mathrm{d}\mu\,\mathrm{d}x. \tag{5.9}$$

For a general test function $\Psi \in \mathrm{C}^0(\overline{\Omega} \times \mathbb{R}^2)$ we consider the limit of

$$\psi^N(t) := \int_\Omega \Psi(x, \overline{e}^N(t, x), \overline{p}^N(t, x))\,\mathrm{d}x = \tfrac{1}{N}\sum_{j=1}^{N} \Psi_j^N(e_j^N(t), a\,\mathrm{sign}(e_j(t)))$$

for $N \to \infty$. Here we used the abbreviation $\Psi_j^N(e, p) := \frac{1}{N}\int_{(j-1)/N}^{j/N} \Psi(y, e, p)\,\mathrm{d}y$ and the definition of $(\overline{e}^N, \overline{p}^N) = \mathscr{P}_N(\boldsymbol{e}^N)$. Substituting $e_j^N(t)$ via (5.7) we find

$$\begin{aligned}\psi^N(t) = \frac{1}{N} &\sum_{\{\, j \mid h_j^N \leqq h_-^N(t)\,\}} \Psi_j^N\Big(-a + \frac{1}{k}(\sigma^N(t)+h_j^N), -a\Big) \\ &+ \frac{1}{N}\sum_{\{\, j \mid h_j^N \geqq h_-^N(t)\,\}} \Psi_j^N\Big(a + \frac{1}{k}(\sigma^N(t)+h_j^N), a\Big).\end{aligned}$$

Recalling $h_j^N = \mu_j^N - G(j/N)$, where all the μ_j^N are independently chosen according to the density distribution f, we can pass to the limit $N \to \infty$. First observe that $h_\pm^N(t)$ converge to $h_\pm^\infty(t)$ defined by

$$h_-^\infty(t) = \sup\{\, h \mid F_G(h) < \xi^\infty(t)\,\} \quad \text{and} \quad h_+^\infty(t) = \inf\{\, h \mid F_G(h) > \xi^\infty(t)\,\},$$

where $F_G(h) := \int_\Omega \int_{\eta=-\infty}^{h} f(\eta+G(x))\,\mathrm{d}\eta\,\mathrm{d}x \in [0, 1]$ and ξ^∞ is defined in (5.8). Note that F_G is a probability distribution with compact support since f has compact support and G is bounded. Subsequently, it suffices to take any $h^\infty(t) \in [h_-^\infty(t), h_+^\infty(t)]$. Using $\sigma^N \to \sigma^\infty$ and the law of large numbers on μ_j^N (see (5.9)) we find $\psi^N(t) \to \psi^\infty(t)$ with

$$\psi^\infty(t) = \int_\Omega \int_{-\infty}^{h^\infty(t)} \Psi(x, -a+(\sigma^\infty(t)+h)/k, -a)\, f(h+G(x))\, \mathrm{d}h\, \mathrm{d}x$$
$$+ \int_\Omega \int_{h^\infty(t)}^{\infty} \Psi(x, a+(\sigma^\infty(t)+h)/k, a)\, f(h+G(x))\, \mathrm{d}h\, \mathrm{d}x.$$

The Young measure ν is defined via $\int_\Omega \int_{\mathbb{R}^2} \Psi(x, e, p)\nu(t, x, \mathrm{d}e, \mathrm{d}p)\, \mathrm{d}x = \psi^\infty(t)$:

$$\int_{\mathbb{R}^2} \widehat{\Psi}(e, p)\nu(t, x, \mathrm{d}e, \mathrm{d}p) = \int_{\mathbb{R}} \widehat{\Psi}\big(a\, \mathrm{sign}(\mu - \widehat{\mu}(t, x)) + (\sigma^\infty(t) + \mu - G(x))/k,\, a\, \mathrm{sign}(\mu - \widehat{\mu}(t, x))\big)\, f(\mu)\, \mathrm{d}\mu,$$

where $\widehat{\mu}(t, x)$ is any solution of $\xi^\infty(t) = F_G(\mu - G(x))$, for example

$$\widehat{\mu}(t, x) = h^\infty(t) + G(x). \tag{5.10}$$

Using $\int_{\mathbb{R}} \mathrm{sign}(\widehat{\mu} - \mu) f(\mu)\, \mathrm{d}\mu = 2F(\widehat{\mu}) - 1$ and the test functions $\widehat{\Psi}(e, p) = e$ and $\widehat{\Psi}(e, p) = p$, we obtain the weak limits $\overline{e}(t)$ and $\overline{p}(t)$, respectively, via

$$\begin{aligned}\overline{e}(t, x) &= \int_{\mathbb{R}} \big(a\, \mathrm{sign}(\mu - \widehat{\mu}(t, x)) + (\sigma^\infty(t) + \mu - G(x))/k\big) f(\mu)\, \mathrm{d}\mu \\ &= a(2F(\widehat{\mu}(t, x)) - 1) + (\sigma^\infty(t) - G(x))/k, \\ \overline{p}(t, x) &= a(2F(\widehat{\mu}(t, x)) - 1).\end{aligned} \tag{5.11}$$

Step 4: For the convergence of energy we use

$$\boldsymbol{E}_0^N(\boldsymbol{e}^N(t)) = \boldsymbol{E}_1^N(\boldsymbol{e}^N(t)) + \boldsymbol{E}_2^N(\boldsymbol{e}^N(t)), \quad \text{where}$$
$$\boldsymbol{E}_1^N(\boldsymbol{e}^N) = \frac{1}{N} \sum_1^N \frac{k}{2}(e_j^N - a\, \mathrm{sign}(e_j^N))^2 \quad \text{and} \quad \boldsymbol{E}_2^N(\boldsymbol{e}^N) = -\frac{1}{N} \sum_1^N h_j^N e_j^N.$$

Using the explicit form (5.7) of e_j^N we obtain

$$\boldsymbol{E}_1^N(\boldsymbol{e}^N(t)) = \tfrac{1}{N} \sum_{j=1}^N \tfrac{1}{2k} \big(\sigma^N(t) - G(j/N) + \mu_j^N\big)^2 \to \int_\Omega \tfrac{1}{2k} \big(\sigma^\infty(t) - G(x)\big)^2 \mathrm{d}x + \Gamma_f,$$

where Γ_f is defined in (5.4). For $\boldsymbol{E}_2^N$ we proceed as for $\psi^N(t)$ and obtain

$$\begin{aligned}&\boldsymbol{E}_2^N(\boldsymbol{e}^N(t)) \\ &= -\frac{1}{N} \sum_{h_j^N \le h_-^N(t)} h_j^N \big(-a + \frac{1}{k}(\sigma^N(t) + h_j^N)\big) - \frac{1}{N} \sum_{h_j^N \ge h_-^N(t)} h_j^N \big(a + \frac{1}{k}(\sigma^N(t) + h_j^N)\big) \\ &\to -\int_{\Omega \times \mathbb{R}} (\mu - G(x)) \Big(a\, \mathrm{sign}(\mu - \widehat{\mu}(t, x)) + (\sigma^\infty(t) - G(x) + \mu)/k\Big) f(\mu)\, \mathrm{d}\mu\, \mathrm{d}x.\end{aligned}$$

Using (5.11) the limit of $\boldsymbol{E}_1^N$ can be identified as a function of $(\overline{e}, \overline{p})$, namely

$$\boldsymbol{E}_1^N(\boldsymbol{e}^N(t)) \;\to\; \int_\Omega \frac{k}{2}(\overline{e}(t,x)-\overline{p}(t,x))^2\,\mathrm{d}x + \Gamma_f.$$

To identify the limit $\boldsymbol{E}_2^N(\boldsymbol{e}^N(t))$ we define $\widetilde{F}(\mu) = \frac{1}{2}\int_\mathbb{R} y\,\mathrm{sign}(\mu-y) f(y)\,\mathrm{d}y$, and denote by $\mu = \widehat{\mu}(\eta) \in [-\infty, \infty]$ any solution of $F(\mu) = \eta \in [0, 1]$. We claim:

(a) For $\eta \in [0, 1]$ we have $\mathscr{F}^*(\eta) = \widetilde{F}(\widehat{\mu}(\eta))$.
(b) For all $\mu, \eta \in \mathbb{R}$ we have: $\mu \in \partial\mathscr{F}^*(\eta) \iff \eta = F(\mu)$.

Indeed, the standard Legendre–Fenchel theory gives

$$\eta = \mathscr{F}'(\mu) = F(\mu) \;\Leftrightarrow\; \mu \in \partial\mathscr{F}^*(\eta) \;\Leftrightarrow\; \mu\eta = \mathscr{F}(\mu) + \mathscr{F}^*(\eta).$$

Thus, differentiating $\eta = F(\widehat{\mu}(\eta))$ yields $1 = f(\widehat{\mu}(\eta))\widehat{\mu}'(\eta)$. Moreover, the definition of $\widetilde{F}$ easily gives $\widetilde{F}'(\mu) = \mu f(\mu)$. Thus, the function $J : \eta \mapsto \widetilde{F}(\widehat{\mu}(\eta))$ satisfies $J'(\eta) = \widehat{\mu}(\eta)$ which leads to $J''(\eta) = \widehat{\mu}'(\eta) = 1/f(\widehat{\mu}(\eta))$. The properties of the Legendre transform give $(\mathscr{F}^*)''(\eta) = 1/\mathscr{F}''(\widehat{\mu}(\eta)) = 1/f(\widehat{\mu}(\eta)) = J''(\eta)$.

Finally, using $\widetilde{F}(\pm\infty) = 0$ we obtain $J(0) = J(1) = 0$. The definition of $\mathscr{F}$ gives $\mathscr{F}(\mu) = \max\{0, \mu\} + m(\mu)$ with $0 \leqq m(\mu) \to 0$ for $|\mu| \to \infty$, which implies $\mathscr{F}^*(0) = \mathscr{F}^*(1) = 0$. Since J and $\mathscr{F}$ coincide at $\eta = 0$ and 1 and have the same second derivative, they are the same on all of $[0, 1]$. Thus, (a) and (b) are established.

Based on these properties of the function $\widetilde{F}$ we can now write

$$\boldsymbol{E}_2^N(\boldsymbol{e}^N(t)) \to \int_\Omega 2a\widetilde{F}(\widehat{\mu}(t,x)) + G(x)\overline{e}(t,x)\,\mathrm{d}x - 2\Gamma_f.$$

Then, by using the representation of $\overline{p}$ in (5.11), the definition of H via $\mathscr{F}^*$, and the equivalence $\mu \in \partial H(p) \;\Leftrightarrow\; p = a(1-2F(\mu))$ we find $H(\overline{p}(t,x)) = 2a\widetilde{F}(\widehat{\mu}(t,x))$. The convergence $\boldsymbol{E}^N(t, \boldsymbol{e}^N(t)) \to \mathscr{E}(t, \overline{e}(t), \overline{p}(t))$ is therefore shown.

Step 5: To show convergence of the dissipation we use the piecewise monotonicity (3.7d) of ℓ, that is, there exist times $0 = t_0 < t_1 < \cdots < t_L = t$ such that ℓ is monotone on $[t_{l-1}, t_l]$. As a consequence the solutions $\boldsymbol{e}^N$ and $\overline{p}$ are monotone on these intervals. The definition of the functionals $\mathrm{Diss}_{\boldsymbol{D}_N}$ and $\mathrm{Diss}_\mathscr{D}$ then gives

$$\mathrm{Diss}_{\boldsymbol{D}_N}(\boldsymbol{e}^N, [0,t]) = \sum_{l=1}^L \boldsymbol{D}_N(\boldsymbol{e}^N(t_{l-1}), \boldsymbol{e}^N(t_l)),$$
$$\mathrm{Diss}_\mathscr{D}(\overline{p}, [0,t]) = \sum_{l=1}^L \mathscr{D}(\overline{p}(t_{l-1}), \overline{p}(t_l)).$$

Thus, it suffices to show convergence for only these time increments. Without loss of generality we consider the case $\ell(t_{l-1}) < \ell(t_l)$. With $\rho_N \to \rho_\infty = 2ka^2$ we have

$$\begin{aligned}\boldsymbol{D}_N(\boldsymbol{e}^N(t_{l-1}),\boldsymbol{e}^N(t_l)) &= \frac{1}{N}\sum_1^N \rho_N\big(\operatorname{sign}(e_j^N(t_l))-\operatorname{sign}(e_j^N(t_{l_1}))\big)\\ &= \tfrac{\rho_N}{N}(m^N(t_l)-m^N(t_{l-1})) \;\to\; \rho_\infty(\xi^\infty(t_l)-\xi^\infty(t_{l-1}))\\ &= \textstyle\int_\Omega ka(\overline{p}(t_l,x)-\overline{p}(t_{l-1},x))\,\mathrm{d}x = \mathscr{D}(\overline{p}(t_{l-1}),\overline{p}(t_l)).\end{aligned}$$

Thus, $\mathrm{Diss}_{\boldsymbol{D}_N}(\boldsymbol{e}^N,[0,t]) \to \mathrm{Diss}_{\mathscr{D}}(\overline{p},[0,t])$ is established as well.

Step 6: It remains to show that $(\overline{e},\overline{p})$ is the unique energetic solution for the macroscopic ERIS $(\mathscr{Q},\mathscr{E},\mathscr{D})$. We first consider the energy balance. For all N we have the microscopic energy balance

$$\boldsymbol{E}_N(t,\boldsymbol{e}^N(t)) + \mathrm{Diss}_{\boldsymbol{D}_N}(\boldsymbol{e}^N,[0,t]) = \boldsymbol{E}_N(0,\boldsymbol{e}_0^N) + \int_0^t \sigma^N(s)\dot{\ell}(s)\,\mathrm{d}s.$$

Since all four terms converge to the desired limits for N we immediately obtain the energy balance (E) for the limit $(\overline{e},\overline{p})$ with respect to the ERIS $(\mathscr{Q},\mathscr{E},\mathscr{D})$.

To establish the stability condition

$$\mathscr{E}(t,\overline{e}(t),\overline{p}(t)) \leqq \mathscr{E}(t,\widetilde{e},\widetilde{p}) + \mathscr{D}(\overline{p}(t),\widetilde{p}) \quad \text{for all } (\widetilde{e},\widetilde{p})\in\mathscr{Q},$$

we use the stability of $\boldsymbol{e}^N(t)$ with respect to $(\mathbb{R}^N,\boldsymbol{E}_N,\boldsymbol{D}_N)$. We test the stability using states $\widetilde{\boldsymbol{e}}^N$ defined like $\boldsymbol{e}^N(t)$ but with a different function $\widetilde{G}$ replacing G. For arbitrary $\widetilde{G}\in \mathrm{H}^1(\Omega)$ with $\int_\Omega \widetilde{G}(x)\,\mathrm{d}x = 0$ we define the new bias vector $(\widetilde{h}_j^N)_j\in\mathbb{R}^N$ via

$$\widetilde{h}_j^N = \mu_j^N - \widetilde{G}(j/N) + \widetilde{\lambda}^N, \text{ where } \sum_1^N \widetilde{h}_j^N = 0.$$

It is essential here that the random biases μ_j^N are the same as the ones used for the construction of $\boldsymbol{e}^N$. We define $F_{\widetilde{G}}$ via $F_{\widetilde{G}}(h) = \int_\Omega F(h+\widetilde{G}(x))\,\mathrm{d}x$. Then, for every pair $(\widetilde{\xi},\widetilde{h})$ satisfying

$$1-\widetilde{\xi} = F_{\widetilde{G}}(\widetilde{h}) \quad\text{and}\quad |\widetilde{\sigma}+\widetilde{h}| \leqq ka, \quad \text{where } \widetilde{\sigma} = k\ell(t) - ak(2\widetilde{\xi}-1),$$

there exists a sequence of thresholds $\widetilde{h}^N$ such that

$$\widetilde{e}_j^N = a\operatorname{sign} e_j^N + \frac{1}{k}(\widetilde{\sigma}^N+\widetilde{h}_j^N) \text{ and } \operatorname{sign}\widetilde{e}_j^N = \begin{cases} 1 & \text{if } \widetilde{h}_j^N \geqq \widetilde{h}^N,\\ -1 & \text{if } \widetilde{h}_j^N < \widetilde{h}^N,\end{cases}$$
$$\widetilde{\sigma}^N \to \widetilde{\sigma}, \quad \widetilde{h}^N \to \widetilde{h}, \quad (2\widetilde{m}^N-1)/N \to \widetilde{\xi},$$

where $\widetilde{m}^N = \big(N + \sum_1^N \operatorname{sign}\widetilde{e}_j^N\big)/2$, $\widetilde{\sigma}^N = k\ell(t) - ak(2\widetilde{m}^N-1)/N$.

Repeating the calculations in Step 3 we obtain $\mathscr{P}_N(\widetilde{\boldsymbol{e}}^N) \rightharpoonup (\widetilde{e},\widetilde{p})$ in $\mathscr{Q}$, where

$$0 = k(\widetilde{e}-\widetilde{p}) + \widetilde{G} - \widetilde{\sigma} \;\text{ and }\; \widetilde{p}(x) = a(1-2F(\widetilde{h}+\widetilde{G}(x))). \tag{5.12}$$

Repeating the calculations in Step 4, while carefully distinguishing between the still relevant h_j^N and the artificial $\widetilde{h}_j^N$, which differ only by $\widetilde{G}(j/N)-\widetilde{\lambda}^N-G(j/N)+\lambda^N$, we find the convergence $\boldsymbol{E}_N(t,\widetilde{\boldsymbol{e}}^N)\to\mathscr{E}(t,\widetilde{e},\widetilde{p})$.

Moreover, we are able to calculate the limit of $\boldsymbol{D}_N(\boldsymbol{e}^N(t),\widetilde{\boldsymbol{e}}^N)$ as follows (using $h^N=h_\pm^N(t)$ and neglecting $\lambda^N,\widetilde{\lambda}^N\to 0$):

$$
\begin{aligned}
\boldsymbol{D}_N(\boldsymbol{e}^N(t),\widetilde{\boldsymbol{e}}^N) &= \frac{\rho_N}{N}\sum_{j=1}^N |\operatorname{sign} e_j^N(t)-\operatorname{sign}\widetilde{e}_j^N|\\
&= \frac{\rho_N}{N}\Big(\#\{\, j \mid h^N+G(j/N)\leqq \mu_j^N < \widetilde{h}^N+\widetilde{G}(j/N)\,\}\\
&\quad +\#\{\, j \mid \widetilde{h}^N+\widetilde{G}(j/N)\leqq \mu_j^N < h^N+G(j/N)\,\}\Big)\\
&\quad \to\ \rho_\infty\int_\Omega\Big(\big[F(\widetilde{h}+\widetilde{G}(x))-F(h^\infty+G(x))\big]^+\\
&\quad +\big[F(h^\infty+G(x))-F(\widetilde{h}+\widetilde{G}(x))\big]^+\Big)\,\mathrm{d}x\\
&= 2ka^2\int_\Omega |F(h^\infty+G(x))-F(\widetilde{h}+\widetilde{G}(x))|\,\mathrm{d}x\\
&= 2ka^2\int_\Omega\left|\frac{1}{2a}(a-\overline{p}(t,x))-\frac{1}{2a}(a-\widetilde{p}(x))\right|\mathrm{d}x = ka\int_\Omega\big|\overline{p}(t,x)-\widetilde{p}(x)\big|\,\mathrm{d}x\\
&= \mathscr{D}(\overline{p}(t),\widetilde{p}),
\end{aligned}
$$

where $[a]^+=\max\{0,a\}$. Hence, we can pass to the limit in the stability condition $\boldsymbol{E}_N(t,\boldsymbol{e}^N(t))\leqq\boldsymbol{E}_N(t,\widetilde{\boldsymbol{e}}^N)+\boldsymbol{D}_N(\boldsymbol{e}^N(t),\widetilde{\boldsymbol{e}}^N)$ and obtain $\mathscr{E}(t,\overline{e}(t),\overline{p}(t))\leqq\mathscr{E}(\widetilde{e},\widetilde{p})+\mathscr{D}(\overline{p}(t),\widetilde{p})$, where the comparison states $(\widetilde{e},\widetilde{p})$ are the ones constructed in (5.12). Via the free choice of $\widetilde{G}$ we are able to generate a dense set of $\widetilde{p}$ in $\mathrm{L}^2(\Omega;[-a,a])$. However, the associated strains $\widetilde{e}$ are the equilibrium strains. By the quadratic nature of $\mathscr{E}$, we easily find $\mathscr{E}(t,\widehat{e},\widetilde{p})\geqq\mathscr{E}(t,\widetilde{e},\widetilde{p})$ for all $\widehat{e}\in\mathrm{L}^2(\Omega)$. Thus, the stability of $(\overline{e}(t),\overline{p}(t))$ is established, and $(\overline{e},\overline{p}):[0,T]\to\mathscr{Q}$ is shown to be an energetic solution for $(\mathscr{Q},\mathscr{E},\mathscr{D})$. □

Notice that the crucial assumption regarding the presence of quenched disorder was used in Step 3 of the above proof, see (5.9). In fact, much less than the assumed randomness in (3.7a) (allowing for the application of the strong law of large numbers) is sufficient to derive (5.9). We need only a type of weak ergodicity that could, for instance, be also generated by quasiperiodic functions.

5.3. Double Asymptotics $(\varepsilon,\delta)\to(0,0)$

Finally, we prove that in the case of bi-quadratic potential the limit does not change if one performs the double asymptotics $(\varepsilon,\delta)\to(0,0)$ under the constraint that δ tends to 0 faster than ε. The result is a consequence of the estimates obtained in Theorem 4.5, which allow one to show that the L^2 difference between the viscous solutions and the discrete solutions tends to 0 with $(\varepsilon,\delta)\to(0,0)$. Since the latter converge weakly, it follows that the former also converge weakly.

Theorem 5.2. *Assume that* (3.7) *is satisfied. Then, there exists a constant* κ_* *such that the following holds. Consider the solutions* $\boldsymbol{e}^{\delta,N} : [0,T] \to \mathbb{R}^N$ *of the viscous problem* (4.8)*, where* $h_j^N = \mu_j^N - G(j/N)$ *with initial conditions* $\boldsymbol{e}^{\delta,N}(0)$ *that are ordered equilibria and satisfy*

$$\mathscr{P}_N(\boldsymbol{e}^{\delta,N}(0)) \rightharpoonup (\overline{e}_0, \overline{p}_0) \text{ in } \mathscr{Q} \quad \text{and} \quad \boldsymbol{E}^N(0, \boldsymbol{e}^{\delta,N}(0)) \to \mathscr{E}(0, \overline{e}_0, \overline{p}_0)$$

as $(\varepsilon, \delta) \to 0$ *with* $0 < \delta < \kappa_* \varepsilon$*. Then with probability 1 with respect to the random biases* μ_j^N *we have*

$$\mathscr{P}_N(\boldsymbol{e}^{\delta,N}(t)) \rightharpoonup (\overline{e}(t), \overline{p}(t)) \quad \text{in } \mathscr{Q} \quad \text{for all } t \in [0,T] \text{ as } (\varepsilon,\delta) \to 0 \quad \text{with } 0 < \delta < \kappa_*\varepsilon,$$

where $(\overline{e}, \overline{p})$ *is the unique solution of the elastoplasticity system* (5.5) *with initial data* $(\overline{e}(0), \overline{p}(0)) = (\overline{e}_0, \overline{p}_0)$.

Proof. The crucial observation is that the definition of the norms $|\cdot|_p$ in $\mathbb{R}^N$ and in $\mathrm{L}^p(\Omega)$ together with the embedding $\mathscr{P}_N$ lead to an additional factor $1/N^{1-1/p}$. For $(\widetilde{e}, \widetilde{p}) = \mathscr{P}_N(\widetilde{\boldsymbol{e}}^N)$ and $(\widehat{e}, \widehat{p}) = \mathscr{P}_N(\widehat{\boldsymbol{e}}^N)$ we have

$$\|\widetilde{e} - \widehat{e}\|_{\mathrm{L}^2(\Omega)} \leqq \frac{1}{N^q} |\widetilde{\boldsymbol{e}}^N - \widehat{\boldsymbol{e}}^N|_p \quad \text{for } p \in [1,\infty] \quad \text{with } q = \min\{1/2, 1/p\},$$
$$\|\widetilde{p} - \widehat{p}\|_{\mathrm{L}^2(\Omega)} = \frac{2a}{\sqrt{N}} \big(\#\{\, j \mid \operatorname{sign} \widetilde{e}_j^N \neq \operatorname{sign} \widehat{e}_j^N \,\}\big)^{1/2}.$$

If $\delta \leqq \kappa_*/N = \kappa_*\varepsilon$, where κ_* is the same as in Theorem 4.5, estimate (4.14) (with $p = \infty$ and $p = 1$ for R_1 and R_2, respectively) yields

$$\|\overline{\boldsymbol{e}}^{\delta,N}(t) - \overline{\boldsymbol{e}}^{0,N}\|_{\mathrm{L}^2(\Omega)} \leqq C\big(\delta + 1/N^{1/2}\big).$$

Moreover, the number of different signs between $\boldsymbol{e}^{\delta,N}(t)$ and $\boldsymbol{e}^{0,N}(t)$ is bounded by N_* (independently of δ and N), which leads to the estimate

$$\|\mathscr{P}_N(\boldsymbol{e}^{\delta,N}(t)) - \mathscr{P}_N(\boldsymbol{e}^{0,N}(t))\|_{\mathrm{L}^2(\Omega)} \leqq C_2\big(\delta + 1/N^{1/2}\big) \leqq C_3/N^{1/2} = C_3\varepsilon^{1/2},$$

where we have used $\delta \leqq \kappa_*/N = \kappa_*\varepsilon$ again. Combining this with the convergence obtained in Theorem 5.1 the desired convergence result is established. □

6. General Potentials

In the previous sections we have restricted our analysis by assuming in (3.7) that: (a) $\Phi = \Phi_{\text{biq}}$, (b) the loading $G(x) = \int_0^x g_{\text{ext}}(y)\,\mathrm{d}y$ is time independent, and (c) ℓ is piecewise monotone. Here we discuss the necessary changes in the results if these assumptions are dropped. More precisely, we argue that, in the case of a general double-well potential and rather general time-dependent body forces, the sequence of limits, first $\delta \to 0$ and then $\varepsilon = 1/N \to 0$, leads to basically the same general picture modulo appropriate modification of the hardening potential and the dissipation potential in the limiting model.

6.1. Microscopic Model

To replace $G(x)$ by a general time-dependent function $G(t, x)$ we need to generalize the concept of *ordered states*. Indeed, since the loading may now depend on time, a state that is ordered for t_1 may no longer be ordered for $t_2 > t_1$. Therefore we need to interpret the order condition locally in $(t, x) \in [0, T] \times \Omega$. This is possible, since $G(t, x)$, $\ell(t)$, and $\sigma(t)$ vary only on the macroscopic scale while the bias coefficients fluctuate on the microscopic scale and are independent of time.

Moreover, since the general double-well potential Φ does not allow us to define a plastic strain $p = a\,\mathrm{sign}(e)$ as in the bi-quadratic case, we need to use the microscopic phase indicator variable $z_j \in \{-1, 0, 1\}$ as in Section 2. The threshold $\widehat{\mu}(t, x)$ is now active in a microscopically large but macroscopically small region, which can be defined as follows $|j - xN| \leqq \sqrt{N}$. For j in this domain, the condition $\mu_j^N > \widehat{\mu}(t, x)$ then implies $e_j^N(t) \geqq e_+$ and $z_j^N(t) = 1$, whereas $\mu_j^N < \widehat{\mu}(t, x)$ implies $e_j^N(t) \leqq e_-$ and $z_j^N(t) = -1$.

In the formal proof which follows, the important issue will be to control the evolution of the threshold $\widehat{\mu}(t, x)$. Looking at the dynamics of the discrete automaton in Definition 4.2 we see that phase changes should occur only if the strain is critical. In terms of the macroscopic stress $\overline{\sigma}(t, x) = \sigma(t) - G(t, x)$, we need to have $\sigma_+ = \widehat{\mu} + \overline{\sigma}$, if $\dot{\widehat{\mu}} < 0$, and $\sigma_- = \widehat{\mu} + \overline{\sigma}$, if $\dot{\widehat{\mu}} > 0$. Moreover, the threshold value $\widehat{\mu}(t, x)$ must always satisfy $\overline{\sigma} + \widehat{\mu} \in [\sigma_-, \sigma_+]$.

6.2. Macroscopic Energy

As in the special case of bi-quadratic energy, we begin with formally computing the limiting continuum energy and determining the hardening potential.

Notice that the relation (3.3) provides a strong correlation between e_j and μ_j and thus controls the joint Young measures ν generated by $(\overline{e}^N, \overline{z}^N)$, which takes the form

$$\int_{\mathbb{R}^2} \widehat{\Psi}(e, z)\nu(t, x, \mathrm{d}e, \mathrm{d}z) = \int_{\mathbb{R}} \widehat{\Psi}\big(\mathrm{sign}(\mu - \widehat{\mu}(t, x)), \psi_{\mathrm{sign}(\mu-\widehat{\mu}(t,x))}(\overline{\sigma}(t, x) + \mu)\big) f(\mu)\,\mathrm{d}\mu.$$

In particular, we can define the macroscopic constitutive relations

$$\widehat{E}(\widetilde{\sigma}, \widetilde{\mu}) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}} \psi_{\mathrm{sign}(\mu-\widetilde{\mu})}(\widetilde{\sigma} + \mu) f(\mu)\,\mathrm{d}\mu, \quad \widehat{Z}(\widetilde{\mu}) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}} \mathrm{sign}(\mu - \widetilde{\mu}) f(\mu)\,\mathrm{d}\mu, \tag{6.1}$$

such that the limits $\overline{e}$ and $\overline{z}$ satisfy

$$\overline{e}(t, x) = \widehat{E}(\overline{\sigma}(t, x), \widehat{\mu}(t, x)) \quad \text{and} \quad \overline{z}(t, x) = \widehat{Z}(\widehat{\mu}(t, x)).$$

By $\sigma = \widehat{S}(e, \mu)$ we denote the unique solution σ of $e = \widehat{E}(\sigma, \mu)$. We can now compute the effective potential as a function of $\overline{e}$ and $\widehat{\mu}$ via

$$\widehat{\Phi}(\overline{e}, \widehat{\mu}) = \int_M \Big(\Phi\big(\psi_{\mathrm{sign}(\mu-\widehat{\mu})}(\widehat{S}(\overline{e}, \widehat{\mu}) + \mu)\big) - \mu\psi_{\mathrm{sign}(\mu-\widehat{\mu})}(\widehat{S}(\overline{e}, \widehat{\mu}) + \mu)\Big) f(\mu)\,\mathrm{d}\mu.$$

The joint Young measure $\widehat{\nu}_{(\overline{e},\widehat{\mu})}$ generated by (e_j, μ_j) and associated with the macroscopic pair $(\overline{e}, \widehat{\mu})$ has the form

$$\int_{\mathbb{R}^2} \widehat{\Psi}(e, \mu)\nu_{(\overline{e},\widehat{\mu})}(\mathrm{d}e, \mathrm{d}\mu) = \int_M \widehat{\Psi}\big(\psi_{\mathrm{sign}(\mu-\widehat{\mu})}(\widehat{S}(\overline{e}, \widehat{\mu})+\mu), \mu\big) f(\mu)\,\mathrm{d}\mu,$$

where $\widehat{\Psi} \in \mathrm{C}_0(\mathbb{R}^2)$ is an arbitrary test function. In particular, it can by checked that the definitions of $\widehat{S}$ and $\widehat{\Phi}$ are compatible in the sense that $\widehat{S}(\overline{e}, \widehat{\mu}) = \partial_{\overline{e}}\widehat{\Phi}(\overline{e}, \widehat{\mu})$.

To calculate the partial derivative of $\widehat{\Phi}$ with respect to $\widehat{\mu}$ we introduce the functions

$$\varphi_\pm(\sigma) = \psi_\pm(\sigma)\sigma - \Phi(\psi_\pm(\sigma)), \tag{6.2}$$

which satisfy the relations

$$\varphi'_\pm(\sigma) = \psi_\pm(\sigma), \quad \varphi_+(\sigma) = \sup_{e \geqq e_+} \sigma e - \Phi(e), \quad \text{and} \quad \varphi_-(\sigma) = \sup_{e \leqq e_-} \sigma e - \Phi(e).$$

For the derivative we obtain (after some elementary calculations involving the chain rule)

$$\begin{aligned}\partial_{\widehat{\mu}}\widehat{\Phi}(\overline{e}, \widehat{\mu}) &= \frac{\partial}{\partial\widehat{\mu}}\Big[\int_{-\infty}^{\widehat{\mu}} \big(\Phi(\psi_{-1}(\widehat{S}(\overline{e}, \widehat{\mu})+\mu)) - \mu\psi_{-1}(\widehat{S}(\overline{e}, \widehat{\mu})+\mu)\big) f(\mu)\,\mathrm{d}\mu \\ &\qquad + \int_{\widehat{\mu}}^{\infty} \big(\Phi(\psi_1(\widehat{S}(\overline{e}, \widehat{\mu})+\mu)) - \mu\psi_1(\widehat{S}(\overline{e}, \widehat{\mu})+\mu)\big) f(\mu)\,\mathrm{d}\mu\Big] \\ &= \big(\varphi_+(\widehat{S}(\overline{e}, \widehat{\mu})+\widehat{\mu}) - \varphi_-(\widehat{S}(\overline{e}, \widehat{\mu})+\widehat{\mu})\big) f(\widehat{\mu}).\end{aligned}$$

Notice that the disorder threshold $\widehat{\mu}$ enters our formulas as a parametrization and that the energy representation in terms of elastic and plastic variables is still implicit. To abolish the auxiliary variable $\widehat{\mu}(t, x)$ and to replace it by the continuous internal variable $\overline{z}(t, x) = \widehat{Z}(\widehat{\mu})$, we assume that the latter relation is invertible. We write $\widehat{\mu} = \widetilde{\mu}(\overline{z})$ and apply the chain rule in (6.1) to obtain

$$\widetilde{\mu}'(\overline{z}) = \frac{-1}{2f(\widetilde{\mu}(\overline{z}))} < 0.$$

We can now define the stored energy density $\overline{\Phi}$ and the stress $\overline{S}$ via

$$\overline{\Phi}(\overline{e}, \overline{z}) = \widehat{\Phi}(\overline{e}, \widetilde{\mu}(\overline{z})) \quad \text{and} \quad \overline{S}(\overline{e}, \overline{z}) = \widehat{S}(\overline{e}, \widetilde{\mu}(\overline{z})),$$

which still satisfy the relation $\partial_{\overline{e}}\overline{\Phi} = \overline{S}$. Moreover, we find the identities

$$\begin{aligned}\partial_{\overline{z}}\overline{\Phi}(\overline{e}, \overline{z}) &= \partial_{\widehat{\mu}}\widehat{\Phi}\widetilde{\mu}' = \varphi_-(\overline{S}(\overline{e}, \overline{z})+\widetilde{\mu}(\overline{z})) - \varphi_+(\overline{S}(\overline{e}, \overline{z})+\widetilde{\mu}(\overline{z})), \\ \partial^2_{\overline{z}}\overline{\Phi}(\overline{e}, \overline{z}) &= \big(\psi_-(\overline{S}(\overline{e}, \overline{z})+\widetilde{\mu}(\overline{z}))-\psi_+(\overline{S}(\overline{e}, \overline{z})+\widetilde{\mu}(\overline{z}))\big)\big(\partial_{\overline{z}}\overline{S}(e, \overline{z})+\tfrac{1}{f(\widetilde{\mu}(\overline{z}))}\big) > 0.\end{aligned} \tag{6.3}$$

Next we show that the function $(\overline{e}, \overline{z}) \mapsto \overline{\Phi}(\overline{e}, \overline{z})$ is convex, which is an important property for proving existence and uniqueness of solutions for the associated plasticity problem. For this we introduce the auxiliary functions

$$\widetilde{E}(\sigma, \overline{z}, \mu) = \psi_{\mathrm{sign}(\mu-\widetilde{\mu}(\overline{z}))}(\sigma+\mu) \;\text{ and }\; E(\sigma, \overline{z}) = \int_{\mathbb{R}} \widetilde{E}(\sigma, \overline{z}, \mu) f(\mu)\,\mathrm{d}\mu,$$

which satisfy $\partial_{\overline{z}} E(\sigma, \overline{z}) = \psi_-(\sigma + \widetilde{\mu}(\overline{z})) - \psi_+(\sigma + \widetilde{\mu}(\overline{z}))$. We then have $\sigma = \overline{S}(\overline{e}, \overline{z})$ if and only if $\overline{e} = E(\sigma, \overline{z}) = \widehat{E}(\sigma, \widetilde{\mu}(\overline{z}))$. Moreover, we define

$$\overline{E}(\overline{e}, \overline{z}, \mu) \stackrel{\text{def}}{=} \widetilde{E}(\overline{S}(\overline{e}, \overline{z}), \overline{z}, \mu)$$

and find the relations

$$\overline{e} = \int_{\mathbb{R}} \overline{E}(\overline{e}, \overline{z}, \mu) f(\mu) \,\mathrm{d}\mu \quad \text{and} \quad \phi(\overline{E}(\overline{e}, \overline{z}, \mu)) - \mu = \overline{S}(\overline{e}, \overline{z}). \tag{6.4}$$

Then, the stored-energy density takes the form

$$\overline{\Phi}(\overline{e}, \overline{z}) = \int_{\mathbb{R}} \Big(\Phi(\overline{E}(\overline{e}, \overline{z}, \mu)) - \mu \overline{E}(\overline{e}, \overline{z}, \mu) \Big) f(\mu) \,\mathrm{d}\mu. \tag{6.5}$$

Lemma 6.1. *The derivatives of $\overline{\Phi}$ take the following form*

$$\partial_{\overline{e}} \overline{\Phi} = \overline{S}, \quad \partial_{\overline{z}} \overline{\Phi} = \varphi_+(\overline{S}(\overline{e}, \overline{z}) + \widetilde{\mu}(\overline{z})) - \varphi_-(\overline{S}(\overline{e}, \overline{z}) + \widetilde{\mu}(\overline{z})),$$

$$\mathrm{D}^2 \overline{\Phi} = \begin{pmatrix} \partial_{\overline{e}} \overline{S} & \Delta \partial_{\overline{e}} \overline{S} \\ \Delta \partial_{\overline{e}} \overline{S} & \Delta^2 \partial_{\overline{e}} \overline{S} + \frac{\Delta}{f(\widetilde{\mu}(\overline{z}))} \end{pmatrix}$$

where $\partial_{\overline{e}} \overline{S} = \frac{1}{\partial_\sigma E(\overline{S}(\overline{e}, \overline{z}), \overline{z})} > 0$ *and* $\Delta = \psi_+(\overline{S}(\overline{e}, \overline{z}) + \widetilde{\mu}(\overline{z})) - \psi_-(\overline{S}(\overline{e}, \overline{z}) + \widetilde{\mu}(\overline{z})) > 0$. *Hence, $\overline{\Phi}$ is uniformly convex.*

Proof. The formula for $\partial_{\overline{e}} \overline{\Phi}$ follows by differentiation under the integral and using (6.4). The formula for $\partial_{\overline{z}} \overline{\Phi}$ follows by using $\widetilde{\mu}'(z) = 1/f(\widetilde{\mu}(z))$ and $\overline{E}(e, z, \mu) = \psi_\pm(\overline{S}(e, z)) + \mu$ for $\mu > \widetilde{\mu}(z)$ and $\mu < \widetilde{\mu}(z)$, respectively.

Differentiating $e = E(\overline{S}(e, z), z)$ with respect to e and using the definition of E we obtain the formula for $\partial_{\overline{e}} \overline{S} = \partial^2_{\overline{e}} \overline{\Phi}$. For the mixed derivative we can use $\varphi'_\pm(\sigma) = \psi_\pm(\sigma)$ to find $\partial_{\overline{e}} \big(\partial_{\overline{z}} \overline{\Phi} \big)$. For $\partial^2_{\overline{z}} \overline{\Phi}$ we differentiate $e = E(\overline{S}(e, z), z)$ with respect to z and find $\partial_z \overline{S}(e, z) = \frac{-\partial_z E}{\partial_\sigma E} = \Delta \partial_e \overline{S}$. Together this gives

$$\partial^2_{\overline{z}} \overline{\Phi} = (\varphi'_+ - \varphi'_-) \big(\Delta \partial_e \overline{S} + \widetilde{\mu}(\overline{z}) \big) = \Delta \big(\Delta \partial_e \overline{S} + 1/f(\widetilde{\mu}(\overline{z})) \big),$$

which is the desired result. □

The above calculations can be done explicity for the bi-quadratic potential Φ_{biq}, see (2.1). We have $\psi_\pm(\sigma) = \sigma/k \pm a$ and find

$$E(\sigma, \overline{z}) = \int_{\mathbb{R}} \big(\sigma/k + a \,\mathrm{sign}(\mu - \widetilde{\mu}(\overline{z})) \big) f(\mu) \,\mathrm{d}\mu = \frac{\sigma}{k} + a\overline{z}.$$

Hence, $\overline{S}(\overline{e}, \overline{z}) = k(\overline{e} - a\overline{z})$, which results in

$$\overline{E}(\overline{e}, \overline{z}, \mu) = \overline{e} - a\overline{z} + \frac{\mu}{k} + a \,\mathrm{sign}(\mu - \widetilde{\mu}(\overline{z})).$$

Inserting this into the definition (6.5) of $\overline{\Phi}$ (with $\Phi = \Phi_{\mathrm{biq}}$) we can use the crucial identity $\Phi_{\mathrm{biq}}(\overline{E}(\overline{e}, \overline{z}, \mu)) = \frac{k}{2}(\overline{e} - a\overline{z} + \frac{\mu}{k})^2$. This follows from the stress relation

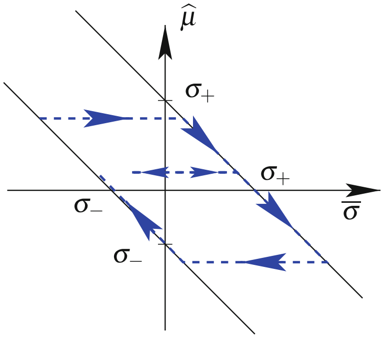

Fig. 8. Evolution of the play operator generated by (6.6)

$\overline{S}(\overline{e},\overline{z}) + \widetilde{\mu}(\overline{z}) \in [\sigma_-,\sigma_+] = [-ka,ka]$, which implies $\text{sign}(\mu-\widetilde{\mu}(\overline{z})) = \text{sign}\,\overline{E}$. Hence, on the one hand we have $\int_{\mathbb{R}} \Phi_{\text{biq}}(\overline{E}(\overline{e},\overline{z},\mu)) f(\mu)\,\mathrm{d}\mu = \frac{k}{2}(\overline{e}-a\overline{z})^2 + 2\Gamma_f$, while on the other hand we have

$$\int_{\mathbb{R}} (-\mu)\overline{E}(\overline{e},\overline{z},\mu) f(\mu)\,\mathrm{d}\mu = -\Gamma_f + a\widetilde{F}(\widetilde{\mu}(z)) = -\Gamma_f + H(z/a).$$

This gives the desired formula in (5.4).

6.3. Macroscopic Dissipation Potential

We now turn to the analysis of the dynamics of $\overline{z}$, which is strongly linked to that of $\widehat{\mu}$ via $\overline{z} = \widehat{Z}(\widehat{\mu})$. From the above we know that $\overline{\sigma} + \widehat{\mu} \in [\sigma_-,\sigma_+]$ and that $\sigma_+ = \widehat{\mu} + \overline{\sigma}$, if $\dot{\widehat{\mu}} < 0$, and $\sigma_- = \widehat{\mu} + \overline{\sigma}$, if $\dot{\widehat{\mu}} > 0$. These conditions can be formulated as a play operator in the form

$$0 \in \partial\widehat{R}(\dot{\widehat{\mu}}(t,x)) + \widehat{\mu}(t,x) + \overline{\sigma}(t,x), \tag{6.6}$$

where the 1-homogeneous friction potential $\widehat{R} : \mathbb{R} \to \mathbb{R}$ is given via

$$\widehat{R}(\dot{\mu}) = -\sigma_{-\,\text{sign}(\dot{\mu})}\dot{\mu} = \begin{cases} -\sigma_-\dot{\mu} & \text{for } \dot{\mu} \geqq 0, \\ -\sigma_+\dot{\mu} & \text{for } \dot{\mu} \leqq 0. \end{cases}$$

This is a classical hysteresis operator that provides for each $\overline{\sigma}$ a unique solution $\widehat{\mu}$, see [6,27,64] and also Fig. 8. Note that $\widehat{\mu} + \overline{\sigma}$ always lie in the interval $[\sigma_-,\sigma_+]$. Moreover, $\widehat{\mu}$ can change only if $\widehat{\mu} + \overline{\sigma}$ is either σ_- or σ_+.

To define the macroscopic dissipation potential we introduce the two quantities

$$\rho_+ \overset{\text{def}}{=} \int_{e_-}^{\psi_+(\sigma_+)} \sigma_+ - \phi(e)\,\mathrm{d}e > 0 \quad\text{and}\quad \rho_- \overset{\text{def}}{=} \int_{\psi_-(\sigma_-)}^{e_+} \phi(e) - \sigma_-\,\mathrm{d}e > 0. \tag{6.7}$$

Recalling $\varphi_\pm$ defined in (6.2) we have the following identities, see also Fig. 9:

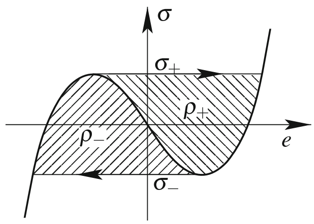

Fig. 9. The areas ρ_+ and ρ_- correspond to energies dissipated during jumps

Lemma 6.2. *For the areas enclosed by the the graph of* ϕ *and the hysteresis loop we have*

$$\rho_+ = \varphi_+(\sigma_+) - \varphi_-(\sigma_+) > 0 \quad \textit{and} \quad \rho_- = \varphi_-(\sigma_-) - \varphi_+(\sigma_-) > 0,$$

Moreover, we have the force relation $\overline{S}(\overline{e}, \overline{z}) + \widetilde{\mu}(\overline{z}) = \sigma_\pm \implies \partial_{\overline{z}}\overline{\Phi}(\overline{e}, \overline{z}) = \mp\rho_\pm$.

Proof. The integral formulae follow easily using $e_\mp = E_\pm(\sigma_\pm)$ and the definition of $\varphi_\pm$ in (6.2). The second statement follows directly from (6.3). □

The above computations show that the critical thresholds $-\sigma_\pm$ for $\overline{\sigma}+\widehat{\mu}$ are reached if and only if $\partial_z\overline{\Phi}(\overline{e}, \overline{z})$ reaches the critical values $\rho_\pm$. Hence, the play operator in (6.6) is equivalent to

$$0 \in \partial\overline{R}(\dot{\overline{z}}) + \partial_{\overline{z}}\overline{\Phi}(\overline{e}, \overline{z}) \quad \text{with } \overline{R}(\overline{v}) \stackrel{\text{def}}{=} \rho_{\text{sign}(\overline{v})}|\overline{v}|. \tag{6.8}$$

6.4. General Elasto-Plastic System

We can now formulate the general macroscopic equations in terms of the variables $\overline{e}$ and $\overline{z}$. Consider the solutions $\boldsymbol{e}^{N,\delta} : [0, T] \to \mathbb{R}^N$ of (2.7). Under the above hypotheses we expect that the embedding $(\overline{e}^{N,\delta}, \overline{z}^{N,\delta}) : [0, T] \to \mathrm{L}^2(\Omega)^2$ converge in the limit "$\lim_{N\to\infty} \lim_{\delta\to 0}$" (weakly in $\mathrm{L}^2(\Omega)^2$) to the solutions $(\overline{e}, \overline{z})$ of the macroscopic elastoplasticity system:

$$0 = \partial_{\overline{e}}\overline{\Phi}(\overline{e}(t, x), \overline{z}(t, x)) - G(t, x) + \sigma(t) \quad \text{for } x \in \Omega, \quad \int_\Omega e(t, x)\,\mathrm{d}x = \ell(t); \tag{6.9a}$$

$$0 \in \partial\overline{R}(\dot{\overline{z}}(t, x)) + \partial_{\overline{z}}\overline{\Phi}(\overline{e}(t, x), \overline{z}(t, x)). \tag{6.9b}$$

The convergence proof must follow the proof of Theorem 4.5 for the limit $\delta \to 0$ and the proof of Theorem 5.1 for $N \to \infty$. While the former convergence is tedious and lengthy it does not need any substantial new ideas. For the second limit we easily see that by construction and the definition $\overline{\sigma}(t, x) = \sigma(t) - G(t, x)$ the macroscopic equilibrium equation (6.9a) is a direct consequence of (4.1).

For the flow rule (6.9b) one can start from (6.6), which is stated in terms of $\widehat{\mu}$. Since $\widehat{\mu} = \widetilde{\mu}(\overline{z})$, we have the identity $\dot{\widehat{\mu}} = \widetilde{\mu}'(\overline{z})\dot{\overline{z}}$. Since $\widetilde{\mu}'$ is assumed to be strictly negative and the limit problem is rate independent, we can replace $\dot{\widehat{\mu}}$ by $-\dot{\overline{z}}$ in any 0-homogeneous subdifferential. At first sight, $\widehat{R}$ and $\overline{R}$ are not directly related. However, since we are dealing with a simple play operator, we only have to match the thresholds. While (6.6) corresponds to the bounds $\sigma_- \leqq \widehat{\mu} + \overline{\sigma} \leqq \sigma_+$, the flow rule (6.9b) corresponds to $-\rho_- \leqq -\partial_{\overline{z}}\overline{\Phi} \leqq \rho_+$. Now we can apply the relations derived in Lemma 6.2 to obtain system (6.9).

6.5. Other Scalings

In this subsection we briefly discuss how one can study the case when the order of the limits is reversed and we first perform a limit $\varepsilon \to 0$, and then the limit $\delta \to 0$ (see also [53]).

Choose a finite $\delta > 0$. In the case $\mu_j = 0$ for all j (that is $r = 0$) the formal pointwise limit $N \to \infty$ leads to the following continuous system

$$\delta \dot{e}(t,x) = -\phi(e(t,x)) - \int_0^x g_{\text{ext}}(t,y)\,\mathrm{d}y + \sigma(t), \quad \int_0^1 e(t,x)\,\mathrm{d}x = \ell(t).$$

Introducing the displacement $u(t,x) = \int_0^x e(t,\xi)\,\mathrm{d}\xi$ and taking the derivative with respect to x we obtain the classical quasistatic visco-elastic problem in space dimension 1:

$$0 = \big(\Phi'(u_x) + \delta \dot{u}_x\big)_x + g_{\text{ext}}(t,x), \quad u(t,x) = 0, \quad \text{and} \quad u(t,1) = \ell(t). \tag{6.10}$$

In general we cannot expect the convergence of solutions of (2.7) to solutions of (6.10), because of the nonconvexity of Φ.

The limiting behavior may be analyzed by introducing distribution functions $F(t,x,\cdot) \in \mathrm{L}^1(\mathbb{R}\times\mathbb{R})$ that account for the fluctuations of the strains e_j^N and the biases μ_j^N via

$$\int_{\mathbb{R}\times\mathbb{R}} F(t,x,\mu,E)\psi(\mu,E)\,\mathrm{d}(\mu,E) = \lim_{N\to\infty} \frac{1}{\#J(x,N)} \sum_{j\in J(x,N)} \psi(\mu_j^N, e_j^N(t)),$$

where $J(x,N) = \{\, j \in \{1,\ldots,N\} \mid |j - Nx| < N^{1/2} \,\}$. The fluctuations of the initial strain $(e_j^N(0))_j$ may be chosen independently of the bias (μ_j^N) and they do not disappear in finite time because of the viscosity $\delta > 0$. Assuming that the above limits exist, we obtain the following transport equation:

$$\delta \partial_t F(t,x,\mu,e) + \big(-\phi(e) + \mu - G(t,x) + \sigma(t)\big)\partial_e F(t,x,\mu,e) = 0, \tag{6.11a}$$

$$\int_\Omega \int_{\mathbb{R}} \int_{\mathbb{R}} e\, F(t,x,\mu,e)\,\mathrm{d}(x,\mu,e) = \ell(t), \quad \int_{\mathbb{R}\times\mathbb{R}} F(t,x,\mu,e)\,\mathrm{d}e = f(\mu). \tag{6.11b}$$

The first constraint in (6.11b) gives the total length of the deformed body, while the second says that the quenched disorder has the bias distribution f, which is independent of t and x. System (6.11) may also be seen as transport equation for a Young measure $\nu_{t,x} \in \mathrm{Prob}(\mathbb{R}\times\mathbb{R})$ and can be treated as in [3,32,60,61].

The problem can be simplified substantially if we choose initial data such that $F(0,\cdot)$ degenerates to a δ-distribution. This property is preserved by the dynamics and leads to solutions $e = \widetilde{e}(t, x, \mu)$ and $F(t, x, \mu, e) = \delta_{\widetilde{e}(t,x,\mu)}(e) f(\mu)$. Then, (6.11) reduces to a transport equation for $\widetilde{e}$:

$$\delta \partial_t \widetilde{e}(t, x, \mu) = -\phi(\widetilde{e}(t, x, \mu)) + \mu - G(t, x) + \sigma(t), \tag{6.12}$$

$$\int_\Omega \int_\mathbb{R} \widetilde{e}(t, x, \mu) f(\mu) \, \mathrm{d}\mu \, \mathrm{d}x = \ell(t). \tag{6.13}$$

The convergence of the ODE-system in $\mathbb{R}^N$ is now trivial, as the discrete setting can be embedded via functions that are piecewise constant in $x \in \Omega$. Moreover, the right-hand side is locally Lipschitz continuous on $\mathrm{L}^\infty(\Omega \times \mathbb{R})$, and classical continuous dependence on the initial data yields convergence.

The limit $\delta \to 0^+$ forces the solutions to stay in equilibria for all $t \in [0, T]$. This means that for small δ the solution should satisfy $0 \approx -\phi(e(t, x, \mu)) + \mu - G(t, x) + \sigma(t)$. Thus, it should be possible to establish the second convergence for $\delta \to 0^+$ and to obtain the same plasticity limit as in the case $\lim_{\varepsilon \to 0} \lim_{\delta \to 0}$. Again we face the problem that the limiting system is governed by steady states which are non-unique because of the non-monotonicity of ϕ. In the ODE case we were able to derive the corresponding jump rules by hand (see (DA3)), but in the general case the problem remains open.

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