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A CLASS OF EXACT SOLUTIONS IN THE PROBLEM OF ASYMMETRIC GRAVITATIONAL COMPRESSION OF A BODY*

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A class of exact solutions is constructed within the confines of Newtonian mechanics for equations of motion of a gravitating gas (dust) at zero pressure. The class comprises five arbitrary functions of a single Lagrangian variable and defines the dynamics of a three-dimensional asymmetric body.

Corresponding solutions were obtained earlier in the general theory of relativity and investigated in terms of the accompanying synchronous metric /1,2/. But in Newtonian mechanics it is necessary to determine, besides the accompanying metric, also the law of motion. It is, however, possible, without resorting to formal assumptions about the metric tensor structure, to formulate directly natural physical conditions that define the form of the law of motion, which yield an equivalent metric. Moreover, the Newtonian gravity potential is obtained in its final form.

An important property of this class of solutions is that in the case of spherical symmetry of the body boundary the external gravitation field is spherically symmetric, although inside the body motion of dust may take place, which as a whole may not have group symmetries. Note that in electrostatic approximation with constant ratio of the density of charge to that of mass, the described class of solutions, with a suitable change of the gravitational constant, defines the dynamics of a charged body of dust.

1. Let us consider in the Cartesian inertial system of coordinates (x^1, x^2, x^3) the law of motion of the medium

$$x^1 = r\sin\theta\cos\varphi + x_0^1$$
$$x^2 = r\sin\theta\sin\varphi + x_0^2, \quad x^3 = r\cos\theta + x_0^3 \tag{1.1}$$

where functions r, x_0^i $(i = 1,2,3)$ depend only on m, t (m is the mass of a sphere of radius r with its center at point x_0^i, t is the time, m, θ, φ are Lagrangian variables). This law corresponds to the decomposition of the motion of particles of medium that lie on this sphere $m = \mathrm{const}$ in spherically symmetric and translational.

The equations of motion of dust in their proper gravitational field are of the form

$$x^{\cdot\cdot i} + k\Phi,^i = 0, \quad \Phi,^i_i = 4\pi\rho, \quad \Delta\rho = \chi; \quad \Delta = \det(x^i,_p) \tag{1.2}$$

where k is the gravitational constant, χ is a function of Lagrangian variables ξ^p, and the comma and dot denote partial derivatives with respect to x^i and ξ^p, and t, respectively. Let us consider the corollary of Eqs.(1.2) that do not contain the gravitational potential Φ

$$\Delta x^{\cdot\cdot i},_p \xi^p,_i + 4\pi k\chi = 0, \quad x^{\cdot i},_{[p} x_{i,q]} = \omega_{pq}(\xi) \tag{1.3}$$

Using the law of motion of the medium of form (1.1) and solving Eqs.(1.3), we obtain

$$r^2 r^{\cdot\cdot} + km = 0, \quad x_0'^1 = a(m)\,r, \quad x_0'^2 = b(m)\,r, \quad x_0'^3 = c(m)\,r \tag{1.4}$$

where the prime denotes a derivative with respect to m with $\omega_{pq} = 0$ and density

$$\rho = \frac{1 + 3\lambda m}{4\pi r^2 (r' + \lambda r)} \tag{1.5}$$
$$\lambda = a\sin\theta\cos\varphi + b\sin\theta\sin\varphi + c\cos\theta$$

The first of Eqs.(1.4) corresponds to radial motion of the medium in the gravitational field of a particle of mass m. The solution (see, e.g. /3/) contains two arbitrary functions of m related to the input data. The presence of three arbitrary functions a, b, c generally disturbs the medium motion symmetry. Axial symmetry corresponds to the case of $a = b = 0$.

2. The solution in /1,2/ related to potential motion, and the space metric was sought in the form

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$$ds^2 = g_{pq} d\xi^p d\xi^q = A^2 dm^2 + B^2 d\zeta d\bar{\zeta} \tag{2.1}$$

where A, B are functions of t, m and of the Lagrangian complex variable ζ. When $\omega_{pq} = 0$, Eqs.(1.3) can be represented in terms of g_{pq} in the form

$$(e_q^q)^{\cdot} + e_{pq} e^{pq} + 4\pi k\rho = 0, \quad e_{pq} = {}^1/_2 g^{\cdot}_{pq}, \quad \nabla_{[p} e^{\cdot}_{q]s} = 0 \tag{2.2}$$

Solving Eqs.(2.2) together with the compatibility conditions $R^r_{pqs} = 0$ and disregarding the degenerate case of $B' = 0$, which contains a combination of one-dimensional motion of dust with plane waves and of axisymmetric motion with uniform transformation, we obtain

$$B = r(f\zeta\bar{\zeta} + g\zeta + \bar{g}\bar{\zeta} + h)^{-1}, \quad A = rB'B^{-1}, \quad r^2 r^{\cdot\cdot} + km = 0 \tag{2.3}$$

where f, g, h are functions of m connected by the relation $fh - g\bar{g} = {}^1/_4$.

Note that equations of adiabatic motion of a gravitating perfect gas, in particular that of dust, generally admit solutions associated with the decomposition of gas motion into an arbitrary plane or one-dimensional motion with plane waves and another with transverse deformation.

In the case of axial symmetry the method, dependent on the solution of parallel transport equations $\nabla_p x^i,_q = 0$, of constructing the law of motion of dust in metric (2.3) yields ultimately the law of motion (1.1). Using the law of motion of form (1.1), (1.4) it is generally possible to indicate a transform that will reduce the metric in variables m, θ, φ to the form of (2.1) and be related to the linear-fractional transform of the complex plane that corresponds to the stereographic projection of the sphere $m = \text{const}$:

$$e^{i\varphi} \operatorname{ctg} \theta/2 = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \quad \alpha\delta - \gamma\beta = 1, \quad \gamma\beta' - \alpha'\delta = \bar{\gamma}\bar{\beta}' - \bar{\alpha}'\bar{\delta}, \quad \bar{\alpha}'\bar{\beta} - \bar{\beta}'\bar{\alpha} = \delta'\gamma - \gamma'\delta \tag{2.4}$$

$$f = {}^1/_2(\alpha\bar{\alpha} + \gamma\bar{\gamma}), \quad g = {}^1/_2(\alpha\bar{\beta} + \gamma\bar{\delta}), \quad h = {}^1/_2(\beta\bar{\beta} + \delta\bar{\delta})$$
$$a + ib = 2(\alpha'\beta - \beta'\alpha), \quad c = 2(\alpha\delta' - \beta\gamma')$$

Axial symmetry corresponds $\beta = \gamma = 0$ with $\alpha, \beta, \gamma, \delta$ representing complex functions of m.

3. The obtained class of solutions of the equations of motion of dust, which contains five arbitrary functions of one Lagrangian variable, can be used in the case of special input data for defining motions of an isolated body of finite mass. Owing to the absence of stresses in the body, it is theoretically always possible to "cut out" a suitable body whose points conform to the law of motion (1.1). The Newtonian gravitational potential is determined using the smooth solution of the Poisson—Laplace equations with suitable asymptotics at infinity.

As an example we shall consider problem of graviational compression of a body of dust with spherical boundary $m = M$. We fix the origin of the system of coordinates x^i at the body center of mass. This condition uniquely determines the three arbitrary functions of t contained in x_0^i (1.4). Then the law of motion assumes the form

$$x^1 = r\sin\theta\cos\varphi - \int_m^M ar\,d\mu, \quad x^2 = r\sin\theta\sin\varphi - \int_m^M br\,d\mu, \quad x^3 = r\cos\theta = \int_m^M cr\,d\mu \tag{3.1}$$

with the outer sphere center at the center of the body mass, i.e. $x_0^i(M, T) = 0$.

Taking this into account it is possible, using the equations of motion (1.2) or directly the formula for the volume gravitational potential related to density distribution, to determine potential Φ. Omitting the arbitrary time function inside the body, we obtain

$$\Phi = -\frac{m}{r} - \int_m^M \frac{d\mu}{r} - x^i \int_m^M \frac{x_{0i}'\mu\,d\mu}{r^3} + \int_m^M \frac{x_0^i x_{0i}'\mu\,d\mu}{r^3} \tag{3.2}$$

which, as a function of x^i can be smoothly continued in the void and be used for the unique determination of the spherically symmetric external potential $\Phi = -M/|x|$.

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