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Abstract Strong local minimizers with surfaces of gradient discontinuity appear in variational problems when the energy density function is not rank-one convex. In this paper we show that the stability of such surfaces is related to the stability outside the surface via a single jump relation that can be regarded as an interchange stability condition. Although this relation appears in the setting of equilibrium elasticity theory, it is remarkably similar to the well-known *normality* condition that plays a central role in classical plasticity theory.

Keywords Martensitic · Phase transitions · Quasi convexity · Plasticity · Normality · Elastic stability

Mathematics Subject Classification 74N20 · 74G65 · 49J10

1 Introduction

In studies of the necessary conditions for singular minimizers containing surfaces of gradient discontinuity, various local jump conditions have been proposed. A partial list of such conditions include Weierstrass–Erdmann relations (traction continuity and Maxwell condition) (Erdmann 1877; Eshelby 1970), quasiconvexity on a phase boundary (Gurtin 1983; Ball and Marsden 1984), the Grinfeld instability condition (Grinfeld

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1982; Grabovsky et al. 2010), and a roughening instability condition (Grabovsky and Truskinovsky 2011). While some of these conditions have been known for a long time, a systematic study of their interdependence has not been conducted, and a full understanding of which conditions are primary and which are derivative is still missing.

The absence of a hierarchy is mostly due to the fact that strong and weak local minima must be treated differently and that variations leading to some of the known necessary conditions represent an intricate combination of strong and weak perturbations. In particular, if the goal is to find local necessary conditions of a *strong local minimum*, the use of weak variations gives rise to redundant information. For instance, Euler–Lagrange equations in the weak form should not be a part of the minimal (essential) local description of strong local minima.

In this paper, we study strong local minimizers, and our goal is to derive an *irreducible* set of necessary conditions at a point of discontinuity using only strong variations of the interface that are complementary to the known strong variations at non-singular points (Ball 1976). More specifically, our main theorem states that all known local conditions associated with gradient discontinuities follow from quasiconvexity on both sides of the discontinuity plus a single interface inequality that we call the *interchange stability condition*. Even though this condition is fully explicit and deceptively simple, it has not been specifically singled out, to the authors' best knowledge, except for a cursory mention by Hill (1986) of the corresponding *equality*, which we call the *elastic normality condition*. To emphasize a relation between this condition and strong variations, we show that it is responsible for the Gâteaux differentiability of the energy functional along special multiscale directions. We call them *material interchange variations* and show that they are devoid of any weak components. We also explain why the elastic normality condition, which R. Hill associated exclusively with weak variations, plays such an important role in the study of strong local minima.

The paper is organized as follows. In Sect. 2 we introduce the interchange stability condition and formulate our main result. In Sect. 3 we link interchange stability with a strong variation that we interpret as a material exchange. We then interpolate between this strong variation and a special weak variation that independently produces the normality condition. Our main theorem is proved in Sect. 4, where we also establish the interdependencies between the known local necessary conditions of strong local minima. An illustrative example of locally stable interfaces associated with simple laminates is discussed in detail in Sect. 5. In Sect. 6 we establish a link between the notions of elastic and plastic normality and show in which sense the elastic normality condition can be interpreted as actual orthogonality with respect to an appropriately defined yield surface. We then illustrate the general construction by studying the case of an antiplane shear in isotropic material with a double-well energy.

2 Preliminaries

Consider the variational functional most readily associated with continuum elasticity theory:

$$E(\boldsymbol{y}) = \int_{\Omega} U(\nabla \boldsymbol{y}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} - \int_{\partial\Omega_N} \boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{y}\mathrm{d}S(\boldsymbol{x}). \tag{2.1}$$

Here, Ω is an open subset of $\mathbb{R}^d$, $\boldsymbol{y} : \Omega \to \mathbb{R}^m$, and $\partial\Omega_N$ is the Neumann part of the boundary. We can absorb the boundary integral into the volume integral by finding a divergence-free $m \times d$ matrix field $\boldsymbol{\tau}(\boldsymbol{x})$ such that $\boldsymbol{t} = \boldsymbol{\tau}\boldsymbol{n}$ on $\partial\Omega_N$, which suggests that the variational functional

$$E(\boldsymbol{y}) = \int_\Omega L(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \tag{2.2}$$

can be used in place of (2.1). We assume that $L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})$ is a continuous and bounded-from-below function on $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{M}$, where $\mathbb{M}$ is the set of all $m \times d$ matrices.

We use the following definition of strong local minimum.

Definition 2.1 The Lipschitz function $\boldsymbol{y} : \Omega \to \mathbb{R}^m$ satisfying boundary conditions is a strong local minimizer if there exists $\delta > 0$ such that for every $\boldsymbol{\phi} \in C_0^1(\Omega; \mathbb{R}^m)$ for which $\max_{\boldsymbol{x}\in\Omega} |\boldsymbol{\phi}| < \delta$ we have $E(\boldsymbol{y} + \boldsymbol{\phi}) \geq E(\boldsymbol{y})$.

In this paper we focus on special singular local minimizers containing a jump discontinuity of $\nabla \boldsymbol{y}(\boldsymbol{x})$ across a C^1 surface $\Sigma \subset \Omega$ in the sense that for every point $\boldsymbol{x} \in \Sigma$ there exist $m \times d$ matrices $\boldsymbol{F}_+(\boldsymbol{x})$ and $\boldsymbol{F}_-(\boldsymbol{x})$ such that for any $\boldsymbol{z} \in \mathbb{R}^d$

$$\lim_{\epsilon\to 0} \nabla \boldsymbol{y}(\boldsymbol{x} + \epsilon \boldsymbol{z}) = \bar{\boldsymbol{F}}(\boldsymbol{z}) = \begin{cases} \boldsymbol{F}_+(\boldsymbol{x}) & \text{if } \boldsymbol{z} \cdot \boldsymbol{n} > 0, \\ \boldsymbol{F}_-(\boldsymbol{x}) & \text{if } \boldsymbol{z} \cdot \boldsymbol{n} < 0, \end{cases} \tag{2.3}$$

where $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ is the unit normal to Σ.[1] We further assume that $\boldsymbol{y} \in C^2(\overline{\Omega} \setminus \Sigma; \mathbb{R}^m)$, which imposes a kinematic compatibility constraint on the jump of the deformation gradient (Hadamard 1908):

$$[\![\boldsymbol{F}]\!] = \boldsymbol{a} \otimes \boldsymbol{n}, \tag{2.4}$$

where $[\![\cdot]\!]$ denotes the jump between the $+$ and the $-$ sides:

$$[\![\boldsymbol{F}]\!] = \boldsymbol{F}_+(\boldsymbol{x}) - \boldsymbol{F}_-(\boldsymbol{x}), \quad \boldsymbol{x} \in \Sigma,$$

and $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}) \in \mathbb{R}^m$ is called a *shear vector*.

Material stability of the deformation $\boldsymbol{y}(\boldsymbol{x})$ at the point $\boldsymbol{x}_0$ is understood as stability with respect to local variations of the form

$$\boldsymbol{y}(\boldsymbol{x}) \mapsto \boldsymbol{y}(\boldsymbol{x}) + \epsilon\boldsymbol{\phi}\left(\frac{\boldsymbol{x} - \boldsymbol{x}_0}{\epsilon}\right), \quad \boldsymbol{\phi} \in C_0^\infty(B, \mathbb{R}^m), \tag{2.5}$$

where B is the unit ball.[2] The corresponding energy variation δE is defined by

$$\delta E = \lim_{\epsilon\to 0} \frac{E\left(\boldsymbol{y}(\boldsymbol{x}) + \epsilon\boldsymbol{\phi}\left(\frac{\boldsymbol{x} - \boldsymbol{x}_0}{\epsilon}\right)\right) - E(\boldsymbol{y})}{\epsilon^d}. \tag{2.6}$$

[1] The choice of the orientation of the unit normal is unimportant as long as it is smooth. By convention, the unit normal points to the region labeled "+".

[2] The choice of unit ball as a support of the test function $\boldsymbol{\phi}$ is arbitrary. $\boldsymbol{\phi}$ can be supported in any bounded domain of $\mathbb{R}^d$, see Ball (1976).

The condition of material stability is different at regular points, where $\nabla \boldsymbol{y}(\boldsymbol{x})$ is continuous, and at the jump discontinuity, defined in (2.3). The condition of material stability at a regular point is obtained by changing the variables $\boldsymbol{x} = \boldsymbol{x}_0 + \epsilon \boldsymbol{z}$ in (2.6) (Ball 1976):

$$\delta E = \int_B \{L(\boldsymbol{x}_0, \boldsymbol{y}(\boldsymbol{x}_0), \nabla \boldsymbol{y}(\boldsymbol{x}_0) + \nabla \boldsymbol{\phi}(\boldsymbol{z})) - L(\boldsymbol{x}_0, \boldsymbol{y}(\boldsymbol{x}_0), \nabla \boldsymbol{y}(\boldsymbol{x}_0))\} \mathrm{d}\boldsymbol{z}. \tag{2.7}$$

To be closer to standard notations, we redefine $W(\boldsymbol{F}) = L(\boldsymbol{x}_0, \boldsymbol{y}(\boldsymbol{x}_0), \boldsymbol{F})$ and write the necessary condition of material stability in the form of the quasiconvexity condition (Ball 2002):

$$\fint_B W(\nabla \boldsymbol{y}(\boldsymbol{x}_0) + \nabla \boldsymbol{\phi}(\boldsymbol{z})) \mathrm{d}\boldsymbol{z} \geq W(\nabla \boldsymbol{y}(\boldsymbol{x}_0)), \tag{2.8}$$

where $\fint_B$ denotes the average over B. We say that $\boldsymbol{F} \in \mathbb{M}$ is *strongly locally stable* if

$$\fint_B W(\boldsymbol{F} + \nabla \boldsymbol{\phi}(\boldsymbol{z})) \mathrm{d}\boldsymbol{z} \geq W(\boldsymbol{F}) \tag{2.9}$$

for any $\boldsymbol{\phi} \in C_0^\infty(B, \mathbb{R}^m)$.

When the point $\boldsymbol{x}_0$ lies at the jump discontinuity, we can again change the variables $\boldsymbol{x} = \boldsymbol{x}_0 + \epsilon \boldsymbol{z}$ and write (Gurtin 1983)

$$\delta E = \int_B \{W(\bar{\boldsymbol{F}}(\boldsymbol{z}) + \nabla \boldsymbol{\phi}(\boldsymbol{z})) - W(\bar{\boldsymbol{F}}(\boldsymbol{z}))\} \mathrm{d}\boldsymbol{z}, \tag{2.10}$$

where $\bar{\boldsymbol{F}}(\boldsymbol{z})$ is defined in (2.3). The associated necessary condition can be written in the form of the quasiconvexity condition at the surface of the jump discontinuity (Gurtin 1983; Ball and Marsden 1984)

$$\fint_{B_{\boldsymbol{n}}^+} W(\boldsymbol{F}_+(\boldsymbol{x}_0) + \nabla \boldsymbol{\phi}) \mathrm{d}\boldsymbol{z} + \fint_{B_{\boldsymbol{n}}^-} W(\boldsymbol{F}_-(\boldsymbol{x}_0) + \nabla \boldsymbol{\phi}) \mathrm{d}\boldsymbol{z} \geq W(\boldsymbol{F}_+(\boldsymbol{x}_0)) + W(\boldsymbol{F}_-(\boldsymbol{x}_0)), \tag{2.11}$$

where $B_{\boldsymbol{n}}^{\pm} = \{\boldsymbol{z} \in B : \boldsymbol{z} \cdot \boldsymbol{n} \gtrless 0\}$. We say that the pair $\boldsymbol{F}_{\pm} \in \mathbb{M}$ satisfying (2.4) determines a *strongly locally stable interface* $\Pi_{\boldsymbol{n}} = \{\boldsymbol{z} \in \mathbb{R}^d : \boldsymbol{z} \cdot \boldsymbol{n} = 0\}$ if

$$\fint_{B_{\boldsymbol{n}}^+} W(\boldsymbol{F}_+ + \nabla \boldsymbol{\phi}) \mathrm{d}\boldsymbol{z} + \fint_{B_{\boldsymbol{n}}^-} W(\boldsymbol{F}_- + \nabla \boldsymbol{\phi}) \mathrm{d}\boldsymbol{z} \geq W(\boldsymbol{F}_+) + W(\boldsymbol{F}_-) \tag{2.12}$$

for any $\boldsymbol{\phi} \in C_0^\infty(B, \mathbb{R}^m)$. It is clear that the strong local stability of the interface $\Pi_{\boldsymbol{n}}$ implies the strong local stability (2.9) of $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$.

It will be convenient to reformulate the conditions of strong local stability in terms of the properties of global minimizers of localized variational problems. Thus, according to (2.9), $\boldsymbol{F}$ is strongly locally stable if and only if $\boldsymbol{y}(\boldsymbol{z}) = \boldsymbol{F}\boldsymbol{z}$ is a minimizer in the localized variational problem

$$\inf_{\substack{\boldsymbol{y}|_{\partial B}=\boldsymbol{F}\boldsymbol{z} \\ \boldsymbol{y}\in W^{1,\infty}(B;\mathbb{R}^m)}} \fint_B W(\nabla \boldsymbol{y}(\boldsymbol{z}))\mathrm{d}\boldsymbol{z}. \tag{2.13}$$

The value of the infimum in (2.13) coincides with the quasiconvex envelope $QW(\boldsymbol{F})$ of $W(\boldsymbol{F})$, i.e., the largest quasiconvex function that does not exceed W (Dacorogna 1982). It is then clear that $\boldsymbol{F} \in \mathbb{M}$ is strongly locally stable if and only if $QW(\boldsymbol{F}) = W(\boldsymbol{F})$.

Similarly, we say that the pair $\boldsymbol{F}_\pm$ satisfying (2.4) determines a strongly locally stable interface if $\bar{\boldsymbol{y}}(\boldsymbol{z})$ solves the localized variational problem

$$\inf_{\substack{\boldsymbol{y}|_{\partial B}=\bar{\boldsymbol{y}}(\boldsymbol{z}) \\ \boldsymbol{y}\in W^{1,\infty}(B;\mathbb{R}^m)}} \int_B W(\nabla \boldsymbol{y}(\boldsymbol{z}))\mathrm{d}\boldsymbol{z}, \tag{2.14}$$

where $\bar{\boldsymbol{y}}$ is defined by

$$\bar{\boldsymbol{y}}(\boldsymbol{z}) = \begin{cases} \boldsymbol{F}_+\boldsymbol{z} & \text{if } \boldsymbol{z}\cdot\boldsymbol{n} > 0, \\ \boldsymbol{F}_-\boldsymbol{z} & \text{if } \boldsymbol{z}\cdot\boldsymbol{n} < 0. \end{cases} \tag{2.15}$$

We are now in a position to formulate our main claim that the strong local stability (2.9) of $\boldsymbol{F}_\pm$, together with a single additional condition, which we call the *interchange stability*, implies a strong local stability of the interface.

Theorem 2.2 *Let $W(\boldsymbol{F})$ be a continuous, bounded-from-below function that is of class C^2 in a neighborhoodof $\{\boldsymbol{F}_+, \boldsymbol{F}_-\} \subset \mathbb{M}$. Assume that the pair $\boldsymbol{F}_\pm$ satisfies the kinematic compatibility condition $[\![\boldsymbol{F}]\!] = \boldsymbol{a}\otimes\boldsymbol{n}$ for some $\boldsymbol{a} \in \mathbb{R}^m$ and $\boldsymbol{n} \in \mathbb{S}^{d-1}$. Then the surface of the jump discontinuity $\Pi_{\boldsymbol{n}} = \{\boldsymbol{z} \in \mathbb{R}^d : \boldsymbol{z}\cdot\boldsymbol{n} = 0\}$ is strongly locally stable if and only if the following conditions are satisfied:*

(S) Material stability in the bulk: $QW(\boldsymbol{F}_\pm) = W(\boldsymbol{F}_\pm)$,
(I) Interchange stability: $\mathfrak{N} = \langle[\![\boldsymbol{P}]\!], [\![\boldsymbol{F}]\!]\rangle \le 0$,

where $\boldsymbol{P}(\boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{F})$ and $\langle[\![\boldsymbol{P}]\!], [\![\boldsymbol{F}]\!]\rangle = \mathrm{Tr}\,([\![\boldsymbol{P}]\!][\![\boldsymbol{F}]\!]^T)$.

Before we turn to the proof of Theorem 2.2, it will be instructive to look closely at the meaning of the scalar quantity $\mathfrak{N}$ entering the algebraic condition (I).

3 Interchange Driving Force

While it is natural that condition (2.12) of the strong local stability of the interface implies the strong local stability of each individual deformation gradient $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$, a less obvious claim of Theorem 2.2 is that the only *joint* stability constraint on the kinematically compatible pair $(\boldsymbol{F}_+, \boldsymbol{F}_-)$ is provided by condition (I). A natural challenge is to identify the variation producing this condition.

We observe that conventional variations, linking both sides of a jump discontinuity and leading to a Maxwell condition (Eshelby 1970) or a roughening instability condition (Grabovsky and Truskinovsky 2011), represent combinations of weak and strong variations. This creates unnecessary coupling and obscures the strong character of the minimizer under consideration. Physically it is clear that if the "materials" on both sides of the interface are stable and if we cannot interchange one material with another without increasing the energy, then the whole configuration should be stable.

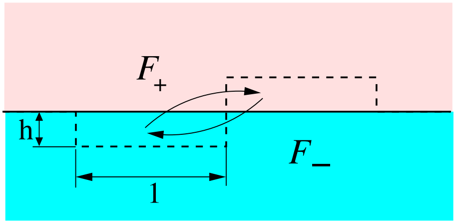


Fig. 1 Interchange variation at surface of gradient discontinuity

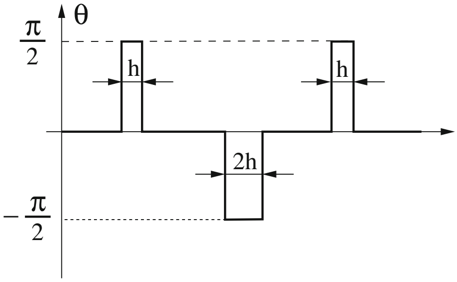


Fig. 2 Strong double-dipole variation of interface *normal*. The angle θ between the original and perturbed normals is plotted as a function of length in the tangential direction

The idea of material interchange is illustrated schematically in Fig. 1, where the two adjacent rectangular domains are flipped and then translated. At $h \to 0$ this construction can be viewed as an interface generalization of the Weierstrass needle variation since the interface normal undergoes finite change, only on a set of small surface area, as shown in Fig. 2.

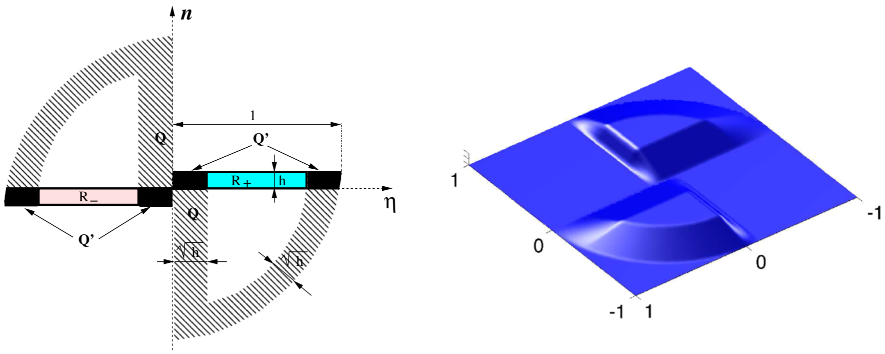
Notice that if taken literally, the schematics shown in Fig. 1 is incompatible with a gradient of any admissible variation. This technical problem is fixed using the appropriate mollifiers, as it would be for the classical single strip variation (e.g., Dacorogna 1989). However, in the material interchange variation, the main energy contributions of the two strips in Fig. 3a cancel out, and the energy balance needs to be examined more carefully than in the classical case of a single strip. For this reason we present an explicit construction of an admissible variation whose gradient mimics the configuration shown in Fig. 1.

We define a family of Lipschitz cutoff functions $\{\zeta_h(t) : h \in (0, 1)\}$ on $[0, +\infty)$ such that $\zeta_h(t) = 1$ when $0 \leq t \leq 1 - \sqrt{h}$, while $\zeta_h(t) = 0$ when $t \geq 1$. Let $\rho(t)$ be another Lipschitz cutoff function, with $\rho(t) = 1$ when $t > 1$ and $\rho(t) = 0$ when $t < 0$. Suppose that $\boldsymbol{\eta}$ is a unit vector in $\mathbb{R}^d$ such that $\boldsymbol{\eta} \perp \boldsymbol{n}$. We define the test function $\boldsymbol{\Phi}_h(z)$ – to be used in (2.11) – as follows:

$$\boldsymbol{\Phi}_h = \boldsymbol{\Phi}_h^+ + \boldsymbol{\Phi}_h^-, \quad \boldsymbol{\Phi}_h^-(z) = \boldsymbol{\Phi}_h^+(-z), \quad \boldsymbol{\Phi}_h^+(z) = h\phi\left(\frac{z \cdot \boldsymbol{n}}{h}\right)\rho\left(\frac{z \cdot \boldsymbol{\eta}}{\sqrt{h}}\right)\zeta_h(|z|)\boldsymbol{a}, \tag{3.1}$$

where

$$\phi(s) = \begin{cases} 1 - s, & 0 < s < 1, \\ 1, & s \leq 0, \\ 0, & s \geq 1. \end{cases}$$



(a) Value map of $|\nabla\Phi_h|$ in the $(\boldsymbol{\eta}, \boldsymbol{n})$-plane. **(b)** Graph of $\Phi_h(\boldsymbol{z})$ over the $(\boldsymbol{\eta}, \boldsymbol{n})$-plane.

Fig. 3 Interchange variation $\boldsymbol{\Phi}_h(\boldsymbol{z}) = \Phi_h(\boldsymbol{z})\boldsymbol{a}$. **a** Value map of $|\nabla\Phi_h|$ in $(\boldsymbol{\eta}, \boldsymbol{n})$-plane. **b** Graph of $\Phi_h(\boldsymbol{z})$ over $(\boldsymbol{\eta}, \boldsymbol{n})$-plane

Observe that $\boldsymbol{\Phi}_h(\boldsymbol{z}) = \Phi_h(\boldsymbol{z})\boldsymbol{a}$, where the graph of $\Phi_h(\boldsymbol{z})$ is given in Fig. 3b. We remark that the variation (3.1) belongs to the class of multiscale variations proposed in Grabovsky and Truskinovsky (2011): it uses a small scale h and another small scale ϵ from (2.5).

The interpretation of the function $\mathfrak{N}$ as an interchange driving force is immediately clear from the following theorem.

Theorem 3.1 *Suppose* $\{\boldsymbol{F}_+, \boldsymbol{F}_-\} \subset \mathbb{M}$ *satisfy* (2.4). *Let* $\boldsymbol{\Phi}_h$ *be given by* (3.1). *Then*

$$\lim_{h\to 0}\frac{1}{h}\int_B \{W(\nabla\bar{\boldsymbol{y}} + \nabla\boldsymbol{\Phi}_h(\boldsymbol{z})) - W(\nabla\bar{\boldsymbol{y}})\}\mathrm{d}\boldsymbol{z} = -\frac{\omega_{d-1}}{2}\mathfrak{N}, \tag{3.2}$$

where $\bar{\boldsymbol{y}}(\boldsymbol{z})$ *is given by* (2.15) *and* ω_k *is the* k*-dimensional volume of the unit ball in* $\mathbb{R}^k$.

Proof In order to compute the energy increment

$$\Delta E(h) = \int_B \{W(\nabla\bar{\boldsymbol{y}} + \nabla\boldsymbol{\Phi}_h(\boldsymbol{z})) - W(\nabla\bar{\boldsymbol{y}})\}\mathrm{d}\boldsymbol{z}, \tag{3.3}$$

we use the Weierstrass function

$$W^\circ(\boldsymbol{F}, \boldsymbol{H}) = W(\boldsymbol{F} + \boldsymbol{H}) - W(\boldsymbol{F}) - \langle \boldsymbol{P}(\boldsymbol{F}), \boldsymbol{H}\rangle. \tag{3.4}$$

We can then rewrite the energy increment $\Delta E(h)$ as

$$\Delta E(h) = \int_B W^\circ(\nabla\bar{\boldsymbol{y}}, \nabla\boldsymbol{\Phi}_h)\mathrm{d}\boldsymbol{z} + \int_B \langle \boldsymbol{P}(\nabla\bar{\boldsymbol{y}}), \nabla\boldsymbol{\Phi}_h\rangle\mathrm{d}\boldsymbol{z}.$$

We easily compute

$$\int_B \langle \boldsymbol{P}(\nabla \bar{\boldsymbol{y}}), \nabla \boldsymbol{\Phi}_h \rangle \mathrm{d}\boldsymbol{z} = -[\![\boldsymbol{P}]\!]\boldsymbol{n} \cdot \int_{\Pi_{\boldsymbol{n}} \cap B} \boldsymbol{\Phi}_h \mathrm{d}S = -h\omega_{d-1}\mathfrak{N} + O(h^{3/2}). \quad (3.5)$$

Observe that $\boldsymbol{\Phi}_h^+(\boldsymbol{z})$ is nonzero only on $\boldsymbol{z} \cdot \boldsymbol{\eta} > 0$, while $\boldsymbol{\Phi}_h^-(\boldsymbol{z})$ is nonzero only on $\boldsymbol{z} \cdot \boldsymbol{\eta} < 0$. Therefore,

$$\int_B W^\circ(\nabla \bar{\boldsymbol{y}}, \nabla \boldsymbol{\Phi}_h)\mathrm{d}\boldsymbol{z} = \int_B W^\circ(\nabla \bar{\boldsymbol{y}}, \nabla \boldsymbol{\Phi}_h^+)\mathrm{d}\boldsymbol{z} + \int_B W^\circ(\nabla \bar{\boldsymbol{y}}, \nabla \boldsymbol{\Phi}_h^-)\mathrm{d}\boldsymbol{z}.$$

To estimate the right-hand side, we identify three regions where $\nabla \boldsymbol{\Phi}_h \neq \boldsymbol{0}$ (Fig. 3a):

$$R_\pm = \{\boldsymbol{z} \in B : \pm\boldsymbol{z} \cdot \boldsymbol{\eta} > \sqrt{h},\ 0 < \pm\boldsymbol{z} \cdot \boldsymbol{n} < h,\ |\boldsymbol{z}| < 1 - \sqrt{h}\},$$

$$\begin{aligned} Q = &\{\boldsymbol{z} \in B : |\boldsymbol{z}| > 1 - \sqrt{h},\ (\boldsymbol{z} \cdot \boldsymbol{\eta})(\boldsymbol{z} \cdot \boldsymbol{n}) < 0\} \\ &\cup \{\boldsymbol{z} \in B : |\boldsymbol{z} \cdot \boldsymbol{\eta}| < \sqrt{h},\ (\boldsymbol{z} \cdot \boldsymbol{\eta})(\boldsymbol{z} \cdot \boldsymbol{n}) < 0,\ |\boldsymbol{z}| < 1 - \sqrt{h}\}, \end{aligned}$$

and

$$\begin{aligned} Q' = &\{\boldsymbol{z} \in B : |\boldsymbol{z} \cdot \boldsymbol{\eta}| < \sqrt{h},\ |\boldsymbol{z} \cdot \boldsymbol{n}| < h,\ (\boldsymbol{z} \cdot \boldsymbol{\eta})(\boldsymbol{z} \cdot \boldsymbol{n}) > 0\} \\ &\{\boldsymbol{z} \in B : |\boldsymbol{z}| > 1 - \sqrt{h},\ |\boldsymbol{z} \cdot \boldsymbol{n}| < h,\ (\boldsymbol{z} \cdot \boldsymbol{\eta})(\boldsymbol{z} \cdot \boldsymbol{n}) > 0\}. \end{aligned}$$

To estimate $\nabla \boldsymbol{\Phi}_h$, we write $\boldsymbol{\Phi}_h^\pm = \boldsymbol{\phi}_h^\pm c_h^\pm$, where

$$\boldsymbol{\phi}_h^\pm(\boldsymbol{z}) = h\phi\left(\pm\frac{\boldsymbol{z} \cdot \boldsymbol{n}}{h}\right)\boldsymbol{a}, \quad c_h^\pm(\boldsymbol{z}) = \rho\left(\pm\frac{\boldsymbol{z} \cdot \boldsymbol{\eta}}{\sqrt{h}}\right)\zeta_h(|\boldsymbol{z}|).$$

It is easy to see that

$$|\boldsymbol{\phi}_h^\pm| = O(h), \quad |c_h^\pm| = O(1), \quad |\nabla \boldsymbol{\phi}_h^\pm| = O(1), \quad |\nabla c_h^\pm| = O\left(\frac{1}{\sqrt{h}}\right).$$

We see that $c_h^\pm = 1$ and $\nabla \boldsymbol{\phi}_h^\pm = \mp[\![\boldsymbol{F}]\!]$ in the regions $R_\pm$, $\boldsymbol{\phi}_h^\pm = h\boldsymbol{a}$ in region Q. Thus,

$$\nabla \boldsymbol{\Phi}_h^\pm(\boldsymbol{z}) = \begin{cases} \mp[\![\boldsymbol{F}]\!], & \boldsymbol{z} \in R_\pm, \\ h\boldsymbol{a} \otimes \nabla c_h^\pm, & \boldsymbol{z} \in Q, \\ c_h^\pm \nabla \boldsymbol{\phi}_h^\pm + \boldsymbol{\phi}_h^\pm \otimes \nabla c_h^\pm, & \boldsymbol{z} \in Q', \\ \boldsymbol{0}, & \text{elsewhere.} \end{cases}$$

Therefore, $\nabla \boldsymbol{\Phi}_h = O(\sqrt{h})$ in region Q and $\nabla \boldsymbol{\Phi}_h = O(1)$ in region Q'. We also see that $|Q| = O(\sqrt{h})$, $|Q'| = O(h^{3/2})$, while $|R_\pm| = h\omega_{d-1}/2 + O(h^{3/2})$. Thus, we estimate

$$\int_{Q\cup Q'} W^\circ(\nabla\bar{\boldsymbol{y}}, \nabla\boldsymbol{\Phi}_h)\mathrm{d}\boldsymbol{z} = O(h^{3/2}),$$

while

$$\int_{R_\pm} W^\circ(\nabla\bar{\boldsymbol{y}}, \nabla\boldsymbol{\Phi}_h^\pm)\mathrm{d}\boldsymbol{z} = \frac{h\omega_{d-1}}{2} W^\circ(\boldsymbol{F}_\pm, \mp[\![\boldsymbol{F}]\!]) + O(h^{3/2}).$$

We conclude that

$$\lim_{h\to 0}\frac{1}{h}\int_B W^\circ(\nabla\bar{\boldsymbol{y}}, \nabla\boldsymbol{\Phi}_h)\mathrm{d}\boldsymbol{z} = \frac{\omega_{d-1}}{2}\left(W^\circ(\boldsymbol{F}_+, -[\![\boldsymbol{F}]\!]) + W^\circ(\boldsymbol{F}_-, [\![\boldsymbol{F}]\!])\right) = \frac{\omega_{d-1}}{2}\mathfrak{N}. \tag{3.6}$$

Combining (3.5) and (3.6) we obtain (3.2). □

While Theorem 3.1 associates the interchange stability (I) with strong variations, the function $\mathfrak{N}$ is also known to be linked with stability with respect to weak variations. Indeed, after being projected onto the shear vector $\boldsymbol{a}$, the traction continuity condition $[\![\boldsymbol{P}]\!]\boldsymbol{n} = 0$, which can be viewed as a weak form of Euler–Lagrange equations, gives the following *normality condition* (Hill 1986):

$$\mathfrak{N} = [\![\boldsymbol{P}]\!]\boldsymbol{n}\cdot\boldsymbol{a} = \langle[\![\boldsymbol{P}]\!], [\![\boldsymbol{F}]\!]\rangle = 0. \tag{3.7}$$

To understand the origin of (3.7), consider the energy increment corresponding to classical weak variations:

$$\bar{\boldsymbol{y}} \mapsto \bar{\boldsymbol{y}}(\boldsymbol{z}) + \epsilon\boldsymbol{\phi}(\boldsymbol{z}), \quad \boldsymbol{\phi} \in C_0^1(B;\mathbb{R}^m). \tag{3.8}$$

We obtain

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_B \{W(\nabla\bar{\boldsymbol{y}} + \epsilon\nabla\boldsymbol{\phi}) - W(\nabla\bar{\boldsymbol{y}})\}\mathrm{d}\boldsymbol{z} = -\int_{\Pi_{\boldsymbol{n}}} [\![\boldsymbol{P}]\!]\boldsymbol{n}\cdot\boldsymbol{\phi}\mathrm{d}S(\boldsymbol{z}). \tag{3.9}$$

Formula (3.9) shows that if $\boldsymbol{\phi}(\boldsymbol{z}) = \boldsymbol{a}\phi(\boldsymbol{z})$, then the vanishing of the first variation implies the normality condition (3.7).

The crucial observation is that our strong variation $\boldsymbol{\Phi}_h$ given by (3.1) is also a scalar multiple of $\boldsymbol{a}$. This suggests the idea that both our weak and strong variations can be regarded as two limits of a single continuum of variations $\{t\boldsymbol{\Phi}_h : t \in [0, 1]\}$ interpolating between them. Indeed, it is easy to see that if $t \to 0$ for fixed h, then the two-parameter family of functions $t\boldsymbol{\Phi}_h$ converges to the weak variation (3.8), and when $h \to 0$ at $t = 1$, we obtain the strong variation (3.1). It is instructive to examine the effect of this family of variations for intermediate values of t. To compute the corresponding asymptotics of the energy increment, we can use the method used in the proof of Theorem 3.1, which is applicable for any $t > 0$. Let

$$\Delta E(t, h) = \int_B (W(\nabla\bar{\boldsymbol{y}} + t\nabla\boldsymbol{\Phi}_h(\boldsymbol{z})) - W(\nabla\bar{\boldsymbol{y}}))\mathrm{d}\boldsymbol{z}.$$

Rewriting the energy increment in terms of the Weierstrass function we obtain

$$\Delta E(t,h) = \int_B W^\circ(\nabla \bar{\boldsymbol{y}}, t\nabla \boldsymbol{\Phi}_h)\mathrm{d}z + t\int_B \langle \boldsymbol{P}(\nabla \bar{\boldsymbol{y}}), \nabla \boldsymbol{\Phi}_h\rangle \mathrm{d}z.$$

Thus,

$$\lim_{t\to 0} \frac{\Delta E(t,h)}{t} = \int_B \langle \boldsymbol{P}(\nabla \bar{\boldsymbol{y}}), \nabla \boldsymbol{\Phi}_h\rangle \mathrm{d}z.$$

When $t > 0$ is fixed, we repeat the steps in the proof of Theorem 3.1 and obtain

$$\lim_{h\to 0} \frac{1}{h}\int_B W^\circ(\nabla \bar{\boldsymbol{y}}, t\nabla \boldsymbol{\Phi}_h)\mathrm{d}z = \frac{\omega_{d-1}}{2}(W^\circ(\boldsymbol{F}_+, -t[\![\boldsymbol{F}]\!]) + W^\circ(\boldsymbol{F}_-, t[\![\boldsymbol{F}]\!])).$$

Hence, using formula (3.5) we obtain

$$\lim_{h\to 0} \frac{\Delta E(t,h)}{h} = \frac{\omega_{d-1}}{2}(W^\circ(\boldsymbol{F}_+, -t[\![\boldsymbol{F}]\!]) + W^\circ(\boldsymbol{F}_-, t[\![\boldsymbol{F}]\!]) - 2t\mathfrak{N}) \tag{3.10}$$

and

$$\lim_{h\to 0}\lim_{t\to 0} \frac{\Delta E(t,h)}{th} = -\omega_{d-1}\mathfrak{N}.$$

From (3.10) we obtain

$$\lim_{t\to 0}\lim_{h\to 0} \frac{\Delta E(t,h)}{th} = -\omega_{d-1}\mathfrak{N},$$

which shows that the h and t limits commute.

The existence of an explicit interpolation between weak and strong variations suggests that we should examine the behavior of the normalized energy along the connecting path. To this end, consider the expression

$$D(t) = \lim_{h\to 0} \frac{2\Delta E(t,h)}{h\omega_{d-1}} = W^\circ(\boldsymbol{F}_+, -t[\![\boldsymbol{F}]\!]) + W^\circ(\boldsymbol{F}_-, t[\![\boldsymbol{F}]\!]) - 2t\mathfrak{N}$$

on the interval [0, 1]. To ensure the symmetry of the two limits, we consider the special case $\mathfrak{N} = 0$.

It is clear that the fine structure of the energy landscape along such a path is not universal and depends sensitively on the function $W(\boldsymbol{F})$. For the purpose of illustration, let us consider the energy density

$$W(\boldsymbol{F}) = f(\theta) + \mu\left|\boldsymbol{\varepsilon} - \frac{\theta}{d}\boldsymbol{I}\right|^2, \quad \theta = \mathrm{Tr}\,\boldsymbol{F}, \quad \boldsymbol{\varepsilon} = \frac{\boldsymbol{F}+\boldsymbol{F}^T}{2}.$$

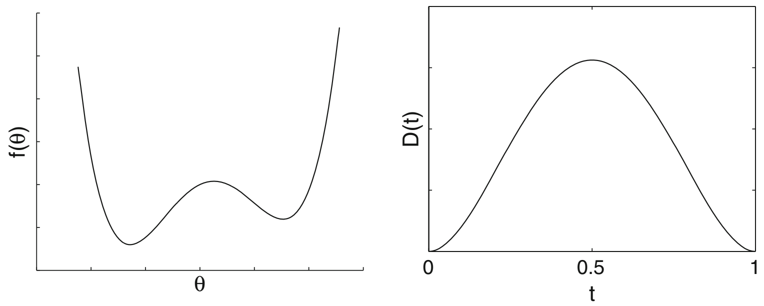


Fig. 4 Connecting the weak ($t = 0$) and the interchange ($t = 1$) variation

Assuming kinematic compatibility (2.4) and normality (3.7) we obtain

$$D(t) = f(\theta_t) + f(\widetilde{\theta}_t) - f(\theta_+) - f(\theta_-) + t(1-t)[\![f'(\theta)]\!][\![\theta]\!],$$

where

$$\theta_t = t\theta_+ + (1-t)\theta_-, \quad \widetilde{\theta}_t = (1-t)\theta_+ + t\theta_-,$$

with $\theta_\pm$ satisfying

$$[\![\Phi']\!][\![\theta]\!] \leq 0, \quad \Phi(\theta) = f(\theta) + \mu\left(1 - \frac{1}{d}\right)\theta^2.$$

One can see that if the function $f(\theta)$ has a double-well structure (Fig. 4a), then the graph of $D(t)$ looks like $t^2(1-t)^2$ (Fig. 4b). The presence of an energy barrier indicates that any combination of the interchange variation and the weak variation (3.8) produces a cruder test of stability than either of the pure variations, which are incapable of detecting the existing instability. This result confirms our intuition that the realms of weak and strong variations are well separated and that the energy landscapes in the strong and weak topologies can be regarded as unrelated (unless all nontrivial features are removed by assuming uniform convexity or quasiconvexity).

We conclude this section by proving an important property of the interchange driving force $\mathfrak{N}$. More specifically, we show that if the deformation gradients $\boldsymbol{F}_\pm$ are strongly locally stable and if they are linked only by the kinematic compatibility condition (2.4), then the interchange driving force $\mathfrak{N}$ is nonnegative.

We first recall the definition of the Maxwell driving force (Erdmann 1877; Eshelby 1970)

$$p^* = [\![W]\!] - \langle\{\{\boldsymbol{P}\}\}, [\![\boldsymbol{F}]\!]\rangle, \tag{3.11}$$

where $\{\{\boldsymbol{P}\}\} = \frac{1}{2}(\boldsymbol{P}_+ + \boldsymbol{P}_-)$.

Theorem 3.2 *Assume that both $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$ are strongly locally stable and satisfy the kinematic compatibility condition* (2.4). *Then*

$$\mathfrak{N} \geq 2|p^*|. \tag{3.12}$$

In particular, $\mathfrak{N} \geq 0$.

The theorem is an immediate consequence of Lemma 3.4 below, which shows that the algebraic inequality (3.12) is a consequence of the Weierstrass condition stated in the next lemma.

Lemma 3.3 *Suppose* $\boldsymbol{F}$ *is strongly locally stable. Then the Weierstrass condition holds:*

$$W^\circ(\boldsymbol{F}, \boldsymbol{u} \otimes \boldsymbol{v}) \geq 0, \text{ for all } \boldsymbol{u} \in \mathbb{R}^m,\ \boldsymbol{v} \in \mathbb{R}^d. \tag{3.13}$$

The proof of the lemma can be found in McShane (1931) and Graves (1939).

Lemma 3.4 *Suppose that both* $\boldsymbol{F}_+$ *and* $\boldsymbol{F}_-$ *satisfy the Weierstrass condition* (3.13) *and the kinematic compatibility condition* (2.4)*. Then inequality* (3.12) *holds.*

Proof Setting $\boldsymbol{F} = \boldsymbol{F}_\pm$ and $\boldsymbol{u} \otimes \boldsymbol{v} = \mp[\![\boldsymbol{F}]\!]$ in (3.13), we obtain

$$W^\circ(\boldsymbol{F}_\pm, \mp[\![\boldsymbol{F}]\!]) = W(\boldsymbol{F}_\mp) - W(\boldsymbol{F}_\pm) \pm \langle \boldsymbol{P}_\pm, [\![\boldsymbol{F}]\!]\rangle = \mp([\![W]\!] - \langle \boldsymbol{P}_\pm, [\![\boldsymbol{F}]\!]\rangle).$$

Writing $\boldsymbol{P}_\pm = \{\!\{\boldsymbol{P}\}\!\} \pm \frac{1}{2}[\![\boldsymbol{P}]\!]$, where $\{\!\{\boldsymbol{P}\}\!\} = (\boldsymbol{P}_+ + \boldsymbol{P}_-)/2$, we obtain

$$W^\circ(\boldsymbol{F}_\pm, \mp[\![\boldsymbol{F}]\!]) = \mp p^* + \frac{\mathfrak{N}}{2} \geq 0. \tag{3.14}$$

Inequality (3.12) follows. □

Theorem 3.2, whose proof is now straightforward, quantifies the extent to which the conditions of the stability of surfaces of jump discontinuities are stronger than conditions of strong local stability of each individual phase.

4 Proof of Main Theorem

We are now in a position to prove Theorem 2.2. The necessity of (*S*) was already observed in Sect. 2, and the necessity of (*I*), even with equality, was shown in Sect. 3. The proof of sufficiency will be split into a sequence of lemmas. Our first step will be to recover the known interface jump conditions. To prove these algebraic relations, only the Weierstrass condition (3.13) will be needed.

Lemma 4.1 *Assume that the pair* $\boldsymbol{F}_\pm$ *satisfies the following three conditions:*

(*K*) *Kinematic compatibility:* $[\![\boldsymbol{F}]\!] = \boldsymbol{a} \otimes \boldsymbol{n}$ *for some* $\boldsymbol{a} \in \mathbb{R}^m$ *and* $\boldsymbol{n} \in \mathbb{S}^{d-1}$;
(*I*) *Interchange stability of interface:* $\mathfrak{N} = \langle [\![\boldsymbol{P}]\!], [\![\boldsymbol{F}]\!]\rangle \leq 0$;
(*W*) *Weierstrass condition:* $W^\circ(\boldsymbol{F}_\pm, \boldsymbol{u} \otimes \boldsymbol{v}) \geq 0$ *for all* $\boldsymbol{u} \in \mathbb{R}^m$ *and* $\boldsymbol{v} \in \mathbb{R}^d$.

Then the following interface conditions must hold:

- *The Maxwell jump condition:*

$$p^* = 0; \tag{4.1}$$

- *Traction continuity:*

$$[\![\boldsymbol{P}]\!]\boldsymbol{n} = \boldsymbol{0}; \tag{4.2}$$

- *Interface roughening condition* (Grabovsky and Truskinovsky 2011)*:*

$$[\![\boldsymbol{P}]\!]^T \boldsymbol{a} = \boldsymbol{0}. \tag{4.3}$$

Proof Combining Lemma 3.4 with (*I*) we conclude that $\mathfrak{N} = 0$, and hence, by (3.12), the Maxwell condition (4.1) holds. To prove the remaining equalities, we set $\boldsymbol{u} = \mp(\boldsymbol{a} + \boldsymbol{\xi})$ and $\boldsymbol{v} = \boldsymbol{n} + \boldsymbol{\eta}$ in the Weierstrass condition (W), where $\boldsymbol{a}$ and $\boldsymbol{n}$ are as in (2.4) and $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are small parameters. Then we obtain a pair of inequalities

$$\omega_{\pm}(\boldsymbol{\xi}, \boldsymbol{\eta}) = W^{\circ}(\boldsymbol{F}_{\pm}, \mp(\boldsymbol{a} + \boldsymbol{\xi}) \otimes (\boldsymbol{n} + \boldsymbol{\eta})) \geq 0 \tag{4.4}$$

that hold for all $\boldsymbol{\xi} \in \mathbb{R}^m$ and all $\boldsymbol{\eta} \in \mathbb{R}^d$. Under our smoothness assumptions on $W(\boldsymbol{F})$, the functions $\omega_{\pm}(\boldsymbol{\xi}, \boldsymbol{\eta})$ are of class C^2 in the neighborhoodof $(\boldsymbol{0}, \boldsymbol{0})$ in the $(\boldsymbol{\xi}, \boldsymbol{\eta})$-space. The Taylor expansion up to first order in $(\boldsymbol{\xi}, \boldsymbol{\eta})$ gives

$$\omega_{\pm}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mp p^* + \frac{\mathfrak{N}}{2} + [\![\boldsymbol{P}]\!]\boldsymbol{n} \cdot \boldsymbol{\xi} + [\![\boldsymbol{P}]\!]^T \boldsymbol{a} \cdot \boldsymbol{\eta} + O(|\boldsymbol{\xi}|^2 + |\boldsymbol{\eta}|^2). \tag{4.5}$$

Then inequalities (4.4), together with $\mathfrak{N} = 0$ and (4.1), imply (4.2) and (4.3). □

Next we prove a differentiability lemma that guarantees the existence of rank-one directional derivatives of quasiconvex and rank-one convex envelopes at marginally stable deformation gradients (Grabovsky and Truskinovsky 2013). This result requires no additional growth conditions, as in the envelope regularity theorems from Ball et al. (2000).

Lemma 4.2 *Let $V(\boldsymbol{F})$ be a rank-one convex function such that $V(\boldsymbol{F}) \leq W(\boldsymbol{F})$. Let*

$$\mathcal{A}_V = \{\boldsymbol{F} \in \mathcal{O} : W(\boldsymbol{F}) = V(\boldsymbol{F})\},$$

where $\mathcal{O}$ is an open subset of $\mathbb{M}$ on which $W(\boldsymbol{F})$ is of class C^1. Then for every $\boldsymbol{F} \in \mathcal{A}_V$, $\boldsymbol{u} \in \mathbb{R}^m$, and $\boldsymbol{v} \in \mathbb{R}^d$

$$\lim_{t \to 0} \frac{V(\boldsymbol{F} + t\boldsymbol{u} \otimes \boldsymbol{v}) - V(\boldsymbol{F})}{t} = \langle \boldsymbol{P}(\boldsymbol{F}), \boldsymbol{u} \otimes \boldsymbol{v} \rangle. \tag{4.6}$$

In particular,

$$V(\boldsymbol{F} + \boldsymbol{u} \otimes \boldsymbol{v}) \geq W(\boldsymbol{F}) + \langle \boldsymbol{P}(\boldsymbol{F}), \boldsymbol{u} \otimes \boldsymbol{v} \rangle.$$

Proof By our assumption, $v(t) = V(\boldsymbol{F} + t\boldsymbol{u} \otimes \boldsymbol{v})$ is convex on $\mathbb{R}$. Recall from the theory of convex functions that

$$q(t) = \frac{v(t) - v(0)}{t}$$

is monotone increasing on each of the intervals $(-\infty, 0)$, $(0, +\infty)$. Therefore, the limits

$$v'(0^{\pm}) = \lim_{t\to 0^{\pm}} \frac{v(t) - v(0)}{t}$$

exist. Moreover, the convexity of $v(t)$ implies that $v'(0^-) \le v'(0^+)$. Let $w(t) = W(\boldsymbol{F} + t\boldsymbol{u} \otimes \boldsymbol{v})$. By assumption, $v(t) \le w(t)$. We also have $v(0) = w(0)$ since $\boldsymbol{F} \in \mathcal{A}_V$. When $t > 0$,

$$\frac{v(t) - v(0)}{t} \le \frac{w(t) - w(0)}{t}.$$

Therefore, $v'(0^+) \le w'(0)$. Similarly, when $t < 0$, we obtain $v'(0^-) \ge w'(0)$. Thus,

$$w'(0) \le v'(0^-) \le v'(0^+) \le w'(0).$$

We conclude that the limit on the left-hand side of (4.6) exists and is equal to

$$w'(0) = \langle \boldsymbol{P}(\boldsymbol{F}), \boldsymbol{u} \otimes \boldsymbol{v}\rangle.$$

Thus, the convex function $v(t)$ is differentiable at $t = 0$, and $v = w(0) + w'(0)t$ is a line tangent to its graph at $t = 0$. The convexity of $v(t)$ then implies that $v(t) \ge w(0) + w'(0)t$ for all $t \in \mathbb{R}$. □

In general, one does not expect explicit formulas for the values of the quasiconvex envelope QW in terms of W. In that respect, the following Lemma 4.3 provides a nice exception to the rule.

Lemma 4.3 *Assume that the pair $\boldsymbol{F}_{\pm}$ satisfies all conditions of Theorem 2.2. Then*

$$QW(t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-) = RW(t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-) = tW(\boldsymbol{F}_+) + (1-t)W(\boldsymbol{F}_-) \quad (4.7)$$

for all $t \in [0, 1]$, where $RW(\boldsymbol{F})$ is the rank-one convex envelope of $W(\boldsymbol{F})$ (Dacorogna 1989).

Proof By assumption (*S*), we have $QW(\boldsymbol{F}_{\pm}) = W(\boldsymbol{F}_{\pm}) \ge RW(\boldsymbol{F}_{\pm}) \ge QW(\boldsymbol{F}_{\pm})$. Therefore,

$$QW(\boldsymbol{F}_{\pm}) = W(\boldsymbol{F}_{\pm}) = RW(\boldsymbol{F}_{\pm}). \quad (4.8)$$

By the rank-one convexity of $QW(\boldsymbol{F})$ (Morrey and Charles 1966; Ball 1976; Dacorogna 1989) and assumptions (*K*) and (*S*), we have

$$QW(t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-) \le tQW(\boldsymbol{F}_+) + (1-t)QW(\boldsymbol{F}_-) = tW(\boldsymbol{F}_+) + (1-t)W(\boldsymbol{F}_-). \quad (4.9)$$

To prove the opposite inequality, we apply Lemma 4.2 and obtain

$$QW(t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-) \ge W(\boldsymbol{F}_-) + \langle \boldsymbol{P}_-, t[\![\boldsymbol{F}]\!]\rangle = tW(\boldsymbol{F}_+) + (1-t)W(\boldsymbol{F}_-) - tp^* - \frac{t}{2}\mathfrak{N}. \quad (4.10)$$

Thus, Lemma 4.1, (4.9), and (4.10) result in the formula for $QW(t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-)$ from (4.7). We also have, in view of assumption (*K*) and (4.8), that

$$tW(\boldsymbol{F}_+) + (1-t)W(\boldsymbol{F}_-) = QW(t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-) \le RW(t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-) \le tRW(\boldsymbol{F}_+) + (1-t)RW(\boldsymbol{F}_-) = tW(\boldsymbol{F}_+) + (1-t)W(\boldsymbol{F}_-).$$

Formula (4.7) is now proved. □

We are now ready to establish inequality (2.12), proving Theorem 2.2. Let $\boldsymbol{\phi}_0 \in W_0^{1,\infty}(B;\mathbb{R}^m)$ be an arbitrary test function. Let $Q_{\boldsymbol{n}}$ be a cube with side 2 centered at the origin and having a face with normal $\boldsymbol{n}$. Consider a $Q_{\boldsymbol{n}}$-periodic function $\bar{\boldsymbol{F}}_{\text{per}}(z)$ on $\mathbb{R}^d$ given on its period $Q_{\boldsymbol{n}}$ by

$$\bar{\boldsymbol{F}}_{\text{per}}(z) = \begin{cases} \boldsymbol{F}_+ & \text{if } \boldsymbol{z}\cdot\boldsymbol{n} > 0, \\ \boldsymbol{F}_- & \text{if } \boldsymbol{z}\cdot\boldsymbol{n} < 0. \end{cases}$$

Assumption (*K*) implies that

$$\bar{\boldsymbol{F}}_{\text{per}}(z) = \{\!\{\boldsymbol{F}\}\!\} + \nabla(\psi(\boldsymbol{z}\cdot\boldsymbol{n})\boldsymbol{a}), \quad \{\!\{\boldsymbol{F}\}\!\} = \frac{1}{2}(\boldsymbol{F}_+ + \boldsymbol{F}_-),$$

where $\psi(\zeta)$ is a two-periodic sawtooth function such that $\psi(\zeta) = |\zeta|/2$, $\zeta \in [-1, 1]$. Let $\boldsymbol{\phi}_{\text{per}}(z)$ be a $Q_{\boldsymbol{n}}$-periodic function such that

$$\boldsymbol{\phi}_{\text{per}}(z) = \begin{cases} \boldsymbol{\phi}_0(z), & z \in B, \\ \boldsymbol{0}, & z \in Q_{\boldsymbol{n}} \setminus B. \end{cases}$$

This function is Lipschitz continuous since $\boldsymbol{\phi}_0 \in W_0^{1,\infty}(B;\mathbb{R}^m)$ and $B \subset Q_{\boldsymbol{n}}$. The function $QW(\boldsymbol{F})$ is quasiconvex, and the function $\boldsymbol{\phi}_{\text{per}}(z) + \psi(\boldsymbol{z}\cdot\boldsymbol{n})\boldsymbol{a}$ is $Q_{\boldsymbol{n}}$-periodic. Therefore (Dacorogna 1989),

$$QW(\{\!\{\boldsymbol{F}\}\!\}) \le \fint_{Q_{\boldsymbol{n}}} QW(\{\!\{\boldsymbol{F}\}\!\} + \nabla(\boldsymbol{\phi}_{\text{per}} + \psi(\boldsymbol{z}\cdot\boldsymbol{n})\boldsymbol{a}))\mathrm{d}z \le \fint_{Q_{\boldsymbol{n}}} W(\bar{\boldsymbol{F}}_{\text{per}}(z) + \nabla\boldsymbol{\phi}_{\text{per}})\mathrm{d}z.$$

By Lemma 4.3,

$$QW(\{\!\{\boldsymbol{F}\}\!\}) = \{\!\{W\}\!\} = \fint_{Q_{\boldsymbol{n}}} W(\bar{\boldsymbol{F}}_{\text{per}}(z))\mathrm{d}z.$$

Hence,

$$\int_{Q_{\boldsymbol{n}}} W(\bar{\boldsymbol{F}}_{\text{per}}(z))\mathrm{d}z \le \int_{Q_{\boldsymbol{n}}} W(\bar{\boldsymbol{F}}_{\text{per}}(z) + \nabla\boldsymbol{\phi}_{\text{per}})\mathrm{d}z.$$

Inequality (2.12) is proved since $\boldsymbol{\phi}_0(z)$ is supported on B.

Theorem 2.2 answers the question studied in Šilhavý (2005) by giving a complete characterization of all possible pairs of deformation gradient values $\boldsymbol{F}_\pm$ that can occur on a stable phase boundary. The global minimality of $\bar{\boldsymbol{y}}(z)$, given by (2.15), also implies that any other interface conditions, for example, local Grinfeld conditions (Grinfeld 1987; Simpson and Spector 1991; Grabovsky et al. 2010) or roughening stability inequality (Grabovsky and Truskinovsky 2011, Remark 4.2), must be consequences of (*K*), (*I*), and (*S*). In addition, the quasiconvexity at the internal surface of the gradient discontinuity looks similar to the quasiconvexity at the boundary (Ball and Marsden 1984). However, the resemblance is deceptive. The quasiconvexity at the external boundary cannot be reduced to the quasiconvexity in the interior (e.g., Simpson and Spector 1987).

5 Stability in Bulk Versus Stability on Interface

In this section we establish a relation between *particular* solutions of the variational problems (2.13) and (2.14) that elucidates the role played in the theory by the normality condition.

Consider the set $\mathfrak{B}$ of all $\boldsymbol{F} \in \mathbb{M}$ that are not strongly locally stable; we called this set the *elastic binodal* in Grabovsky and Truskinovsky (2013). For such $\boldsymbol{F}$ the infimum in the variational problem (2.13) may be reachable only by minimizing sequences characterized by their Young measures (Tartar 1983; Kinderlehrer and Pedregal 1991). Suppose that for some $\boldsymbol{F} \in \mathfrak{B}$ a Young measure solution of (2.13) is a simple laminate (Pedregal 1993):

$$\nu = \theta \delta_{\boldsymbol{F}_+} + (1-\theta)\delta_{\boldsymbol{F}_-}, \quad \boldsymbol{F} = \theta \boldsymbol{F}_+ + (1-\theta)\boldsymbol{F}_-, \quad 0 < \theta < 1. \tag{5.1}$$

The set $\mathfrak{B}_1$ of all such $\boldsymbol{F} \in \mathfrak{B}$ will be called the *simple laminate region*. There is a direct connection between the simple laminate region $\mathfrak{B}_1$ and locally stable interfaces.

Theorem 5.1 *A strongly locally stable interface determined by $\boldsymbol{F}_\pm$ corresponds to a straight line segment $\{\theta \boldsymbol{F}_+ + (1-\theta)\boldsymbol{F}_- : \theta \in (0,1)\} \subset \mathfrak{B}_1$, so that the laminate Young measure (5.1) solves (2.13) with $\boldsymbol{F} = \theta \boldsymbol{F}_+ + (1-\theta)\boldsymbol{F}_-$. Conversely, every point $\boldsymbol{F} \in \mathfrak{B}_1$ corresponding to a laminate Young measure (5.1) determines a strongly locally stable interface.*

Proof If $\Pi_{\boldsymbol{n}} = \{z \in \mathbb{R}^d : z \cdot \boldsymbol{n} = 0\}$ is a strongly locally stable interface determined by $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$, then the pair $\boldsymbol{F}_\pm$ satisfies conditions (*K*), (*I*), and (*S*). By Lemma 4.3, the gradient Young measure (5.1) attains a minimum in (2.13) for $\boldsymbol{F} = \theta \boldsymbol{F}_+ + (1-\theta)\boldsymbol{F}_-$, $\theta \in (0,1)$, and thus $\boldsymbol{F} \in \mathfrak{B}_1$. If $\mathfrak{B}_1$ has a nonempty interior, then formula (4.7) says that the graph of the quasiconvex envelope $QW(\boldsymbol{F})$ over $\mathfrak{B}_1$ is formed by straight line segments joining $(\boldsymbol{F}_+, W(\boldsymbol{F}_+))$ and $(\boldsymbol{F}_-, W(\boldsymbol{F}_-))$. In other words, the graph of $QW(\boldsymbol{F})$ is a ruled surface.

Conversely, if the gradient Young measure (5.1) attains a minimum in (2.13), then $\boldsymbol{F}_\pm$ satisfies the kinematic compatibility condition (2.4) and

$$QW(\theta \boldsymbol{F}_+ + (1-\theta)\boldsymbol{F}_-) = \theta W(\boldsymbol{F}_+) + (1-\theta)W(\boldsymbol{F}_-). \tag{5.2}$$

The difference between (5.2) and (4.7) is that (5.2) is assumed to hold for a single fixed value of $\theta \in (0, 1)$. Therefore, both the material stability (*S*) at $\boldsymbol{F}_{\pm}$ and the interchange stability (*I*) need to be established. □

Lemma 5.2 *Assume that the pair* $\boldsymbol{F}_{\pm}$ *satisfies* (2.4) *and that* (5.2) *holds for some* $\theta \in (0, 1)$. *Then* $QW(\boldsymbol{F}_{\pm}) = W(\boldsymbol{F}_{\pm})$ *and* $\mathfrak{N} = 0$.

Proof The proof is based on the following general property of convex functions. □

Lemma 5.3 *Let* $\phi(t)$ *be a convex function on* $[0, 1]$. *Suppose that* $\phi(\theta) = \theta\phi(1) + (1-\theta)\phi(0)$ *for some* $\theta \in (0, 1)$. *Then*

$$\phi(t) = t\phi(1) + (1-t)\phi(0) \tag{5.3}$$

for all $t \in [0, 1]$.

Proof Let $y = \phi(\theta) + m(t-\theta)$ be the equation of the supporting line to the graph of the convex function $\phi(t)$, i.e.,

$$\phi(t) \geq \phi(\theta) + m(t-\theta) = \theta\phi(1) + (1-\theta)\phi(0) + m(t-\theta).$$

Substituting $t = 0$ and $t = 1$ we find that $m = \phi(1) - \phi(0)$. Hence, $\phi(t) \geq t\phi(1) + (1-t)\phi(0)$. The opposite inequality follows from convexity. The lemma is proved. □

To prove Lemma 5.2, we recall that $QW(\boldsymbol{F}) \leq W(\boldsymbol{F})$ for all $\boldsymbol{F} \in \mathbb{M}$. By (5.2) and the rank-one convexity of $QW(\boldsymbol{F})$, we have

$$\begin{aligned}\theta W(\boldsymbol{F}_+) + (1-\theta)W(\boldsymbol{F}_-) &= QW(\theta\boldsymbol{F}_+ + (1-\theta)\boldsymbol{F}_-)\\ &\leq \theta QW(\boldsymbol{F}_+) + (1-\theta)QW(\boldsymbol{F}_-) \leq \theta W(\boldsymbol{F}_+) + (1-\theta)W(\boldsymbol{F}_-),\end{aligned}$$

which is possible if and only if $QW(\boldsymbol{F}_{\pm}) = W(\boldsymbol{F}_{\pm})$. Then, defining

$$\phi(t) = QW(t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-)$$

and applying Lemma 5.3 we obtain (4.7). We can also apply Lemma 4.2, with $\boldsymbol{F} = \boldsymbol{F}_{\pm}$, $V(\boldsymbol{F}) = QW(\boldsymbol{F})$. Formula (4.6) allows us to differentiate (4.7) at $t = 0$ and $t = 1$:

$$\langle \boldsymbol{P}_-, [\![\boldsymbol{F}]\!]\rangle = [\![W]\!], \quad \langle \boldsymbol{P}_+, [\![\boldsymbol{F}]\!]\rangle = [\![W]\!].$$

Subtracting the two equalities we obtain $\mathfrak{N} = 0$. □

Thus, we have shown that every $\boldsymbol{F}$ in the simple laminate region $\mathfrak{B}_1$ gives rise to the pair $\{\boldsymbol{F}_+, \boldsymbol{F}_-\}$ satisfying all conditions of Theorem 2.2. Theorem 2.2 then implies that the interface $\Pi_{\boldsymbol{n}}$ determined by $\boldsymbol{F}_{\pm}$ is strongly locally stable. Theorem 5.1 is now proved. □

Remark 5.4 The system of algebraic Eqs. (2.4), (4.1), (4.2), and (4.3) defines a codimension 1 surface $\mathfrak{J} \subset \mathbb{M}$ called the *jump set*. More precisely, the jump set is the projection of the solution set $(\boldsymbol{F}_+, \boldsymbol{F}_-, \boldsymbol{a}, \boldsymbol{n}) \in \mathbb{M} \times \mathbb{M} \times \mathbb{R}^m \times \mathbb{S}^{d-1}$ onto the first component $\mathbb{M}$.[3] We showed in Grabovsky and Truskinovsky (2011) that under some nondegeneracy assumptions, the jump set must lie in the closure of the binodal region $\mathfrak{B}$. In fact, all points on the jump set are marginally stable and detectable through the nucleation of an infinite layer in an infinite space (Grabovsky and Truskinovsky 2013). It follows that the existence of a strongly locally stable interface has significant consequences for the geometry of $\mathfrak{B}$. The presence of stable interfaces implies that a part of the jump set must coincide with a part of the *binodal*, the boundary of $\mathfrak{B}$. The rank-one lines joining $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$, both of which lie on the binodal, cover the simple lamination region $\mathfrak{B}_1 \subset \mathfrak{B}$.

6 Analogy with Plasticity Theory

In this section we show that the algebraic equation

$$\mathfrak{N} = 0,$$

interpreted previously as a condition of interchange equilibrium, is conceptually similar to the well-known *normality condition* in plasticity theory (Lubliner 1990; Fosdick and Volkmann 1993).

To establish a link between the two frameworks, we now show that a microstructure in elasticity theory plays the role of a "mechanism" in plasticity theory. Consider a loading program with affine Dirichlet boundary conditions $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{F}(t)\boldsymbol{x}$. Suppose that $\boldsymbol{F}(t) \in \mathfrak{B}_1$ for an interval of values of the loading parameter t. Then, for every t the deformation gradient $\boldsymbol{F}(t)$ will be accommodated by a laminate (5.1), so that

$$\boldsymbol{F}(t) = \theta(t)\boldsymbol{F}_+(t) + (1 - \theta(t))\boldsymbol{F}_-(t). \tag{6.1}$$

We now interpret representation (6.1) from the point of view of plasticity theory. While the deformations associated with the change of $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$ in each layer of the laminate are elastic, the deformation associated with the change of parameter t, affecting the microstructure and modifying the Young measure ν, can be regarded as inelastic. In fact, it is similar to lattice-invariant shear characterizing elementary slip in crystal plasticity theory. To be more specific, we can decompose the strain rate $\dot{\boldsymbol{F}}$ as follows:

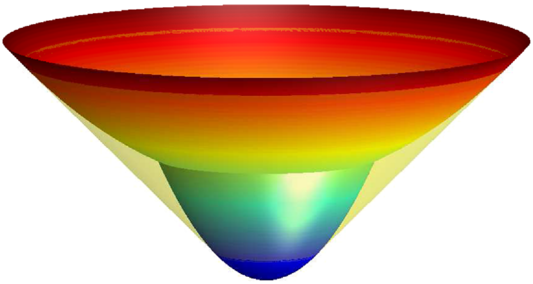
$$\dot{\boldsymbol{F}} = \theta\dot{\boldsymbol{F}}_+ + (1 - \theta)\dot{\boldsymbol{F}}_- + \dot{\theta}[\![\boldsymbol{F}]\!] = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p,$$

where $\dot{\boldsymbol{\varepsilon}}^e = \theta\dot{\boldsymbol{F}}_+ + (1-\theta)\dot{\boldsymbol{F}}_-$ is the elastic strain rate and $\dot{\boldsymbol{\varepsilon}}^p = \dot{\theta}[\![\boldsymbol{F}]\!]$ is the "plastic" strain rate.

Next we notice that in equilibrium the inelastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ defines an *affine* direction along the quasiconvex envelope of the energy (Lemma 4.3). This suggests

[3] Projection onto the second component defines the same surface because of the symmetry of equations under the phase interchange $\boldsymbol{F}_+ \to \boldsymbol{F}_-$, $\boldsymbol{F}_- \to \boldsymbol{F}_+$, $\boldsymbol{n} \to -\boldsymbol{n}$.

Fig. 5 Antiplane shear energy density function and its convex envelope



that there is a stress plateau with which one can associate a notion of the yield stress (Lubliner 1990).

To find an equation for the corresponding yield surface, we choose a special loading path where the elastic fields in the layers do not change $\dot{\boldsymbol{F}}_{\pm} = \mathbf{0}$. Then, differentiating (4.7) in t, we find that the total stress field $\boldsymbol{P}_{\text{tot}}(t) = QW_{\boldsymbol{F}}(\boldsymbol{F}(t))$ lies on the hyperplane:[4]

$$\mathfrak{Y}_{\mathcal{M}} = \{\boldsymbol{P} : \langle \boldsymbol{P}, [\![\boldsymbol{F}]\!] \rangle = [\![W]\!]\}, \tag{6.2}$$

which we interpret as the yield surface associated with the "plastic" mechanism $\mathcal{M} = (\boldsymbol{F}_{+}, \boldsymbol{F}_{-})$. If we now rewrite our elastic normality condition $\mathfrak{N} = 0$ in the form

$$\langle \boldsymbol{P}_{+}, [\![\boldsymbol{F}]\!] \rangle = \langle \boldsymbol{P}_{-}, [\![\boldsymbol{F}]\!] \rangle, \tag{6.3}$$

it becomes apparent that the "plastic" strain rate $\dot{\boldsymbol{\varepsilon}}^{p} = \dot{\theta}[\![\boldsymbol{F}]\!]$ is orthogonal to the yield surface $\mathfrak{Y}_{\mathcal{M}}$.

To strengthen the analogy, we observe that in plasticity theory the yield surface marks the set of minimally stable elastic states (Salman and Truskinovsky 2012). In an elastic framework the states $\boldsymbol{F}_{\pm}$ adjacent to the jump discontinuity are also only marginally stable (Remark 5.4). The fact that in elasticity the normality condition appears as a part of energy *minimization* while in plasticity theory it is usually derived by maximizing plastic *dissipation* is secondary in view of the implied rate-independent nature of plastic dissipation (Lubliner 1990).

The analogy between elastic and plastic normality conditions becomes more transparent if we consider a simple example. Suppose that our material is isotropic and the deformation is antiplane shear. Take the energy density in the form

$$W(\boldsymbol{F}) = \min\left\{\frac{\mu_{+}}{2}|\boldsymbol{F}|^{2} + w_{+}, \frac{\mu_{-}}{2}|\boldsymbol{F}|^{2} + w_{-}\right\}, \tag{6.4}$$

where $\boldsymbol{F} \in \mathbb{R}^{2}$ and the shear moduli of the "phases" $\mu_{\pm}$ are positive.

In this scalar example, the quasiconvex and convex envelopes of the energy density coincide, and hence we can write (Fig. 5)

[4] In fact, $\boldsymbol{P}_{\text{tot}}(t) = t\boldsymbol{P}_{+} + (1-t)\boldsymbol{P}_{-}$, as was shown in Ball et al. (2000).

$$QW(\boldsymbol{F}) = CW(\boldsymbol{F}) = \begin{cases} \frac{\mu_+}{2}|\boldsymbol{F}|^2 + w_+ & \text{if } |\boldsymbol{F}| \leq \varepsilon_+, \\ \frac{\mu_-}{2}|\boldsymbol{F}|^2 + w_- & \text{if } |\boldsymbol{F}| \geq \varepsilon_-, \\ |\boldsymbol{F}|\sqrt{-\frac{2[\![w]\!]\mu_+\mu_-}{[\![\mu]\!]}} + \frac{[\![\mu w]\!]}{[\![\mu]\!]} & \text{if } \varepsilon_+ \leq |\boldsymbol{F}| \leq \varepsilon_-, \end{cases}$$

where

$$\varepsilon_+ = \sqrt{\frac{-2[\![w]\!]\mu_-}{[\![\mu]\!]\mu_+}}, \quad \varepsilon_- = \sqrt{-\frac{2[\![w]\!]\mu_+}{[\![\mu]\!]\mu_-}}.$$

Observe also that the binodal region

$$\mathfrak{B} = \{\boldsymbol{F} \in \mathbb{R}^2 : \varepsilon_+ \leq |\boldsymbol{F}| \leq \varepsilon_-\}$$

coincides with the simple laminate region $\mathfrak{B}_1$ since for $\boldsymbol{F}_0 \in \mathfrak{B}$ the gradient Young measures

$$\nu(\boldsymbol{F}) = \theta\delta_{\boldsymbol{F}_+}(\boldsymbol{F}) + (1-\theta)\delta_{\boldsymbol{F}_-}(\boldsymbol{F}), \quad \theta = \frac{|\boldsymbol{F}_0| - \varepsilon_-}{[\![\varepsilon]\!]}, \quad \boldsymbol{F}_\pm = \frac{\varepsilon_\pm}{|\boldsymbol{F}_0|}\boldsymbol{F}_0$$

attain the infimum in (2.13).

By fixing $\boldsymbol{F}_+$ on the circle $\mathcal{C}_+ = \{\boldsymbol{F} \in \mathbb{R}^2 : |\boldsymbol{F}| = \varepsilon_+\}$ we then obtain the unique

$$\boldsymbol{F}_- = \frac{\varepsilon_-}{\varepsilon_+}\boldsymbol{F}_+,$$

which furnishes the "plastic" mechanism $\mathcal{M} = (\boldsymbol{F}_+, \boldsymbol{F}_-)$. The associated yield plane (line) $\mathfrak{Y}_\mathcal{M}$ can be written explicitly as

$$\mathfrak{Y}_\mathcal{M} = \left\{\boldsymbol{P} \in \mathbb{R}^2 : \boldsymbol{P} \cdot \boldsymbol{F}_+ = \frac{2\varepsilon_+[\![w]\!]}{[\![\varepsilon]\!]}\right\}.$$

We observe that as $\boldsymbol{F}_+$ is varied over the circle $\mathcal{C}_+$, the yield lines $\mathfrak{Y}_\mathcal{M}$ form an envelope of the circle

$$\mathcal{P} = \left\{\boldsymbol{P} \in \mathbb{R}^2 : |\boldsymbol{P}| = \frac{2[\![w]\!]}{[\![\varepsilon]\!]}\right\}$$

in stress space (Fig. 6), which is the image of the binodal $\partial\mathfrak{B}$ under the map $\boldsymbol{P}(\boldsymbol{F})$.

Since the stress in each phase of the laminate is always the same,

$$\boldsymbol{P}_+ = \boldsymbol{P}_- = \mu_+\boldsymbol{F}_+ = \mu_-\boldsymbol{F}_-,$$

we can write

$$|\boldsymbol{P}_\pm|^2 = \mu_+^2\varepsilon_+^2 = -\frac{2[\![w]\!]\mu_+\mu_-}{[\![\mu]\!]} = \frac{4[\![w]\!]^2}{[\![\varepsilon]\!]^2}.$$

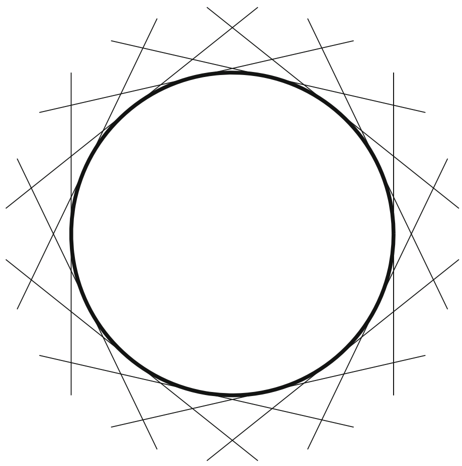


Fig. 6 Envelope of yield lines in antiplane shear

Thus, in an arbitrary loading program the total stress $\boldsymbol{P}(t) = \theta \boldsymbol{P}_+ + (1-\theta)\boldsymbol{P}_-$ will be confined to the yield surface envelope $\mathcal{P}$, provided $\boldsymbol{F}(t) \in \mathfrak{B}$.

Two cautionary notes are in order. First, in contrast to conventional plasticity theory, the regions of stress space both inside and outside the yield surface $\mathcal{P}$ are elastic. This distinguishes our *transformational plasticity*, where hysteresis is infinitely narrow, from classical plasticity, where hysteresis is essential. The second observation is that the yield plane envelope picture seen in the example will hold in general as long as $[\![\boldsymbol{P}]\!] = \boldsymbol{0}$, in particular, it will hold for all scalar problems ($m = 1$). If $[\![\boldsymbol{P}]\!] \neq 0$, our hardening-free plastic analogy breaks down because the total stress in an arbitrary loading program is no longer confined to any surface. In this case, the "plastic" mechanism operates on a set of full measure, and the proposed analogy requires a generalization.

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