J Elast (2016) 123:225–243
DOI 10.1007/s10659-015-9557-y

Legendre-Hadamard Conditions for Two-Phase Configurations

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Received: 6 August 2015 / Published online: 3 December 2015

Abstract We generalize the classical Legendre-Hadamard conditions by using quadratic extensions of the energy around a set of two configurations and obtain new algebraic necessary conditions for nonsmooth strong local minimizers. The implied bounds of stability are easily accessible as we illustrate on a nontrivial example where quasiconvexification is unknown.

Keywords Martensitic phase transitions · Quasiconvexity · Elastic stability · Ellipticity · Phase boundaries · Calculus of variations · Strong local minimizers · Laminates

Mathematics Subject Classification 74N20 · 74G65 · 49J10

1 Introduction

Readily computable, algebraic Legendre-Hadamard (LH) conditions play a crucial role in delimiting stability boundaries for weak local minima in the Calculus of Variations [9]. In the study of strong local minima they have been deemed rather useless because the corresponding stability bounds obtained at points of regularity are usually too crude. In this paper we show that if LH conditions are appropriately generalized at points of reduced regularity, they may be quite useful in establishing stability bounds for strong local minimizers as well. To illustrate the idea we consider in full detail the simplest case when reduced regularity is associated with the presence of static jump discontinuities. In elasticity theory those would be phase boundaries associated with solid-solid phase transformations [2] or the boundaries of shear bands associated with localization phenomena in softening materials [16, 20, 21]. Having these applications in mind we use the language of hyperelasticity theory throughout the paper.

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The principle of minimum energy, underlying the idea of elastic stability [25], points towards the study of strong local minimizers of the energy functional

$$E(\boldsymbol{y}) = \int_{\Omega} W\big(\nabla \boldsymbol{y}(\boldsymbol{x})\big)d\boldsymbol{x} - \int_{\partial\Omega} \boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{y} dS(\boldsymbol{x}), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is an open and bounded domain, the map $\boldsymbol{y} : \Omega \to \mathbb{R}^m$, satisfying appropriate boundary conditions, is the "deformation", and the stored energy density function $W(\boldsymbol{F})$ is of class C^2 on the space $\mathbb{M}$ of all $m \times d$ matrices.

Suppose that for a given minimizer $\boldsymbol{y}(\boldsymbol{x})$ of (1.1), the value of its gradient at a *regular* point $\boldsymbol{x}$ is $\boldsymbol{F}(\boldsymbol{x})$. It is well known that $\boldsymbol{F}(\boldsymbol{x})$ should be inside the domain of quasiconvexity [1] of the energy density $W(\boldsymbol{F})$. However this condition is non-algebraic [18] and is usually impossible to check. We also know that $\boldsymbol{F}(\boldsymbol{x})$ should satisfy the Legendre-Hadamard (LH) condition which is not sufficient, but easy to check. The question discussed in this paper is whether the algebraic conditions of LH type can be formulated for *singular* points $\boldsymbol{x}$ that lie on the surface of jump discontinuity of $\boldsymbol{F}(\boldsymbol{x})$, where the deformation gradient takes two values $\boldsymbol{F}_+(\boldsymbol{x})$ and $\boldsymbol{F}_-(\boldsymbol{x})$. While such singularity can be regarded as the simplest, it is generic in applications where the energy density function is not rank-one convex [15, 17]. We need to mention that algebraic conditions of other types have also been found for surfaces of jump discontinuity [10, 13, 23]. The interrelations between those conditions and the new ones found here warrant an examination which is beyond the scope of the present discussion.

The origin of the classical LH conditions is the Weierstrass inequality localized around a given gradient $\boldsymbol{F}$ (one phase configuration) [6, 15] and the challenge is to generalize the corresponding localization procedure for the case of two gradients $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$ (two-phase configuration). The resulting LH type algebraic conditions will then depend *simultaneously* on the second order Taylor expansions of $W(\boldsymbol{F})$ around $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$. In order to go beyond separate LH conditions at $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$ we exploit the existing rank-one connections between the neighborhoods of $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$ and use the fact that both gradients $\boldsymbol{F}_\pm$ must lie on the *boundary* of the domain of quasiconvexity [11]. This allows us to obtain a new algebraic inequality that is shown to be strictly stronger than the one based on the conventional Weierstrass condition. We present a nontrivial example where the exact stability limits requiring the knowledge of the quasi-convexification of the energy are unknown and our algebraic results are the sharpest computable stability bounds.

The paper is organized as follows. In Sect. 2 we recall the derivation of the classical LH conditions for single-phase configurations and present a set of algebraic equations imposed by stationarity for the two-phase configurations. In Sect. 3 we discuss our first generalization of the LH conditions for two-phase configurations when the quadratic extension of the energy is built around only one of the two gradients. In Sect. 4 we introduce continuous families of equilibrated two phase configurations known as laminates and show how such families can be used to obtain new LH type conditions based on the simultaneous linearization of the energy around two gradients. The example considered in Sect. 5 illustrates the gap between the new and the known necessary conditions. Our conclusions are summarized in Sect. 6, where we also mention the possibility of extending the proposed technique to singularities of higher co-dimensionality involving more than two deformation gradients. Several technical results are presented in the form of Appendices A and B.

2 Preliminaries

We say that $\boldsymbol{F}$ lies in the binodal region [12] if $QW(\boldsymbol{F}) < W(\boldsymbol{F})$, where $QW(\boldsymbol{F})$ is the quasiconvex envelope of $W(\boldsymbol{F})$ [5]. A consequence of quasiconvexity is the Weierstrass convexity condition [19], that says that $W(\boldsymbol{F})$ must be convex along rank-one lines passing through $\boldsymbol{F}$. This condition can be formulated as the nonnegativity of the Weierstrass excess function

$$\omega(\boldsymbol{u},\boldsymbol{v};\boldsymbol{F}) = W(\boldsymbol{F}+\boldsymbol{u}\otimes\boldsymbol{v}) - W(\boldsymbol{F}) - \langle W_{\boldsymbol{F}}(\boldsymbol{F}), \boldsymbol{u}\otimes\boldsymbol{v}\rangle \geq 0, \quad \forall \boldsymbol{u}\in\mathbb{R}^m,\ \boldsymbol{v}\in\mathbb{R}^d. \tag{2.1}$$

Physically, this condition can be interpreted as stability with respect to the nucleation of an infinite compatible slab, e.g., [4, 8]. We observe that if (2.1) holds for $\boldsymbol{F}$, then $\boldsymbol{u}=0$, $\boldsymbol{v}=0$ is the point of minimum of $\omega(\boldsymbol{u},\boldsymbol{v})$ and that by construction $\nabla_{(\boldsymbol{u},\boldsymbol{v})}\omega(0,0;\boldsymbol{F})=0$. The Hessian of $\omega(\boldsymbol{u},\boldsymbol{v};\boldsymbol{F})$ at $\boldsymbol{u}=0$, $\boldsymbol{v}=0$ must be nonnegative semidefinite which is equivalent [14] to nonnegative semidefiniteness of the acoustic tensor of $W(\boldsymbol{F})$

$$\boldsymbol{A}(\boldsymbol{m};\boldsymbol{F}) \geq 0, \quad \forall \boldsymbol{m}\in\mathbb{R}^d,\ |\boldsymbol{m}|=1, \tag{2.2}$$

understood in the sense of quadratic forms. This condition is called the Legendre-Hadamard (LH) condition, and the acoustic tensor $\boldsymbol{A}(\boldsymbol{m};\boldsymbol{F})$ is the quadratic form

$$\boldsymbol{A}(\boldsymbol{m};\boldsymbol{F})\boldsymbol{b}\cdot\boldsymbol{b} = \langle W_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{F})(\boldsymbol{b}\otimes\boldsymbol{m}), \boldsymbol{b}\otimes\boldsymbol{m}\rangle. \tag{2.3}$$

Consider now the boundary of validity of (2.1). At the point where $\boldsymbol{F}$ crosses this boundary $\omega(\boldsymbol{u},\boldsymbol{v};\boldsymbol{F})=0$. This can happen in one of two ways. Either, we deal with the existing zero at $\boldsymbol{u}\otimes\boldsymbol{v}=0$, while an eigenvalue of the acoustic tensor becomes negative, or we encounter a new zero $\boldsymbol{u}\otimes\boldsymbol{v}=\boldsymbol{a}\otimes\boldsymbol{n}$, where $|\boldsymbol{n}|=1$. Physically, the instability in the former case is *local* in phase space, corresponding to the formation inside the homogeneous phase $\boldsymbol{F}$ of a slab, whose normal $\boldsymbol{m}$ is the direction for which the eigenvalues of $\boldsymbol{A}(\boldsymbol{m};\boldsymbol{F})$ become negative, and whose deformation gradient is infinitesimally close to $\boldsymbol{F}$. The instability in the latter case is *global* in phase space, corresponding to the nucleation inside the homogeneous phase $\boldsymbol{F}=\boldsymbol{F}_-$ of a slab representing a genuinely new "phase" with the gradient $\boldsymbol{F}_+=\boldsymbol{F}+\boldsymbol{a}\otimes\boldsymbol{n}$, where the unit vector $\boldsymbol{n}$ is normal to the slab.

In the case of local instability the corresponding boundary in the $\boldsymbol{F}$ space is contained in the codimension 1 variety [12], defined by the system

$$\det \boldsymbol{A}(\boldsymbol{m};\boldsymbol{F})=0, \qquad \nabla_{\boldsymbol{m}}\det\boldsymbol{A}(\boldsymbol{m};\boldsymbol{F})=0, \tag{2.4}$$

where each point is equipped with its own value of $\boldsymbol{m}$, satisfying (2.4). When the instability is global and $\boldsymbol{F}$ crosses the boundary of validity of (2.1), while $\omega(\boldsymbol{a},\boldsymbol{n};\boldsymbol{F})=0$, for some $\boldsymbol{a}\neq 0$, $|\boldsymbol{n}|=1$, the function $\omega(\boldsymbol{u},\boldsymbol{v};\boldsymbol{F})$ is still nonnegative, and, hence, is minimized by $\boldsymbol{u}=\boldsymbol{a}$, $\boldsymbol{v}=\boldsymbol{n}$. It follows that $\nabla_{(\boldsymbol{u},\boldsymbol{v})}\omega(\boldsymbol{a},\boldsymbol{n};\boldsymbol{F})=0$, and that the corresponding Hessian at $\boldsymbol{u}=\boldsymbol{a}$, $\boldsymbol{v}=\boldsymbol{n}$ must be nonnegative semidefinite. Denoting

$$\boldsymbol{P}_\pm = W_{\boldsymbol{F}}(\boldsymbol{F}_\pm), \qquad \mathsf{L}_\pm = W_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{F}_\pm),$$

we obtain the stationarity condition

$$\nabla_{(\boldsymbol{u},\boldsymbol{v})}\omega(\boldsymbol{a},\boldsymbol{n};\boldsymbol{F}) = \left([\![\boldsymbol{P}]\!]\boldsymbol{n}, [\![\boldsymbol{P}]\!]^T\boldsymbol{a}\right) = (0,0), \quad [\![\boldsymbol{P}]\!] = \boldsymbol{P}_+ - \boldsymbol{P}_-.$$

This means that at the points of the surface bounding the region of $\boldsymbol{F}$-space where the Weierstrass condition is satisfied the following system of equations must hold

$$\begin{cases} [\![\boldsymbol{F}]\!] = \boldsymbol{a}\otimes\boldsymbol{n}, \\ \omega(\boldsymbol{a},\boldsymbol{n};\boldsymbol{F}_-) = [\![W]\!] - \langle \boldsymbol{P}_-, [\![\boldsymbol{F}]\!]\rangle = 0, \\ [\![\boldsymbol{P}]\!]\boldsymbol{n} = 0, \\ [\![\boldsymbol{P}]\!]^T\boldsymbol{a} = 0. \end{cases} \tag{2.5}$$

Observe, that the system (2.5) is symmetric with respect to $\boldsymbol{F}_+$, $\boldsymbol{F}_-$, since $[\![W]\!] - \langle \boldsymbol{P}_-, [\![\boldsymbol{F}]\!]\rangle$ differs from $[\![W]\!] - \langle \boldsymbol{P}_+, [\![\boldsymbol{F}]\!]\rangle$ by $\langle [\![\boldsymbol{P}]\!], [\![\boldsymbol{F}]\!]\rangle$, which is zero, due to either of the last two equations in (2.5). It is therefore customary to write the second equation in (2.5), known as the Maxwell condition, in a symmetric form

$$p^* = [\![W]\!] - \big\langle \{\boldsymbol{P}\}, [\![\boldsymbol{F}]\!]\big\rangle = 0, \quad \{\boldsymbol{P}\} = \frac{1}{2}(\boldsymbol{P}_+ + \boldsymbol{P}_-). \tag{2.6}$$

The two values $\boldsymbol{F}_\pm$ of the deformation gradient at a jump discontinuity of the gradient of a strong local minimizer must satisfy the system of equations (2.5), which defines a codimension 1 variety $\mathfrak{J}$ in the $\boldsymbol{F}$-space, called the *jump set*. The system (2.5) can be viewed as a generalization of the Weierstrass-Erdmann corner conditions. The last equation in (2.5), obtained in [11], is not a part of the original Weierstrass-Erdmann system, derived for the scalar case where the last two equations coincide.

3 Stability Conditions with One Phase Fixed

To find stable two-phase configurations we need to search among marginally stable states at the boundary of validity of the Weierstrass test (2.1) because strict rank-one-convexity (Weierstrass stability) excludes jump discontinuities[1] [6, 17]. Such marginally stable states $\boldsymbol{F}_+$ are located on the jump set and have the property that nucleation of layers of $\boldsymbol{F}_-$ is not only energy neutral, but infinitesimal variations in the shear vector and the normal of the layer are felt only to the second order in energy [11]. A straightforward approach to stability of two-phase configurations is to look for conditions ensuring that such variations cannot decrease the energy.

Suppose that either $\boldsymbol{F}_+$ or $\boldsymbol{F}_-$ is about to fail (2.1). The Hessian of $\omega(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{F}_\pm)$ at $\boldsymbol{u} = \boldsymbol{a}$, $\boldsymbol{v} = \boldsymbol{n}$ is a quadratic form $\mathcal{Q}_\pm$ on $\mathbb{R}^m \oplus \mathbb{R}^d$

$$\mathcal{Q}_\pm(\boldsymbol{\xi}, \boldsymbol{\eta}) = \pm[\![\boldsymbol{P}]\!]\boldsymbol{\eta} \cdot \boldsymbol{\xi} + \frac{1}{2}\big\langle \mathsf{L}_\pm(\boldsymbol{a} \otimes \boldsymbol{\eta} \pm \boldsymbol{\xi} \otimes \boldsymbol{n}), \boldsymbol{a} \otimes \boldsymbol{\eta} \pm \boldsymbol{\xi} \otimes \boldsymbol{n}\big\rangle.$$

Its nonnegativity is equivalent to the nonnegative semidefiniteness of $(m + d) \times (m + d)$ matrix

$$\mathbb{C}_\pm = \begin{bmatrix} \boldsymbol{A}_\pm & \pm(\boldsymbol{B}_\pm + [\![\boldsymbol{P}]\!]) \\ \pm(\boldsymbol{B}_\pm + [\![\boldsymbol{P}]\!])^T & \boldsymbol{A}^*_\pm \end{bmatrix}, \tag{3.1}$$

where $m \times d$ matrices $\boldsymbol{B}_\pm$, $d \times d$ symmetric matrices $\boldsymbol{A}^*_\pm$, and $m \times m$ symmetric matrices $\boldsymbol{A}_\pm$ are given by

$$\boldsymbol{B}_\pm \boldsymbol{\eta} \cdot \boldsymbol{\xi} = \big\langle \mathsf{L}_\pm(\boldsymbol{a} \otimes \boldsymbol{\eta}), \boldsymbol{\xi} \otimes \boldsymbol{n}\big\rangle, \qquad \boldsymbol{A}^*_\pm \boldsymbol{\eta} \cdot \boldsymbol{\eta} = \big\langle \mathsf{L}_\pm(\boldsymbol{a} \otimes \boldsymbol{\eta}), \boldsymbol{a} \otimes \boldsymbol{\eta}\big\rangle,$$
$$\boldsymbol{A}_\pm \boldsymbol{\xi} \cdot \boldsymbol{\xi} = \big\langle \mathsf{L}_\pm(\boldsymbol{\xi} \otimes \boldsymbol{n}), \boldsymbol{\xi} \otimes \boldsymbol{n}\big\rangle.$$

Therefore the region of rank one convexity must be contained within the part of the jump set satisfying

$$\mathbb{C}_\pm \geq 0, \quad \text{in the sense of quadratic forms.} \tag{3.2}$$

[1] Microstructure based metastable states would also be disallowed, if the microstructure contains phase boundaries like a finite rank laminate.

Inequalities (3.2), first obtained in [11], can be simplified if the acoustic tensor $\boldsymbol{A}_\pm$ is nonsingular. In that case, minimizing $\mathcal{Q}_\pm(\boldsymbol{\xi}, \boldsymbol{\eta})$ in $\boldsymbol{\xi}$ we obtain

$$\min_{\boldsymbol{\xi}\in\mathbb{R}^m} \mathcal{Q}_\pm(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbb{B}_\pm \boldsymbol{\eta} \cdot \boldsymbol{\eta}, \quad \boldsymbol{\eta} \in \mathbb{R}^d,$$

where

$$\mathbb{B}_\pm = \boldsymbol{A}^*_\pm - \big(\boldsymbol{B}_\pm + [\![\boldsymbol{P}]\!]\big)^T \boldsymbol{A}_\pm^{-1}\big(\boldsymbol{B}_\pm + [\![\boldsymbol{P}]\!]\big) \geq 0. \tag{3.3}$$

4 Stability of Laminates

A more nuanced approach is to consider families of marginally stable two-phase configurations *in the immediate neighborhood* of the selected gradients $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$. One consequence of equations (2.5) is that the line joining $(\boldsymbol{F}_+, W(\boldsymbol{F}_+))$ and $(\boldsymbol{F}_-, W(\boldsymbol{F}_-))$ is tangent to the graph of $W(\boldsymbol{F})$. Points on that line can be interpreted as laminates with variable volume fraction of the limiting phases $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$. This line also represents an upper bound for the quasiconvex envelope, since its direction is rank-one. We can now consider laminates combining phases $\boldsymbol{F}'_+$, $\boldsymbol{F}'_-$, satisfying (2.5) and lying in the vicinity of $\boldsymbol{F}_+$, $\boldsymbol{F}_-$. The corresponding rank-one lines form a lined surface under the following non-degeneracy condition.

Definition 4.1 We say that the pair of point $\boldsymbol{F}_\pm \in \mathfrak{J}$, satisfying (2.5) is *non-degenerate* if the acoustic tensor $\boldsymbol{A}_\pm$ is nonsingular and the quadratic forms $\mathbb{B}_\pm$ vanish only on the scalar multiples of $\boldsymbol{n}$.

Knowing that the classical LH conditions were obtained from variations whose gradients were supported in the vicinity of a single deformation gradient $\boldsymbol{F}$ we can now extend this idea to the two-phase configurations and consider energy variation along the non-degenerate family of laminates $\boldsymbol{F}'_+$, $\boldsymbol{F}'_-$. Even if $\boldsymbol{F}_\pm \in \mathfrak{J}$ is a non-degenerate pair, a neighborhood of $\boldsymbol{F}_-$ (or $\boldsymbol{F}_+$) in $\mathfrak{J}$ may be comprised of several C^1 components.

Lemma 4.2 *Suppose $\boldsymbol{F}_\pm \in \mathfrak{J}$ is a non-degenerate pair in the sense of Definition 4.1 and $\mathfrak{J}_- \subset \mathfrak{J}$ is relatively open C^1 manifold, containing $\boldsymbol{F}_-$. Then there exists a corresponding relatively open C^1 manifold $\mathfrak{J}_+ \subset \mathfrak{J}$, containing $\boldsymbol{F}_+$, such that*

(i) *The vectors $\boldsymbol{N}_\pm = \mathsf{L}_\pm[\![\boldsymbol{F}]\!] - [\![\boldsymbol{P}]\!]$ are normal to $\mathfrak{J}_\pm$ at $\boldsymbol{F}_\pm$.*
(ii) *There exist $\epsilon_\pm > \delta > 0$, such that for every $\boldsymbol{F}'_\pm \in \mathfrak{J}_\pm \cap B(\boldsymbol{F}_\pm, \delta)$ there exists a unique $\boldsymbol{F}'_\mp \in \mathfrak{J}_\mp \cap B(\boldsymbol{F}_\mp, \epsilon_\mp)$ satisfying (2.5).*
(iii) *Straight line segments $\{t\boldsymbol{F}'_+ + (1-t)\boldsymbol{F}'_- : t \in (0,1)\}$, where either $\boldsymbol{F}'_+ \in \mathfrak{J}_+ \cap B(\boldsymbol{F}_+, \delta)$ or $\boldsymbol{F}'_- \in \mathfrak{J}_- \cap B(\boldsymbol{F}_-, \delta)$ foliate an open subset $\mathfrak{O}$ of $\mathbb{M}$.*

The proof can be found in Appendix A. Lemma 4.2 can be used to construct an upper bound $\overline{W}(\boldsymbol{F})$, $\boldsymbol{F} \in \mathfrak{O}$, for the quasiconvex envelope of $W(\boldsymbol{F})$, formed by laminates:

$$\overline{W}(\boldsymbol{F}) = tW\big(\boldsymbol{F}'_+\big) + (1-t)W\big(\boldsymbol{F}'_-\big), \quad \boldsymbol{F} \in \mathfrak{O}, \ \boldsymbol{F} = t\boldsymbol{F}'_+ + (1-t)\boldsymbol{F}'_-. \tag{4.1}$$

We also recall that tangency of the rank-one lines to the graph of $W(\boldsymbol{F})$ leads to the equations

$$\overline{W}(\boldsymbol{F}_\pm) = W(\boldsymbol{F}_\pm), \qquad \overline{W}_{\boldsymbol{F}}(\boldsymbol{F}_\pm) = W_{\boldsymbol{F}}(\boldsymbol{F}_\pm). \tag{4.2}$$

The function $\overline{W}(\boldsymbol{F})$, $\boldsymbol{F} \in \mathfrak{O}$, will be called the *laminate envelope* of $W(\boldsymbol{F})$.

If we now assume that $\boldsymbol{F}_\pm$ belongs to the domain of quasiconvexity, i.e., $W(\boldsymbol{F}_\pm) = QW(\boldsymbol{F}_\pm)$, then the function $\Delta(\boldsymbol{F}) = W(\boldsymbol{F}) - QW(\boldsymbol{F})$ attains its minimum value 0 at $\boldsymbol{F}_\pm$. Hence,

$$QW(\boldsymbol{F}_\pm) = W(\boldsymbol{F}_\pm), \qquad QW_{\boldsymbol{F}}(\boldsymbol{F}_\pm) = W_{\boldsymbol{F}}(\boldsymbol{F}_\pm). \tag{4.3}$$

Therefore, the inequality $\overline{W}(\boldsymbol{F}) \geq QW(\boldsymbol{F})$ for all $\boldsymbol{F} \in \mathfrak{O}$, together with (4.2) and (4.3) implies that $\overline{W}_{\boldsymbol{FF}}(\boldsymbol{F}_\pm) \geq QW_{\boldsymbol{FF}}(\boldsymbol{F}_\pm)$ in the sense of quadratic forms.[2] Recalling that acoustic tensors of quasiconvex functions are nonnegative semidefinite, we obtain the desired LH type condition.

Theorem 4.3 *If $\boldsymbol{F}_\pm$ satisfying (2.5) are stable, then the acoustic tensor of the laminate envelope $\overline{W}(\boldsymbol{F})$ must be nonnegative semidefinite at $\boldsymbol{F}_\pm$.*

The ensuing algebraic inequalities, representing the new necessary condition for stability of the states on the boundary of validity of the Weierstrass test, can be stated explicitly if $\boldsymbol{F}_\pm \in \mathfrak{J}$ is a non-degenerate pair in the sense of Definition 4.1. Then the acoustic tensor of $\overline{W}(\boldsymbol{F})$ is given by

$$\boldsymbol{A}_\pm^{\text{lam}}(\boldsymbol{m}) = \boldsymbol{A}_\pm(\boldsymbol{m}) - \frac{\boldsymbol{N}_\pm \boldsymbol{m} \otimes \boldsymbol{N}_\pm \boldsymbol{m}}{\boldsymbol{A}_\pm(\boldsymbol{n})\boldsymbol{a} \cdot \boldsymbol{a}}. \tag{4.4}$$

Formula (4.4) is a consequence of the formula for $\overline{W}_{\boldsymbol{FF}}$, derived in Appendix B. In the typical case when the acoustic tensor $\boldsymbol{A}(\boldsymbol{m}; \boldsymbol{F})$ of $W(\boldsymbol{F})$ is strictly positive definite (for all $|\boldsymbol{m}| = 1$) on the jump set[3] one can reduce the nonnegativity of $\boldsymbol{A}_\pm^{\text{lam}}(\boldsymbol{m})$ to a pair of scalar inequalities

$$\frac{\boldsymbol{A}_\pm(\boldsymbol{m})^{-1} \boldsymbol{N}_\pm \boldsymbol{m} \cdot \boldsymbol{N}_\pm \boldsymbol{m}}{\boldsymbol{A}_\pm(\boldsymbol{n})\boldsymbol{a} \cdot \boldsymbol{a}} \leq 1. \tag{4.5}$$

The new conditions (4.5) constitute the main result of this paper.

To show that inequalities (4.5) are stronger than (3.3) we observe that when $\boldsymbol{m} = \boldsymbol{n}$, the inequality (4.5) becomes equality and can be restated as the property that the function

$$f_\pm(\boldsymbol{m}) = \boldsymbol{A}_\pm(\boldsymbol{m})^{-1} \boldsymbol{N}_\pm \boldsymbol{m} \cdot \boldsymbol{N}_\pm \boldsymbol{m}$$

achieves its maximal value $\boldsymbol{A}_\pm(\boldsymbol{n})\boldsymbol{a} \cdot \boldsymbol{a}$ at $\boldsymbol{m} = \boldsymbol{n}$. It is easy to check that (2.5) imply $\nabla f_\pm(\boldsymbol{n}) = 0$. We also compute

$$\nabla\nabla f_\pm(\boldsymbol{n}) = -\mathbb{B}_\pm.$$

Thus, the condition of nonpositivity of the Hessian $\nabla\nabla f_\pm(\boldsymbol{n})$ is the same as (3.3). We conclude that (4.5), or equivalently,

$$\max_{|\boldsymbol{m}|=1} f_\pm(\boldsymbol{m}) = f_\pm(\boldsymbol{n}) \tag{4.6}$$

implies (3.3).

In the next section we present a detailed study of an example showing that the new conditions (4.6) can be strictly stronger than the known conditions (3.3).

[2] In this inequality $QW_{\boldsymbol{FF}}(\boldsymbol{F}_\pm)$ is understood as a limiting value of $QW_{\boldsymbol{FF}}(\boldsymbol{F})$, $\boldsymbol{F} \in \mathfrak{O}$.

[3] It holds in every nontrivial example of which we are aware.

5 An Example

To illustrate our general results we now consider a class of “Hadamard materials” [3, 7, 22, 24]

$$W(\boldsymbol{F}) = \frac{1}{2}|\boldsymbol{F}|^2 + h(d), \quad \boldsymbol{F} \in \{\boldsymbol{F} \in \mathbb{M} : \det \boldsymbol{F} > 0\},\ d = \det \boldsymbol{F}, \tag{5.1}$$

where $h(d)$ is smooth, but nonconvex. While the explicit expression for $QW(\boldsymbol{F})$ is apparently unknown, various LH type algebraic conditions of stability, discussed in this paper, can be computed explicitly. This will allow us to compare them at a quantitative level.

5.1 The Classical LH Condition

We compute

$$\boldsymbol{P}(\boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{F}) = \boldsymbol{F} + h'(d)\mathrm{cof}\boldsymbol{F} = \boldsymbol{F} + dh'(d)\boldsymbol{F}^{-T},$$

$$\mathsf{L}(\boldsymbol{F})\boldsymbol{H} = W_{\boldsymbol{FF}}(\boldsymbol{F})\boldsymbol{H} = \boldsymbol{H} + \left(h''(d) + \frac{h'(d)}{d}\right)\langle \mathrm{cof}\boldsymbol{F}, \boldsymbol{H}\rangle \mathrm{cof}\boldsymbol{F} - \frac{h'(d)}{d}(\mathrm{cof}\boldsymbol{F})\boldsymbol{H}^T\mathrm{cof}\boldsymbol{F},$$

$$\boldsymbol{A}(\boldsymbol{m}; \boldsymbol{F}) = \boldsymbol{I} + h''(d)\mathrm{cof}\boldsymbol{F}\boldsymbol{m} \otimes \mathrm{cof}\boldsymbol{F}\boldsymbol{m}.$$

Spinodal region, defined as the region of failure of the classical (single phase) LH condition is then

$$\mathfrak{S} = \left\{\boldsymbol{F} : h''(\det \boldsymbol{F})\mathrm{cof}\big(\boldsymbol{F}^T\boldsymbol{F}\big) < -\boldsymbol{I}\right\}. \tag{5.2}$$

It will be convenient to formulate the classical LH condition in terms of the maximal eigenvalue μ of $\hat{\boldsymbol{G}} = \mathrm{cof}(\boldsymbol{F}^T\boldsymbol{F})$ and $d = \det \boldsymbol{F}$:

$$h''(d)\mu + 1 \geq 0. \tag{5.3}$$

5.2 The Jump Set

We can now solve explicitly the system (2.5) for (5.1). We have $\boldsymbol{F}_+ = \boldsymbol{F}_- + \boldsymbol{a} \otimes \boldsymbol{n}$, so that

$$d_+ = d_- + \mathrm{cof}\boldsymbol{F}_-\boldsymbol{n} \cdot \boldsymbol{a}, \quad d_\pm = \det \boldsymbol{F}_\pm, \tag{5.4}$$

$$[\![\boldsymbol{P}]\!]\boldsymbol{n} = \boldsymbol{a} + \lambda\mathrm{cof}\boldsymbol{F}_-\boldsymbol{n}, \quad \lambda = [\![h']\!],$$

$$[\![\boldsymbol{P}]\!]^T\boldsymbol{a} = |\boldsymbol{a}|^2\boldsymbol{n} + \lambda\mathrm{cof}\boldsymbol{F}_-^T\boldsymbol{a}. \tag{5.5}$$

Substituting $\boldsymbol{a} = -\lambda\mathrm{cof}\boldsymbol{F}_-\boldsymbol{n}$ into the last equation in (2.5) we obtain, taking (5.5) into account, that $\boldsymbol{n}$ must be an eigenvector of $\hat{\boldsymbol{G}}_-$. Let μ_- denote the corresponding eigenvalue. Then $\mathrm{cof}\boldsymbol{F}_-\boldsymbol{n} \cdot \boldsymbol{a} = -\lambda\mu_-$, and hence, formula (5.4) for d_+ becomes $[\![d]\!] = -\lambda\mu_-$. Interchanging $+$ and $-$ we conclude that $\mu_- = \mu_+ = \mu$, i.e. $\boldsymbol{n}$ is a common eigenvector of both $\hat{\boldsymbol{G}}_+$ and $\hat{\boldsymbol{G}}_-$, corresponding to the same eigenvalue μ. Eliminating λ from

$$\lambda = [\![h']\!], \qquad [\![d]\!] = -\lambda\mu$$

we obtain

$$\frac{[\![h']\!]}{[\![d]\!]} = -\frac{1}{\mu} \tag{5.6}$$

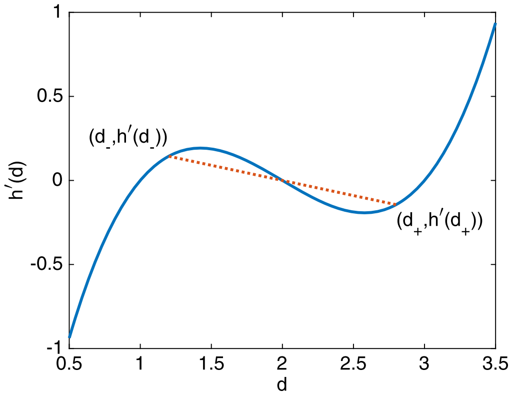

Fig. 1 Equal area property (5.7) defining the function $d_+ = D(d_-)$

We compute

$$p^* = -\frac{1}{2}[\![h']\!][\![d]\!] + [\![h]\!] - h'(d_-)[\![d]\!] = [\![h]\!] - \{\!\{h'\}\!\}[\![d]\!] = 0. \tag{5.7}$$

Traditionally this condition is interpreted geometrically as equality of the areas of the two regions between the secant line joining $(d_-, h'(d_-))$ and $(d_+, h'(d_+))$ and the graph of $h'(d)$. However, such geometric interpretation requires $h(d)$ to have the "double-well" shaped graph.

Definition 5.1 We will say that the function $h(d)$ has the double-well shape if the following 3 properties are satisfied.

(i) There exists $d_0 > 0$, such that $h'(d)$ is concave on $(0, d_0)$ and has a local maximum there,
(ii) $h'(d)$ is convex on $(d_0, +\infty)$ and has a local minimum there
(iii) $\lim_{d\to 0^+} h'(d) = -\infty$.

If $h(d)$ has double-well shape in the sense of Definition 5.1, then the equal area condition (5.7) defines a one-to-one correspondence $d_+ = D(d_-)$, see Fig. 1. In this case (5.6) represents the jump set as the surface in the $\boldsymbol{\mu}$-space of eigenvalues of $\hat{\boldsymbol{G}}$. We compute

$$\operatorname{cof}\boldsymbol{F}_+ = \frac{1}{d_-}\operatorname{cof}\boldsymbol{F}_-\big(d_-\boldsymbol{I} - \lambda\mu(\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n})\big).$$

Therefore,

$$\begin{aligned} \boldsymbol{a} &= -\lambda\operatorname{cof}\boldsymbol{F}_-\boldsymbol{n} = -\lambda\operatorname{cof}\boldsymbol{F}_+\boldsymbol{n}, \\ \hat{\boldsymbol{G}}_+ &= \frac{1}{d_-^2}\big(d_-\boldsymbol{I} - \lambda\mu(\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n})\big)\hat{\boldsymbol{G}}_-\big(d_-\boldsymbol{I} - \lambda\mu(\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n})\big). \end{aligned} \tag{5.8}$$

Hence, $\hat{\boldsymbol{G}}_+$ has the same eigenvectors as $\hat{\boldsymbol{G}}_-$. One of them is $\boldsymbol{n}$, corresponding to their common eigenvalue μ. The remaining eigenvalues $\mu_k^\pm$, $k = 2, \dots, n$ (corresponding to eigenvectors orthogonal to $\boldsymbol{n}$) are related by the formula

$$\frac{\mu_k^+}{d_+^2} = \frac{\mu_k^-}{d_-^2}, \quad k \ge 2. \tag{5.9}$$

5.3 Stability Conditions with One Phase Fixed

We now examine stability of the part of the jump set, satisfying

$$\boldsymbol{A}_{\pm}\boldsymbol{a}\cdot\boldsymbol{a}=\lambda^2\mu\big(1+h''(d_{\pm})\mu\big)>0,$$

so that (3.2) has the form (3.3). We compute

$$\mathbb{B}_{\pm}=\lambda^2\left(\mu\boldsymbol{I}-\left(1\pm\frac{\lambda\mu}{d_{\pm}}\right)^2\hat{\boldsymbol{G}}_{\pm}(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})\right).$$

Recalling that $\lambda\mu=-[\![d]\!]$, we obtain

$$\mathbb{B}_{\pm}=\lambda^2\left(\mu\boldsymbol{I}-\frac{d_{\mp}^2}{d_{\pm}^2}\hat{\boldsymbol{G}}_{\pm}(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})\right).$$

We see that $\mathbb{B}_{\pm}$ has the same eigenvectors as $\hat{\boldsymbol{G}}_{\pm}$. If $\boldsymbol{u}_k$ is an eigenvector of $\hat{\boldsymbol{G}}_{\pm}$ corresponding to an eigenvalue $\mu_k^{\pm}$, then, according to (5.9), the corresponding eigenvalue of $\mathbb{B}_{\pm}$ will be

$$\beta_{\pm}=\lambda^2\big(\mu-\mu_k^{\mp}\big).$$

We conclude that (3.3) is equivalent (under the assumption that $d_+>d_-$) to

$$\mu'_+\le\mu,\qquad \mu'_-\le\mu\frac{d_-^2}{d_+^2},\tag{5.10}$$

where $\mu'_{\pm}$ is the second largest eigenvalue of $\hat{\boldsymbol{G}}_{\pm}$.

5.4 Stability of Laminates

The double-well shape of $h(d)$, in the sense of Definition 5.1, implies that the tangent line at $d=d_+$ (respectively d_-) to the graph of $h'(d)$ lies below (respectively above) the graph of $h'(d)$ at $d=d_-$ (respectively d_+), provided $d_+>d_-$. Taking (5.6) into account, this statement is equivalent to $1+h''(d_{\pm})\mu>0$. It follows that the acoustic tensor $\boldsymbol{A}(\boldsymbol{m};\boldsymbol{F})$ is strictly positive definite for every unit vector $\boldsymbol{m}$ on the part of the jump set, satisfying (5.10). Therefore, the LH condition for laminates reduces to (4.6). We compute

$$f_{\pm}(\boldsymbol{m})=\boldsymbol{A}_{\pm}(\boldsymbol{m})^{-1}\boldsymbol{N}_{\pm}\boldsymbol{m}\cdot\boldsymbol{N}_{\pm}\boldsymbol{m}=\frac{\lambda^2}{|\boldsymbol{m}|^2}\left(\hat{\boldsymbol{G}}_{\pm}\boldsymbol{Q}_{\pm}\boldsymbol{m}\cdot\boldsymbol{Q}_{\pm}\boldsymbol{m}-\frac{h''(d_{\pm})(\hat{\boldsymbol{G}}_{\pm}\boldsymbol{Q}_{\pm}\boldsymbol{m}\cdot\boldsymbol{m})^2}{|\boldsymbol{m}|^2+h''(d_{\pm})\hat{\boldsymbol{G}}_{\pm}\boldsymbol{m}\cdot\boldsymbol{m}}\right),$$

where

$$\boldsymbol{Q}_{\pm}=\big(1+\mu h''(d_{\pm})\big)\boldsymbol{I}\pm\frac{\lambda\mu}{d_{\pm}}(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n}).$$

Let us write $\boldsymbol{m}=x\boldsymbol{n}+\boldsymbol{m}_{\perp}$, where $\boldsymbol{m}_{\perp}\perp\boldsymbol{n}$. Then

$$|\boldsymbol{m}|^2=x^2+|\boldsymbol{m}_{\perp}|^2,\qquad \boldsymbol{Q}_{\pm}\boldsymbol{m}=q_{\pm}x\boldsymbol{n}+\theta_{\pm}\boldsymbol{m}_{\perp},\qquad \hat{\boldsymbol{G}}_{\pm}\boldsymbol{Q}_{\pm}\boldsymbol{m}=q_{\pm}\mu x+\theta_{\pm}\hat{\boldsymbol{G}}_{\pm}\boldsymbol{m}_{\perp},$$

where

$$q_{\pm}=1+\mu h''(d_{\pm}),\qquad \theta_{\pm}=q_{\pm}\pm\frac{\lambda\mu}{d_{\pm}}=\mu h''(d_{\pm})+\frac{d_{\mp}}{d_{\pm}}.$$

Hence,

$$f_{\pm}(\boldsymbol{m}) = \frac{\lambda^2}{1+t}\left(q_{\pm}^2\mu t + \theta_{\pm}^2\nu_{\pm} - \frac{h''(d_{\pm})(q_{\pm}\mu t + \theta_{\pm}\nu_{\pm})^2}{1+t+h''(d_{\pm})(\mu t + \nu_{\pm})}\right),$$

where

$$t = \frac{x^2}{|\boldsymbol{m}_{\perp}|^2}, \qquad \nu_{\pm} = \nu_{\pm}(\boldsymbol{m}_{\perp}) = \frac{\hat{\boldsymbol{G}}_{\pm}\boldsymbol{m}_{\perp}\cdot\boldsymbol{m}_{\perp}}{|\boldsymbol{m}_{\perp}|^2}.$$

Using a symbolic algebra program we compute

$$\gamma_{\pm}(\boldsymbol{m}) = f_{\pm}(\boldsymbol{m}) - f_{\pm}(\boldsymbol{n}) = \lambda^2\frac{q_{\pm}t[\nu_{\pm}(1\pm\frac{\lambda\mu}{d_{\pm}})^2-\mu] + \nu_{\pm}(\theta_{\pm}^2 - q_{\pm}\mu h''(d_{\pm})) - q_{\pm}\mu}{(1+t)(1+t+h''(d_{\pm})(\mu t+\nu_{\pm}))}.$$

Observe that $\nu_{\pm}$ is a positive number that does not exceed the second largest eigenvalue $\mu'_{\pm}$ of $\hat{\boldsymbol{G}}_{\pm}$. Hence, according to (5.10),

$$\nu_{\pm}\left(1\pm\frac{\lambda\mu}{d_{\pm}}\right)^2 - \mu = \nu_{\pm}\frac{d_{\mp}^2}{d_{\pm}^2} - \mu \le \mu'_{\mp} - \mu \le 0.$$

It follows that the numerator of $\gamma_{\pm}(\boldsymbol{m})$ is a decreasing function of t on $[0,+\infty)$. Observing that the classical LH condition guarantees that the denominator of $\gamma_{\pm}(\boldsymbol{m})$ is both positive and increasing on $t\in[0,+\infty)$, we conclude that $\gamma_{\pm}(\boldsymbol{m})$ is a decreasing function of t. Hence, it will be nonpositive for all $t\ge 0$ if and only if $\gamma_{\pm}(\boldsymbol{m}_{\perp})\le 0$, which is equivalent to

$$\mu'_{\pm}\big(\theta_{\pm}^2 - q_{\pm}\mu h''(d_{\pm})\big) - q_{\pm}\mu \le 0,$$

where $\mu'_{\pm}$ is the second largest eigenvalue of $\hat{\boldsymbol{G}}_{\pm}$. Returning to the original notations we rewrite this inequality as follows

$$\frac{1}{\mu'_{\pm}} \ge \frac{(h''(d_{\pm}) + \mu^{-1}d_{\mp}/d_{\pm})^2}{\mu^{-1}+h''(d_{\pm})} - h''(d_{\pm}). \tag{5.11}$$

Taking relation (5.9) into account we conclude that in our example the new LH-type condition (4.6) is equivalent to

$$\frac{d_{\pm}^2}{\mu'_{\pm}} \ge \max\left\{\frac{(h''(d_+)d_+ + \mu^{-1}d_-)^2}{\mu^{-1}+h''(d_+)} - d_+^2h''(d_+), \frac{(h''(d_-)d_- + \mu^{-1}d_+)^2}{\mu^{-1}+h''(d_-)} - d_-^2h''(d_-)\right\}. \tag{5.12}$$

5.5 Quartic Energy

The foregoing analysis shows that energy (5.1) results in essentially two-dimensional problem, i.e., the constraints on the jump set refer only to the largest eigenvalue μ (and its eigenvector $\boldsymbol{n}$) and the second largest eigenvalue $\mu'_{\pm}$ of $\hat{\boldsymbol{G}}_{\pm}$. Hence, without loss of generality, we assume $n=2$. In order to visualize various stability condition discussed in this paper we take

$$h(d) = \frac{1}{4d_0}\big((d_0-d)^2-1\big)^2, \quad d_0 > 1.$$

The coefficient $1/4d_0$ is chosen because it simplifies calculations and results in more transparent formulas. Obviously, the convexification of $h(d)$ agrees with $h(d)$ whenever

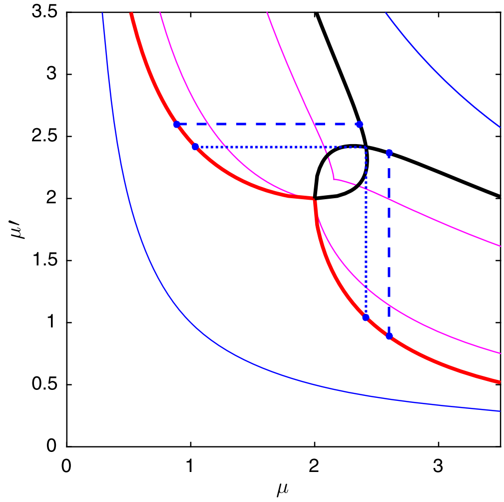

Fig. 2 Jump set for $h(d) = ((d-2)^2 - 1)^2/8$ and various algebraic stability bounds (Color figure online)

$d \notin (d^*_-, d^*_+)$, where $d^*_\pm = d_0 \pm 1$. The curves $d = d^*_\pm$ are the outer thin solid lines shown in blue in Fig. 2. We compute

$$h'(d) = \frac{1}{d_0}\big((d-d_0)^3 - (d-d_0)\big), \qquad h''(d) = \frac{1}{d_0}\big(3(d-d_0)^2 - 1\big), \qquad D(d) = 2d_0 - d.$$

Let μ and μ' be the larger and the smaller eigenvalues of $\hat{\boldsymbol{G}} = \mathrm{cof}(\boldsymbol{F}^T\boldsymbol{F})$, respectively. Since $n = 2$, μ and μ' are also squares of singular values of $\boldsymbol{F}$. Hence, $d = \sqrt{\mu\mu'}$. Thus, the LH condition (5.3) becomes

$$\frac{\mu}{d_0}\big(3\big(\sqrt{\mu\mu'} - d_0\big)^2 - 1\big) + 1 \geq 0. \tag{5.13}$$

We can solve this inequality for μ' explicitly. If $\mu \in (0, d_0)$, then (5.13) will be satisfied for all $0 < \mu' \leq \mu$. If $\mu \geq d_0$, then either

$$\mu' \leq \frac{1}{\mu}\left(d_0 - \frac{\sqrt{1 - d_0/\mu}}{\sqrt{3}}\right)^2, \quad \mu \geq d_0 \tag{5.14}$$

or

$$\mu \geq \mu' \geq \frac{1}{\mu}\left(d_0 + \frac{\sqrt{1 - d_0/\mu}}{\sqrt{3}}\right)^2, \qquad \mu \geq \frac{d_0}{2} + \sqrt{\left(\frac{d_0}{2}\right)^2 + \frac{1}{3}}. \tag{5.15}$$

The curves (5.14), (5.15) bounding the region of failure of the classical LH condition (elastic spinodal) are the inner thin solid lines shown in magenta in Fig. 2.

Using our formulas for $h'(d)$ and $D(d)$ we obtain simple equations for the jump set

$$-\frac{d_0}{\mu_j} = (\sqrt{\mu_1\mu_2} - d_0)^2 - 1, \quad j = 1, 2.$$

The union of the curves $j = 1$ and $j = 2$ constitutes the jump set, shown in Fig. 2 as bold solid lines. They differ by the interchange of μ_1 and μ_2. Therefore, it suffices to study stability of the part of the jump set, given by one of them, which we can write, without loss of generality, as

$$-\frac{d_0}{\mu} = \left(\sqrt{\mu\mu'} - d_0\right)^2 - 1.$$

Solving for μ' we obtain

$$\mu' = \mu'_{\pm} = \frac{1}{\mu}\left(d_0 \pm \sqrt{1 - \frac{d_0}{\mu}}\right)^2, \quad \mu \geq d_0. \tag{5.16}$$

The part of the jump set corresponding to the plus sign in (5.16) is shown in red and lies both below the part corresponding to the minus sign (shown in black) and below the lower part of the elastic spinodal (shown in magenta). Inequality (5.10) is equivalent to the constraint

$$\mu \geq \mu_0 = \frac{d_0}{2} + \sqrt{\left(\frac{d_0}{2}\right)^2 + 1} \tag{5.17}$$

for the equation in (5.16), meaning that the parts of the curves (5.16) corresponding to $\mu \in [d_0, \mu_0)$ are unstable. The corresponding stability limits are is indicated in Fig. 2 by the vertical and horizontal dotted lines, delimiting the sector corresponding to the unstable part of the phase space.

Finally, writing (5.16) of the jump set as

$$d_{\pm} = d_0 \pm \sqrt{1 - \frac{d_0}{\mu}}, \tag{5.18}$$

we compute

$$\mu h''(d_{\pm}) = \frac{2\mu}{d_0} - 3.$$

Hence, (5.11) can be written as

$$\frac{\mu}{\mu'_{\pm}} \geq 2\left(\sqrt{\frac{\mu}{d_0} - 1} \mp \frac{1}{\sqrt{d_0\mu} \pm \sqrt{\frac{\mu}{d_0} - 1}}\right)^2 - \frac{2\mu}{d_0} + 3.$$

Only the "bottom" sign choice gives us new information for $\mu \in [\mu_0, 3d_0/2)$:

$$\mu'_{-} \leq \mu\left(2\left(\sqrt{\frac{\mu}{d_0} - 1} + \frac{1}{\sqrt{d_0\mu} - \sqrt{\frac{\mu}{d_0} - 1}}\right)^2 - \frac{2\mu}{d_0} + 3\right)^{-1}, \quad \mu \in \left[\mu_0, \frac{3d_0}{2}\right). \tag{5.19}$$

We remark that $\mu_0 < 3d_0/2$ only when $d_0 > 2/\sqrt{3}$. The boundaries of validity of inequalities (5.12) are shown in Fig. 2 by dashed vertical and horizontal lines.

6 Conclusions

The main idea behind our general approach is to probe quasi-convexity by considering approximate minimizers of a certain type. At the first step, detailed in this paper, we select a pair of gradients $\boldsymbol{F}_{\pm}$ satisfying (2.5) and construct the laminate envelope $\overline{W}(\boldsymbol{F})$. Then we use a quadratic expansion around the selected gradients to obtain the generalized LH conditions (4.5). At the next step we would need to search for $\overline{\boldsymbol{F}}_{+} = \boldsymbol{F}_3$ satisfying (2.5), where $\overline{\boldsymbol{F}}_{-} = t\boldsymbol{F}_{+} + (1-t)\boldsymbol{F}_{-}$, and $\boldsymbol{F}_{\pm} \in \mathfrak{J}$ achieve equality in (4.5) for some unit vector $\boldsymbol{m} \neq \pm\boldsymbol{n}$. At the same time $W(\boldsymbol{F})$ and its derivatives in (2.5) are replaced by $\overline{W}(\boldsymbol{F})$, using formulas (4.2). This gives us the secondary laminate envelope $\overline{\overline{W}}(\boldsymbol{F})$ and we would need to use again a quadratic expansion around the selected triple of gradients to obtain new generalized LH conditions. The process can be continued utilizing local information about $W(\boldsymbol{F})$ at the set of gradients $\boldsymbol{F}_1, \dots, \boldsymbol{F}_N$. It will result in progressively tighter bounds on the quasiconvex envelope, see also [12]. In this procedure, the selection of the gradients $\boldsymbol{F}_1, \dots, \boldsymbol{F}_N$ is a global step while probing their infinitesimal stability is a local step. By approximating the energy in this way, we can obtain a set of algebraic bounds for strong local minima that can be interpreted as generalized LH conditions. In this paper we limited ourselves only to the first step of this iterative scheme involving only two gradients $\boldsymbol{F}_1, \boldsymbol{F}_2$ and demonstrated in detail how that approach can generate new explicit bounds.

Acknowledgements This material is based upon work supported by the National Science Foundation under Grant No. DMS-1412058. LT was also supported by the French ANR contract EVOCRIT.

Appendix A: Geometry of the Jump Set

A.1 Normal to the Jump Set

Suppose $\boldsymbol{F}_{-}(s) \in \mathfrak{J}_{-}$ is a smooth curve passing through $\boldsymbol{F}_{-}$ at $s = 0$. We would like to show by the implicit function theorem that there are smooth curves $\boldsymbol{a}(s)$, $\boldsymbol{n}(s)$ passing through $\boldsymbol{a}$ and $\boldsymbol{n}$ at $s = 0$ and satisfying

$$\begin{cases} (W_{\boldsymbol{F}}(\boldsymbol{F}_{-}(s) + \boldsymbol{a}(s) \otimes \boldsymbol{n}(s)) - W_{\boldsymbol{F}}(\boldsymbol{F}_{-}(s)))\boldsymbol{n}(s) = \boldsymbol{0}, \\ (W_{\boldsymbol{F}}(\boldsymbol{F}_{-}(s) + \boldsymbol{a}(s) \otimes \boldsymbol{n}(s)) - W_{\boldsymbol{F}}(\boldsymbol{F}_{-}))^T \boldsymbol{a}(s) = \boldsymbol{0}, \\ W(\boldsymbol{F}_{-}(s) + \boldsymbol{a}(s) \otimes \boldsymbol{n}(s)) - W(\boldsymbol{F}_{-}(s)) - W_{\boldsymbol{F}}(\boldsymbol{F}_{-}(s))\boldsymbol{n}(s) \cdot \boldsymbol{a}(s) = 0. \end{cases} \tag{A.1}$$

Differentiating (A.1) with respect to s at $s = 0$ we obtain

$$\begin{cases} \boldsymbol{A}_{+}\dot{\boldsymbol{a}} + (\boldsymbol{B}_{+} + [\![\boldsymbol{P}]\!])\dot{\boldsymbol{n}} + ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})\boldsymbol{n} = \boldsymbol{0}, \\ (\boldsymbol{B}_{+} + [\![\boldsymbol{P}]\!])^T \dot{\boldsymbol{a}} + \boldsymbol{A}_{+}^{*}\dot{\boldsymbol{n}} + ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})^T \boldsymbol{a} = \boldsymbol{0}, \\ -\frac{1}{2}(\boldsymbol{A}_{+}\boldsymbol{a}, \dot{\boldsymbol{a}}) - \frac{1}{2}(\boldsymbol{A}_{+}^{*}\boldsymbol{n}, \dot{\boldsymbol{n}}) + ([\![\boldsymbol{P}]\!] - \{\!\{\mathsf{L}\}\!\}[\![\boldsymbol{F}]\!], \dot{\boldsymbol{F}}_{-}) = 0. \end{cases} \tag{A.2}$$

We now solve the first two equations in (A.2) for $\dot{\boldsymbol{a}}$, $\dot{\boldsymbol{n}}$. We obtain the system

$$\mathbb{C}_{+} \begin{bmatrix} \dot{\boldsymbol{a}} \\ \dot{\boldsymbol{n}} \end{bmatrix} = - \begin{bmatrix} ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})\boldsymbol{n} \\ ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})^T \boldsymbol{a} \end{bmatrix}, \tag{A.3}$$

where $\mathbb{C}_{\pm}$ is given by (3.1). Observe that

$$\begin{bmatrix} ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}})\boldsymbol{n} \\ ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})^T \boldsymbol{a} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{a} \\ -\boldsymbol{n} \end{bmatrix} = 0.$$

By the non-degeneracy assumption the right-hand side of (A.3) is orthogonal to the null-space of $\mathbb{C}_{+}$. Thus, (A.3) has a solution. This solution is determined uniquely by the property

$\dot{\boldsymbol{n}}\cdot\boldsymbol{n}=0$, since the vector $[\mathbf{0},\boldsymbol{n}]$ does not belong to $(\mathbb{R}[\boldsymbol{a},-\boldsymbol{n}])^{\perp}$. Thus, there is a well defined operator denoted $\mathbb{C}_{+}^{-1}:(\mathbb{R}[\boldsymbol{a},-\boldsymbol{n}])^{\perp}\to(\mathbb{R}[\mathbf{0},\boldsymbol{n}])^{\perp}$ with the property $\mathbb{C}_{+}(\boldsymbol{a},\boldsymbol{n})(\mathbb{C}_{+}^{-1}\boldsymbol{z})=\boldsymbol{z}$ for any $\boldsymbol{z}\in(\mathbb{R}[\boldsymbol{a},-\boldsymbol{n}])^{\perp}$. We can write

$$\begin{bmatrix}\dot{\boldsymbol{a}}\\ \dot{\boldsymbol{n}}\end{bmatrix}=-\mathbb{C}_{+}^{-1}\begin{bmatrix}([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})\boldsymbol{n}\\ ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})^{T}\boldsymbol{a}\end{bmatrix}.$$

Substituting this into the third equation in (A.2) we obtain

$$\frac{1}{2}\begin{bmatrix}\boldsymbol{A}_{+}(\boldsymbol{n})\boldsymbol{a}\\ \boldsymbol{A}_{+}^{*}(\boldsymbol{a})\boldsymbol{n}\end{bmatrix}\cdot\mathbb{C}_{+}^{-1}\begin{bmatrix}([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})\boldsymbol{n}\\ ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})^{T}\boldsymbol{a}\end{bmatrix}+\big([\![\boldsymbol{P}]\!]-\{\!\{\mathsf{L}\}\!\}[\![\boldsymbol{F}]\!],\dot{\boldsymbol{F}}_{-}\big)=0.\qquad\text{(A.4)}$$

Observe that $\mathbb{C}_{+}$ is a symmetric matrix and that

$$\begin{bmatrix}\boldsymbol{A}_{+}\boldsymbol{a}\\ \boldsymbol{A}_{+}^{*}\boldsymbol{n}\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{a}\\ -\boldsymbol{n}\end{bmatrix}=0.$$

Therefore,

$$\begin{bmatrix}\boldsymbol{A}_{+}\boldsymbol{a}\\ \boldsymbol{A}_{+}^{*}\boldsymbol{n}\end{bmatrix}\cdot\mathbb{C}_{+}^{-1}\begin{bmatrix}([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})\boldsymbol{n}\\ ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})^{T}\boldsymbol{a}\end{bmatrix}=\mathbb{C}_{+}^{-1}\begin{bmatrix}\boldsymbol{A}_{+}\boldsymbol{a}\\ \boldsymbol{A}_{+}^{*}\boldsymbol{n}\end{bmatrix}\cdot\begin{bmatrix}([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})\boldsymbol{n}\\ ([\![\mathsf{L}]\!]\dot{\boldsymbol{F}}_{-})^{T}\boldsymbol{a}\end{bmatrix}.$$

We easily compute

$$\mathbb{C}_{+}^{-1}\begin{bmatrix}\boldsymbol{A}_{+}\boldsymbol{a}\\ \boldsymbol{A}_{+}^{*}\boldsymbol{n}\end{bmatrix}=\begin{bmatrix}\boldsymbol{a}\\ \mathbf{0}\end{bmatrix}.$$

Hence, (A.4) becomes

$$\left(\frac{1}{2}[\![\mathsf{L}]\!][\![\boldsymbol{F}]\!]+[\![\boldsymbol{P}]\!]-\{\!\{\mathsf{L}\}\!\}[\![\boldsymbol{F}]\!],\dot{\boldsymbol{F}}_{-}\right)=0.$$

In other words, $(\boldsymbol{N}_{-},\dot{\boldsymbol{F}}_{-})=0$, where

$$\boldsymbol{N}_{-}=\mathsf{L}_{-}[\![\boldsymbol{F}]\!]-[\![\boldsymbol{P}]\!].\qquad\text{(A.5)}$$

A.2 Foliation of the Simple Laminate Region

Now, suppose that $\boldsymbol{F}_{-}$ varies over an open subset G of $\mathfrak{J}_{-}$ that is sufficiently small, so that due to the non-degeneracy assumption, the functions $\boldsymbol{a}(\boldsymbol{F}_{-})$ and $\boldsymbol{n}(\boldsymbol{F}_{-})$ are well-defined and smooth on G. Let us show that the line segments joining $\boldsymbol{F}_{-}$ and $\boldsymbol{F}_{+}=\boldsymbol{F}_{-}+\boldsymbol{a}(\boldsymbol{F}_{-})\otimes\boldsymbol{n}(\boldsymbol{F}_{-})$ foliate an open subset of $\mathbb{M}$, as $\boldsymbol{F}_{-}$ ranges over G. We have a map $\boldsymbol{F}:G\times[0,1]\to\mathbb{M}$ given by

$$\boldsymbol{F}=\boldsymbol{F}_{-}+t\boldsymbol{a}(\boldsymbol{F}_{-})\otimes\boldsymbol{n}(\boldsymbol{F}_{-}).\qquad\text{(A.6)}$$

If G is sufficiently small then the only possibility that the map $\boldsymbol{F}$ is not injective is for $\boldsymbol{F}$ to be locally non-injective, i.e., for $d\boldsymbol{F}(\boldsymbol{F}_{-},t)$ to be singular for some $t\in[0,1]$. If this is the case, then there exists $\dot{\boldsymbol{F}}_{-}\in T_{\boldsymbol{F}_{-}}\mathfrak{J}_{-}$ and $t\in[0,1]$ such that

$$\dot{\boldsymbol{F}}_{-}+t(\dot{\boldsymbol{a}}\otimes\boldsymbol{n}+\boldsymbol{a}\otimes\dot{\boldsymbol{n}})=\boldsymbol{a}\otimes\boldsymbol{n},$$

where $[\dot{\boldsymbol{a}}, \dot{\boldsymbol{n}}]$ solves (A.3). If (A.6) is satisfied then there exist $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{\eta} \in \mathbb{R}^d$ such that

$$\dot{\boldsymbol{F}}_- = \boldsymbol{u} \otimes \boldsymbol{n} + \boldsymbol{a} \otimes \boldsymbol{\eta}, \qquad \boldsymbol{u} + t\dot{\boldsymbol{a}} = \lambda \boldsymbol{a}, \qquad \boldsymbol{\eta} + t\dot{\boldsymbol{n}} = (1-\lambda)\boldsymbol{n} \tag{A.7}$$

for some $\lambda \in \mathbb{R}$. Substituting (A.7) into (A.3) we obtain

$$\widetilde{\mathbb{C}}_t \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_+(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_+^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix}, \quad \widetilde{\mathbb{C}}_t = (1-t)\mathbb{C}_+ + t\mathbb{C}_-.$$

The solution $[\boldsymbol{u}, \boldsymbol{\eta}]$ is determined up to a multiple of $[\boldsymbol{a}, -\boldsymbol{n}]$. However, any choice of solution $[\boldsymbol{u}, \boldsymbol{\eta}]$ gives one and the same value $\dot{\boldsymbol{F}}_-$ in (A.7). We therefore, write

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\eta} \end{bmatrix} = \widetilde{\mathbb{C}}_t^{-1} \begin{bmatrix} \boldsymbol{A}_+(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_+^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix},$$

where the inverse of the positive semidefinite matrix $\widetilde{\mathbb{C}}_t$ is computed on $(\mathbb{R}[\boldsymbol{a}, -\boldsymbol{n}])^\perp$, where it is positive definite by assumption of local stability. The resulting $\dot{\boldsymbol{F}}_-$ has to belong to $T_{\boldsymbol{F}_-}\mathfrak{J}_-$, therefore $(\dot{\boldsymbol{F}}_-, \boldsymbol{N}_-) = 0$, i.e.,

$$q(t) = \left(\widetilde{\mathbb{C}}_t^{-1} \begin{bmatrix} \boldsymbol{A}_+(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_+^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix}, \begin{bmatrix} \boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_-^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix} \right) = 0. \tag{A.8}$$

Let us show that (A.8) is impossible. We observe that

$$q(0) = \big(\boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a}, \boldsymbol{a}\big) > 0, \qquad q(1) = \big(\boldsymbol{A}_+(\boldsymbol{n})\boldsymbol{a}, \boldsymbol{a}\big) > 0.$$

For any $t \in (0, 1)$ we have

$$\widetilde{\mathbb{C}}_t > t\mathbb{C}_- > 0,$$

where the inequality is strict on $(\mathbb{R}[\boldsymbol{a}, -\boldsymbol{n}])^\perp$. Therefore,

$$t\widetilde{\mathbb{C}}_t^{-1} < \mathbb{C}_-^{-1}. \tag{A.9}$$

We have

$$\widetilde{\mathbb{C}}_t \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{n} \end{bmatrix} = 2(1-t) \begin{bmatrix} \boldsymbol{A}_+(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_+^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix} + 2t \begin{bmatrix} \boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_-^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{n} \end{bmatrix} - \lambda \begin{bmatrix} \boldsymbol{a} \\ -\boldsymbol{n} \end{bmatrix} = 2(1-t)\widetilde{\mathbb{C}}_t^{-1} \begin{bmatrix} \boldsymbol{A}_+(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_+^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix} + 2t\widetilde{\mathbb{C}}_t^{-1} \begin{bmatrix} \boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_-^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix},$$

where $\lambda \in \mathbb{R}$ is chosen such that the left-hand side is in $(\mathbb{R}[\boldsymbol{a}, -\boldsymbol{n}])^\perp$. Taking inner product with $[\boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a}, \boldsymbol{A}_-^*(\boldsymbol{a})\boldsymbol{n}]$ we obtain

$$\big(\boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a}, \boldsymbol{a}\big) = (1-t)q(t) + t \left(\widetilde{\mathbb{C}}_t^{-1} \begin{bmatrix} \boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_-^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix}, \begin{bmatrix} \boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_-^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix} \right).$$

By (A.9) we have

$$\big(\boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a}, \boldsymbol{a}\big) < (1-t)q(t) + \left(\mathbb{C}_-^{-1} \begin{bmatrix} \boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_-^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix}, \begin{bmatrix} \boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a} \\ \boldsymbol{A}_-^*(\boldsymbol{a})\boldsymbol{n} \end{bmatrix} \right) = (1-t)q(t) + \big(\boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a}, \boldsymbol{a}\big).$$

Thus, $q(t) > 0$ for all $t \in [0, 1]$. It follows that $\mathcal{G} = \boldsymbol{F}(G \times (0, 1))$ is an open subset of $\mathbb{M}$.

Appendix B: Formula for the Second Derivative of $\overline{W}(\boldsymbol{F})$

Let $\boldsymbol{F}_0 \in \mathfrak{O}$ and let $\boldsymbol{F}(s)$ be any curve in $\mathfrak{O}$ passing through $\boldsymbol{F}_0$ when $s = 0$. Then, for sufficiently small s there exist unique $\boldsymbol{F}_-(s) \in \mathfrak{J}$, $\boldsymbol{a}(s) \otimes \boldsymbol{n}(s)$ and $t(s) \in (0, 1)$, such that $\boldsymbol{F}(s) = t(s)\boldsymbol{F}_+(s) + (1 - t(s))\boldsymbol{F}_-(s)$ and (2.5) holds, where $\boldsymbol{F}_+(s) = \boldsymbol{F}_-(s) + \boldsymbol{a}(s) \otimes \boldsymbol{n}(s) \in \mathfrak{J}$. Let $\boldsymbol{\xi} = \dot{\boldsymbol{F}}(0)$. In the calculations below dot over a symbol denotes derivative in s at $s = 0$, while a symbol without an argument s refers to $s = 0$. For example, $\boldsymbol{a}$ denotes $\boldsymbol{a}(0)$.

Differentiating (4.1) at $s = 0$ we obtain

$$\big\langle \overline{W}_{\boldsymbol{F}}(\boldsymbol{F}_0), \boldsymbol{\xi}\big\rangle = t\langle \boldsymbol{P}_+, \dot{\boldsymbol{F}}_+\rangle + (1-t)\langle \boldsymbol{P}_-, \dot{\boldsymbol{F}}_-\rangle - \dot{t}[\![W]\!]. \tag{B.1}$$

To compute $\dot{\boldsymbol{F}}_\pm$ we need to express $\boldsymbol{F}_\pm(s)$ in terms of $\boldsymbol{F}(s)$ and $\boldsymbol{a}(s) \otimes \boldsymbol{n}(s)$:

$$\boldsymbol{F}_-(s) = \boldsymbol{F}(s) - t(s)\boldsymbol{a}(s) \otimes \boldsymbol{n}(s), \qquad \boldsymbol{F}_+(s) = \boldsymbol{F}(s) + \big(1 - t(s)\big)\boldsymbol{a}(s) \otimes \boldsymbol{n}(s). \tag{B.2}$$

Differentiating in s at $s = 0$ we get

$$\dot{\boldsymbol{F}}_+ = \boldsymbol{\xi} - \dot{t}[\![\boldsymbol{F}]\!] + (1-t)[\![\dot{\boldsymbol{F}}]\!], \qquad \dot{\boldsymbol{F}}_- = \boldsymbol{\xi} - \dot{t}[\![\boldsymbol{F}]\!] - t[\![\dot{\boldsymbol{F}}]\!], \tag{B.3}$$

where $[\![\dot{\boldsymbol{F}}]\!]$ is just a compact form of

$$[\![\dot{\boldsymbol{F}}]\!] = \boldsymbol{a} \otimes \dot{\boldsymbol{n}} + \dot{\boldsymbol{a}} \otimes \boldsymbol{n}. \tag{B.4}$$

Substituting (B.3) into (B.1) we obtain

$$\begin{aligned}\big\langle \overline{W}_{\boldsymbol{F}}(\boldsymbol{F}_0), \boldsymbol{\xi}\big\rangle &= \big\langle t\boldsymbol{P}_+ + (1-t)\boldsymbol{P}_-, \boldsymbol{\xi}\big\rangle + \dot{t}\big\{[\![W]\!] - \big\langle t\boldsymbol{P}_+ + (1-t)\boldsymbol{P}_-, [\![\boldsymbol{F}]\!]\big\rangle\big\} \\ &\quad + t(1-t)\big\langle [\![\boldsymbol{P}]\!], [\![\dot{\boldsymbol{F}}]\!]\big\rangle.\end{aligned}$$

Using (2.5) we conclude that

$$\overline{W}_{\boldsymbol{F}}\big(\boldsymbol{F}(s)\big) = t(s)W_{\boldsymbol{F}}\big(\boldsymbol{F}_+(s)\big) + \big(1 - t(s)\big)W_{\boldsymbol{F}}\big(\boldsymbol{F}_-(s)\big). \tag{B.5}$$

Differentiating (B.5) at $s = 0$ and using (B.3) we see that

$$\overline{W}_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{F}_0)\boldsymbol{\xi} = \mathsf{L}_t\boldsymbol{\xi} - \dot{t}\boldsymbol{N}_t + t(1-t)[\![\mathsf{L}]\!][\![\dot{\boldsymbol{F}}]\!], \tag{B.6}$$

where we use the shorthand $X_t = tX_+ + (1-t)X_-$.

In order to express $\dot{\boldsymbol{a}}$, $\dot{\boldsymbol{n}}$ and $\dot{t}$ in terms of $\boldsymbol{\xi}$ we differentiate the last two equations in (2.5) at $s = 0$ and obtain

$$\widetilde{\mathbb{C}}_t \begin{bmatrix} \dot{\boldsymbol{a}} \\ \dot{\boldsymbol{n}} \end{bmatrix} - \dot{t} \begin{bmatrix} [\![\boldsymbol{A}]\!]\boldsymbol{a} \\ [\![\boldsymbol{A}^*]\!]\boldsymbol{n} \end{bmatrix} = -\begin{bmatrix} ([\![\mathsf{L}]\!]\boldsymbol{\xi})\boldsymbol{n} \\ ([\![\mathsf{L}]\!]\boldsymbol{\xi})^T\boldsymbol{a} \end{bmatrix}. \tag{B.7}$$

Differentiating the Maxwell relation we obtain $\langle \boldsymbol{N}_\pm, \dot{\boldsymbol{F}}_\pm\rangle = 0$ (which is expressing the fact that $\boldsymbol{N}_\pm$ are orthogonal to the jump set at $\boldsymbol{F}_\pm$). Using (B.3) to eliminate $\dot{\boldsymbol{F}}_\pm$ we obtain

$$\dot{t}\boldsymbol{A}_-(\boldsymbol{n})\boldsymbol{a} \cdot \boldsymbol{a} = \langle \boldsymbol{N}_-, \boldsymbol{\xi}\rangle - t\big\langle \boldsymbol{N}_-, [\![\dot{\boldsymbol{F}}]\!]\big\rangle, \qquad \dot{t}\boldsymbol{A}_+(\boldsymbol{n})\boldsymbol{a} \cdot \boldsymbol{a} = \langle \boldsymbol{N}_+, \boldsymbol{\xi}\rangle + (1-t)\big\langle \boldsymbol{N}_+, [\![\dot{\boldsymbol{F}}]\!]\big\rangle.$$

We combine the two relations into a more symmetric form by multiplying the first equation by $1 - t$, the second one by t and adding:

$$\dot{t} = \frac{\langle \boldsymbol{N}_t, \boldsymbol{\xi}\rangle + t(1-t)\langle [\![\boldsymbol{N}]\!], [\![\dot{\boldsymbol{F}}]\!]\rangle}{\boldsymbol{A}_t\boldsymbol{a} \cdot \boldsymbol{a}}.$$

Substituting this formula into (B.6) and (B.7) we obtain

$$\langle \overline{W}_{FF}\xi, \xi\rangle = \langle \mathsf{L}_t\xi, \xi\rangle - \frac{\langle N_t, \xi\rangle^2}{A_t a\cdot a} - t(1-t)\left(\langle [\![\mathsf{L}]\!]\xi, [\![\dot{F}]\!]\rangle - \frac{\langle N_t, \xi\rangle\langle [\![N]\!], [\![\dot{F}]\!]\rangle}{A_t a\cdot a}\right), \tag{B.8}$$

and

$$\mathbb{D}(t)\begin{bmatrix}\dot{a}\\ \dot{n}\end{bmatrix} = -\begin{bmatrix}([\![\mathsf{L}]\!]\xi)n\\ ([\![\mathsf{L}]\!]\xi)^T a\end{bmatrix} + \frac{\langle N_t, \xi\rangle}{A_t a\cdot a}\begin{bmatrix}[\![A]\!]a\\ [\![A^*]\!]n\end{bmatrix}, \tag{B.9}$$

respectively, where

$$\mathbb{D}(t) = \widetilde{\mathbb{C}}_t - \frac{t(1-t)}{A_t a\cdot a}\begin{bmatrix}[\![A]\!]a\\ [\![A^*]\!]n\end{bmatrix}\otimes\begin{bmatrix}[\![A]\!]a\\ [\![A^*]\!]n\end{bmatrix}. \tag{B.10}$$

Hence,

$$\left\langle \overline{W}_{FF}(F_0)\xi, \xi\right\rangle = \langle \mathsf{L}_t\xi, \xi\rangle - \frac{\langle N_t, \xi\rangle^2}{A_t a\cdot a} + t(1-t)\mathbb{D}(t)^{-1}Z(t)\cdot Z(t), \tag{B.11}$$

where

$$Z(t) = -\begin{bmatrix}([\![\mathsf{L}]\!]\xi)n\\ ([\![\mathsf{L}]\!]\xi)^T a\end{bmatrix} + \frac{\langle N_t, \xi\rangle}{A_t a\cdot a}\begin{bmatrix}[\![A]\!]a\\ [\![A^*]\!]n\end{bmatrix}.$$

In the last term of (B.11) $\mathbb{D}(t)^{-1}Z(t)$ is understood in the orthogonal complement to $[a, -n]$. This is justified by the following lemma.

Lemma B.1 *Assume that $F_\pm$ is a non-degenerate pair in the sense of Definition 4.1. Assume that (3.2) holds. Then $\mathbb{D}(t)$ is a positive-definite self-adjoint operator on $V = (\mathbb{R}[a, -n])^\perp$.*

Proof It is obvious that $\mathbb{D}(t)$ is self-adjoint and that $\mathbb{D}(t)[a, -n] = 0$. Thus, $\mathbb{D}(t)$ is a self-adjoint operator on V. By assumption of non-degeneracy, $\widetilde{\mathbb{C}}_t$ is a positive-definite self-adjoint operator on V. Thus, in order to prove the lemma we need to show that

$$\widetilde{\mathbb{C}}_t^{-1}X\cdot X < \frac{A_t a\cdot a}{t(1-t)}, \quad X = \begin{bmatrix}[\![A]\!]a\\ [\![A^*]\!]n\end{bmatrix} \tag{B.12}$$

for every $t \in [0, 1]$.

We observe that

$$\widetilde{\mathbb{C}}_t\begin{bmatrix}a\\ n\end{bmatrix} = 2\begin{bmatrix}\widetilde{A}_t a\\ \widetilde{A}_t^* n\end{bmatrix}.$$

We have

$$X = \frac{1}{t}\left(\begin{bmatrix}A_+ a\\ A_+^* n\end{bmatrix} - \begin{bmatrix}\widetilde{A}_t a\\ \widetilde{A}_t^* n\end{bmatrix}\right).$$

Hence,

$$\widetilde{\mathbb{C}}_t^{-1}X\cdot X = \frac{\widetilde{A}_t a\cdot a - 2A_+ a\cdot a + \widetilde{\mathbb{C}}_t^{-1}Y_+\cdot Y_+}{t^2},$$

where

$$Y_+ = \begin{bmatrix}A_+ a\\ A_+^* n\end{bmatrix}.$$

When $t = 0$ or 1, the statement is obvious. For $t \in (0, 1)$ we have

$$\widetilde{\mathbb{C}}_t > (1-t)\mathbb{C}_+,$$

in the sense of quadratic forms on V, so that

$$\widetilde{\mathbb{C}}_t^{-1}\boldsymbol{Y}_+ \cdot \boldsymbol{Y}_+ < \frac{1}{1-t}\mathbb{C}_+^{-1}\boldsymbol{Y}_+ \cdot \boldsymbol{Y}_+ = \frac{\boldsymbol{A}_+\boldsymbol{a}\cdot\boldsymbol{a}}{1-t}.$$

Thus, we obtain

$$\widetilde{\mathbb{C}}_t^{-1}\boldsymbol{X}\cdot\boldsymbol{X} < \frac{\boldsymbol{A}_t\boldsymbol{a}\cdot\boldsymbol{a}}{t(1-t)},$$

as required. □

Finally, let us show that the acoustic tensor of $\overline{W}(\boldsymbol{F})$ at $\boldsymbol{F}_0 = t\boldsymbol{F}_+ + (1-t)\boldsymbol{F}_-$ is nonnegative whenever it is nonnegative at both $\boldsymbol{F}_+$ and $\boldsymbol{F}_-$. Indeed, the last term in (B.11) is nonnegative, by Lemma B.1. It remains to observe that the function

$$\phi(t) = \langle \mathsf{L}_t\boldsymbol{\xi}, \boldsymbol{\xi}\rangle - \frac{\langle \boldsymbol{N}_t, \boldsymbol{\xi}\rangle^2}{\boldsymbol{A}_t\boldsymbol{a}\cdot\boldsymbol{a}}$$

is concave in t, which we can see by differentiating $\phi(t)$ twice:

$$\phi''(t) = -\frac{2(\langle \boldsymbol{N}_+, \boldsymbol{\xi}\rangle \boldsymbol{A}_-\boldsymbol{a}\cdot\boldsymbol{a} - \langle \boldsymbol{N}_-, \boldsymbol{\xi}\rangle \boldsymbol{A}_+\boldsymbol{a}\cdot\boldsymbol{a})^2}{(\boldsymbol{A}_t\boldsymbol{a}\cdot\boldsymbol{a})^3}.$$

Thus, $\phi(t)$ attains its minimum at $t = 0$ or $t = 1$. Setting $t = 0$ and 1 in (B.11) we obtain

$$\overline{W}_{\boldsymbol{FF}}(\boldsymbol{F}_\pm) = \mathsf{L}_\pm - \frac{\boldsymbol{N}_\pm \otimes \boldsymbol{N}_\pm}{\boldsymbol{A}_\pm\boldsymbol{a}\cdot\boldsymbol{a}}. \tag{B.13}$$

Formula (4.4) now follows easily from (B.13).

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