Journal of Elasticity **72:** 145–172, 2003.

About Clapeyron's Theorem in Linear Elasticity ⋆

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Received: 6 August 2002; in revised form: 17 March 2003

Abstract. We examine some elementary interpretations of the classical theorem of CLAPEYRON in linear elasticity theory. As we show, a straightforward application of this theorem in the purely mechanical setting leads to an apparent paradox which can be resolved by referring either to dynamics or to thermodynamics. These richer theories play an essential part in understanding the physical significance of this theorem.

Mathematics Subject Classifications (2000): 74A15, 74B05, 80A17

Key words: elasticity, Clapeyron, dissipation, viscoelasticity, thermoelasticity.

In remembrance of Clifford Truesdell and his scientific program of enlightenment.

1. Introduction

According to Love [11, p. 173], "The potential energy of deformation of a body, which is in equilibrium under given load, is equal to half the work done by the external forces, acting through the displacements from the unstressed state to the state of equilibrium." This is now commonly known as CLAPEYRON's theorem in linear elasticity theory.⋆⋆ In particular, this theorem, taken literally, implies that the elastic stored energy accounts for only half of the energy spent to load the body; the remaining half of the work done to the body by the external forces is unaccounted for and is lost somewhere in achieving the equilibrium state. It is particularly striking that this apparent paradox is reached within the framework of

⋆ The National Science Foundation Grant No. DMS-0102841 is gratefully acknowledged for their support of this research.

⋆⋆ In 1852, Lamé [9] published his volume, *Leçons sur la théorie mathématique de l'élasticité des corps solides*, in which he devoted his seventh lecture to what he termed CLAPEYRON's *Theorem*. (See [13, pp. 565 and 578], for relevant remarks.) Earlier, Lamé and Clapeyron [10] had noted this result in a joint memoir of 1833. Although Emile Clapeyron [3], himself, first published on this theorem in 1858, in a résumé of an original memoir that apparently was never published, it is argued by Todhunter and Pearson [14, p. 419], that the "result of the memoir of 1833 was due entirely to Clapeyron, for Lamé in his *Leçons*, of 1852, ... terms it CLAPEYRON's *Theorem*, and CLAPEYRON here speaks of it as he would do only if it were entirely due to himself."

the purely conservative linear theory of elasticity. Alternatively, however, within elastostatics the common characterization of the work done to reach equilibrium is conceptually ambiguous and a different interpretation may be required.

To illustrate the above concerns, let us first recall that in the linear theory of elasticity the total strain energy of a body that occupies the region $\Omega \subset \mathbb{R}^3$ and supports a, generally, dynamical displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and strain field $\mathbf{e} \equiv (\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}})/2 = \mathbf{e}(\mathbf{x}, t)$ relative to its undistorted state at time $t = 0$ is defined by

$$U[\mathbf{e}](t) \equiv \int_\Omega \frac{1}{2}\rho\mathbb{C}[\mathbf{e}] \cdot \mathbf{e}\,\mathrm{d}v. \tag{1.1}$$

Here, ρ is the mass density of the body and $\mathbb{C}$ is the positive definite and completely symmetric elasticity tensor. Further, the work done during the interval of time $(0, t)$ due to an applied boundary traction field $\mathbf{t}^* = \mathbf{t}^*(\mathbf{x}, t)$ and body force field $\mathbf{b}^* = \mathbf{b}^*(\mathbf{x}, t)$ over the displacement $\mathbf{u}(\mathbf{x}, t)$ is given by

$$W[\mathbf{u}](t) = \int_0^t \left(\int_{\partial\Omega} \mathbf{t}^* \cdot \dot{\mathbf{u}}\,\mathrm{d}a + \int_\Omega \mathbf{b}^* \cdot \dot{\mathbf{u}}\,\mathrm{d}v\right)\mathrm{d}t. \tag{1.2}$$

The corresponding stress field in Ω at time t, $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$, satisfies the generalized HOOKE'S law $\mathbf{T} = \rho\mathbb{C}[\mathbf{e}]$ and is symmetric. Throughout this paper we shall assume, for convenience, that the body is homogeneous, so that ρ and $\mathbb{C}$ are constant. If the loads $\mathbf{t}^*$ and $\mathbf{b}^*$ are 'dead', i.e., independent of time, so that $\mathbf{t}^* = \bar{\mathbf{t}}(\mathbf{x})$ and $\mathbf{b}^* = \bar{\mathbf{b}}(\mathbf{x})$, then for a body that is undistorted at time $t = 0$, (1.2) may be integrated to yield

$$W[\mathbf{u}](t)|_{(\mathbf{t}^*,\mathbf{b}^*)=(\bar{\mathbf{t}},\bar{\mathbf{b}})} = \int_{\partial\Omega} \bar{\mathbf{t}} \cdot \mathbf{u}\,\mathrm{d}a + \int_\Omega \bar{\mathbf{b}} \cdot \mathbf{u}\,\mathrm{d}v \equiv \overline{W}[\mathbf{u}](t). \tag{1.3}$$

This 'dead load work' represents the "work done by the external forces" to which LOVE referred in his quote concerning equilibrium reproduced in the first line of this introduction, above. Of course, in this case the loads are equilibrated so that

$$\int_{\partial\Omega} \bar{\mathbf{t}}\,\mathrm{d}a + \int_\Omega \bar{\mathbf{b}}\,\mathrm{d}v = 0, \qquad \int_{\partial\Omega} \mathbf{x} \times \bar{\mathbf{t}}\,\mathrm{d}a + \int_\Omega \mathbf{x} \times \bar{\mathbf{b}}\,\mathrm{d}v = 0 \tag{1.4}$$

and $\mathbf{u}$ is an equilibrium displacement field, say $\bar{\mathbf{u}}(\mathbf{x})$; the corresponding 'dead load work' is then

$$\overline{W}[\bar{\mathbf{u}}] \equiv \overline{W}[\mathbf{u}](t)|_{\mathbf{u}=\bar{\mathbf{u}}(\mathbf{x})}. \tag{1.5}$$

Suppose that the displacement field $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{x})$ corresponds to an equilibrium state with strain $\bar{\mathbf{e}}(\mathbf{x})$ and stress $\overline{\mathbf{T}}(\mathbf{x})$ satisfying $\overline{\mathbf{T}} = \rho\mathbb{C}[\bar{\mathbf{e}}]$ and

$$\operatorname{div}\overline{\mathbf{T}} + \bar{\mathbf{b}} = \mathbf{0} \quad \text{in } \Omega, \qquad \overline{\mathbf{T}}\mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \partial\Omega, \tag{1.6}$$

where $\mathbf{n}$ is the outer unit normal to Ω on $\partial\Omega$. Without loss of generality, we may eliminate the possibility of an added infinitesimal rigid displacement field in $\bar{\mathbf{u}}(\mathbf{x})$ and render $\bar{\mathbf{u}}(\mathbf{x})$ unique by imposing the normalization conditions

$$\int_\Omega \rho \bar{\mathbf{u}}\, \mathrm{d}v = \mathbf{0}, \qquad \int_\Omega \rho \mathbf{x} \times \bar{\mathbf{u}}\, \mathrm{d}v = \mathbf{0}, \tag{1.7}$$

where the mass density ρ is included only for later convenience. Then, according to the usual derivation of CLAPEYRON's theorem, we see, starting with (1.3) and (1.5) and using (1.6), generalized HOOKE'S law, the symmetry of $\overline{\mathbf{T}}$ and the divergence theorem, that

$$\frac{1}{2}\overline{W}[\bar{\mathbf{u}}] = \frac{1}{2}\left(\int_{\partial\Omega} \overline{\mathbf{T}}\mathbf{n} \cdot \bar{\mathbf{u}}\, \mathrm{d}a + \int_\Omega \bar{\mathbf{b}} \cdot \bar{\mathbf{u}}\, \mathrm{d}v\right) = U[\bar{\mathbf{e}}]. \tag{1.8}$$

Literally following LOVE's statement of CLAPEYRON's theorem, one may infer that elastostatics alone* accounts for only half of the work that is expended to reach equilibrium; the coefficient one-half is a result of the linearity of the theory. In the remainder of this paper, we continue within the linear framework and consider, respectively, in Sections 2, 3 and 4, the richer dynamical theories of elasticity, viscoelasticity and thermoelasticity in order to shed light on this seemingly paradoxical and incomplete conclusion.

SYNOPSIS

In Section 2, we argue that within ideal elasticity theory the quantity $\overline{W}[\bar{\mathbf{u}}]$ of CLAPEYRON's theorem does not reasonably represent the work done by the external forces to reach an elastostatic equilibrium state $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{x})$. We then investigate 'fast' versus 'slow' time dependent loading conditions and conclude that within the assumptions of elastostatics $\overline{W}[\bar{\mathbf{u}}]/2$ is a better representative of the work expended to reach equilibrium. In Sections 3 and 4, we amend ideal elasticity theory so as to include the mechanisms of viscous and thermal dissipation, respectively. Then dead loading becomes compatible with the notion of achieving equilibrium and, we conclude that the quantity $\overline{W}[\bar{\mathbf{u}}]$ of CLAPEYRON's theorem does adequately represent the corresponding work done by the external applied forces. Here we find that half of $\overline{W}[\bar{\mathbf{u}}]$ becomes stored in the body in the form of equilibrium strain energy and the remaining half is dissipated either through the action of viscous dissipation or heat transfer. In Section 5, we offer some conclusions.

* In elastostatics, there is, of course, no time dependence and formally the work done by the loads $\bar{\mathbf{t}}(\mathbf{x})$ and $\bar{\mathbf{b}}(\mathbf{x})$ to reach the equilibrium displacement $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{x})$ from an undistorted state commonly is calculated by using (1.3) and (1.5), as was done in (1.8). As noted earlier, for purely equilibrium theory this may not properly represent the 'work done to reach equilibrium' because this tacitly assumes that the loads are 'dead' and applied over time and, therefore, impulsive. For an ideal elastic body this circumstance is not compatible with the notion of reaching equilibrium, as we shall see in Section 2 and related Appendices A and B.

In Appendices A and B, we show some example calculations to further illustrate the claims of Section 2. It should be noted that throughout the main body of this paper we assume, for convenience, that the traction field is specified on the complete boundary of Ω. However, in the elementary examples of these appendices we prefer to hold one part of the boundary fixed and specify the traction on the complementary part for all time. While these boundary conditions clearly are not consistent with $(1.4)_1$ and $(2.4)_1$, nevertheless they are normal and allowable; moreover, they do not compromise the main purpose of illustrating the difference between dead and retarded loading.

2. Elastodynamics

Here, we shall first consider the consequences of 'dead' loading within elastodynamics regarding work and energy and then show how equilibrium theory is best accounted for by introducing a retarded system of loads.

2.1. 'DEAD' LOADING

Suppose that for all time $t > 0$ the body is 'dead' loaded with the same loads as in the static situation described above, so that $\mathbf{t}^* = \bar{\mathbf{t}}(\mathbf{x})$ and $\mathbf{b}^* = \bar{\mathbf{b}}(\mathbf{x})$ in (1.2). On the boundary of Ω we set

$$\mathbf{Tn} = \bar{\mathbf{t}} \quad \text{on } \partial\Omega,\ \forall t > 0, \tag{2.1}$$

and initially the body is at rest and undistorted so that

$$\mathbf{u}(\mathbf{x}, 0) = \dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega. \tag{2.2}$$

The dynamical equation is

$$\operatorname{div} \mathbf{T} + \bar{\mathbf{b}} = \rho\ddot{\mathbf{u}} \quad \text{in } \Omega,\ \forall t > 0, \tag{2.3}$$

and we recall that $\bar{\mathbf{t}} = \bar{\mathbf{t}}(\mathbf{x})$ and $\bar{\mathbf{b}} = \bar{\mathbf{b}}(\mathbf{x})$ are supposed to be balanced in the sense of (1.4). Of course, $\mathbf{u}$, $\mathbf{e}$ and $\mathbf{T}$ are related through the strain-displacement and stress–strain equations of Section 1. Under these conditions, it readily follows, from (2.1), (2.3) and the symmetry of $\mathbf{T}$, that the linear and angular momentum are conserved. Thus, by integration in time and use of (2.2), it is clear that the resulting motion naturally satisfies the normalization

$$\int_\Omega \rho\mathbf{u}\,\mathrm{d}v = \mathbf{0}, \qquad \int_\Omega \rho\mathbf{x} \times \mathbf{u}\,\mathrm{d}v = \mathbf{0} \quad \forall t \geqslant 0. \tag{2.4}$$

In addition, by forming the inner product of (2.3) with $\dot{\mathbf{u}}$, integrating over Ω and using the symmetry of $\mathbb{C}$ together with (2.1) and (1.1), we readily reach the classical power theorem

$$\int_{\partial\Omega} \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\,\mathrm{d}a + \int_\Omega \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}U[\mathbf{e}](t) + \frac{\mathrm{d}}{\mathrm{d}t}K[\dot{\mathbf{u}}](t), \tag{2.5}$$

where

$$K[\dot{\mathbf{u}}](t) \equiv \int_{\Omega} \frac{1}{2}\rho|\dot{\mathbf{u}}|^2\,\mathrm{d}v \tag{2.6}$$

is the kinetic energy of the body. Then, by integrating (2.5) in time and using (2.2) and (1.3) we obtain the standard balance of mechanical energy

$$\overline{W}[\mathbf{u}](t) = U[\mathbf{e}](t) + K[\dot{\mathbf{u}}](t), \tag{2.7}$$

which is supposed to be valid for all time $t \geqslant 0$.

Now, as a first elementary observation, let us assume that Ω may, at some time $t = \bar{t}$ during the motion, instantaneously support the equilibrium displacement field in the sense that $\mathbf{u}(\mathbf{x}, \bar{t}) = \bar{\mathbf{u}}(\mathbf{x})$; it may be that $\dot{\mathbf{u}}(\cdot, \bar{t}) \neq \mathbf{0}$ so Ω is not coincidently at rest. Be that as it may, nevertheless, (2.4) will be met at $t = \bar{t}$ because of (1.7) and, in addition, because $\overline{W}[\mathbf{u}](\bar{t}) = \overline{W}[\bar{\mathbf{u}}]$ and $U[\mathbf{e}](\bar{t}) = U[\bar{\mathbf{e}}]$, we see from (2.7) that

$$\overline{W}[\bar{\mathbf{u}}] = U[\bar{\mathbf{e}}] + K[\dot{\mathbf{u}}](\bar{t}). \tag{2.8}$$

Then, recalling (1.8), we may conclude that half the work done during the time interval $(0, \bar{t})$ is stored in the body as strain energy and the remaining half satisfies

$$\frac{1}{2}\overline{W}[\bar{\mathbf{u}}] = K[\dot{\mathbf{u}}](\bar{t}); \tag{2.9}$$

it has been spent to produce the instantaneous kinetic energy of the body. Accordingly, under the present circumstances, it is this kinetic energy that must be spontaneously extracted from Ω if the body is to be arrested in the equilibrium state $\mathbf{u}(\mathbf{x}, \bar{t}) = \bar{\mathbf{u}}(\mathbf{x})$. However, there is no mechanism in this conservative ideal elastic system to do so!⋆

Let us now consider an alternative description of how the work may be channeled into strain energy and kinetic energy based upon time-averaging of the corresponding energies. The assumption here is that there is a time $t = t^* > 0$, perhaps one among many, at which the body instantaneously is at rest, i.e., $\dot{\mathbf{u}}(\mathbf{x}, t^*) \equiv 0$ in Ω.⋆⋆ To describe the average motion, we introduce the time-average displacement field as

$$\langle\mathbf{u}\rangle(\mathbf{x}) \equiv \frac{1}{t^*}\int_0^{t^*} \mathbf{u}(\mathbf{x}, t)\,\mathrm{d}t, \tag{2.10}$$

⋆ According to [11, p. 123] (see also [13, p. 537, art. 988]), in 1839 Poncelet [12] was the first to note that "a load suddenly applied may cause a strain twice as great as that produced by a gradual application of the same load." While this observation of Poncelet, which also contains an interesting factor of 2, appeared contemporaneously with the original and later announcements of CLAPEYRON's theorem, there appears to have been no recognition of a possible relationship between the claims of either authors.

⋆⋆ While, generally, there may not be such a time, in the case of periodic motion there is a countable set of such times; a specific one-dimensional example is discussed later in Appendix A. Notice, though, that according to $(2.4)_1$ the *average* of $\dot{\mathbf{u}}(\mathbf{x}, t)$ over Ω is always zero.

with the time-average strain $\langle \mathbf{e} \rangle(\mathbf{x})$ and stress $\langle \mathbf{T} \rangle(\mathbf{x})$ fields defined analogously. Then, it readily follows that

$$\langle \mathbf{e} \rangle = \frac{1}{2}\big(\nabla\langle \mathbf{u} \rangle + (\nabla\langle \mathbf{u} \rangle)^{\mathrm{T}}\big), \qquad \langle \mathbf{T} \rangle = \rho\mathbb{C}[\langle \mathbf{e} \rangle]. \tag{2.11}$$

Moreover, because of (2.2), by time-averaging (2.3) and (2.1) we find

$$\operatorname{div}\langle \mathbf{T} \rangle + \bar{\mathbf{b}} = \mathbf{0} \quad \text{in } \Omega, \qquad \langle \mathbf{T} \rangle\mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \partial\Omega. \tag{2.12}$$

Also, note that the time-average displacement field $\langle \mathbf{u} \rangle(\mathbf{x})$ satisfies the normalization (2.4) and recall that the loads $\bar{\mathbf{t}} = \bar{\mathbf{t}}(\mathbf{x})$ and $\bar{\mathbf{b}} = \bar{\mathbf{b}}(\mathbf{x})$ are balanced in the sense of (1.4). Thus, because of uniqueness and the fact that $\langle \mathbf{u} \rangle(\mathbf{x})$ and $\bar{\mathbf{u}}(\mathbf{x})$ solve the same equilibrium boundary-value problem, we may conclude that

$$\langle \mathbf{u} \rangle(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}), \qquad \langle \mathbf{e} \rangle(\mathbf{x}) = \bar{\mathbf{e}}(\mathbf{x}), \qquad \langle \mathbf{T} \rangle(\mathbf{x}) = \overline{\mathbf{T}}(\mathbf{x}) \quad \text{in } \Omega. \tag{2.13}$$

Now, by time-averaging (2.7), using (1.3), recalling the notation established in (2.10) and applying (2.13), we easily have

$$\langle \overline{W}[\mathbf{u}] \rangle = \overline{W}[\langle \mathbf{u} \rangle] = \overline{W}[\bar{\mathbf{u}}] = \langle U[\mathbf{e}] \rangle + \langle K[\dot{\mathbf{u}}] \rangle. \tag{2.14}$$

In particular, the average of the 'dead load work', $\langle \overline{W}[\mathbf{u}] \rangle$, is equal to the quantity $\overline{W}[\bar{\mathbf{u}}]$ of CLAPEYRON's theorem in (1.8), and our immediate aim is to determine how this average work expended is divided up between the average strain energy $\langle U[\mathbf{e}] \rangle$ and the average kinetic energy $\langle K[\dot{\mathbf{u}}] \rangle$ of the body. To do so, we first introduce the difference displacement field

$$\mathbf{u}'(\mathbf{x}, t) \equiv \mathbf{u}(\mathbf{x}, t) - \bar{\mathbf{u}}(\mathbf{x}), \tag{2.15}$$

with $\mathbf{e}'(\mathbf{x}, t)$ and $\mathbf{T}'(\mathbf{x}, t)$ defined analogously, and observe, using (1.1), the symmetry of $\mathbb{C}$ and (1.8), that

$$\begin{aligned} U[\mathbf{e}](t) &= \int_\Omega \frac{1}{2}\rho\mathbb{C}[\bar{\mathbf{e}} + \mathbf{e}'] \cdot (\bar{\mathbf{e}} + \mathbf{e}')\,\mathrm{d}v \\ &= U[\bar{\mathbf{e}}] + \int_\Omega \rho\mathbb{C}[\bar{\mathbf{e}}] \cdot \mathbf{e}'\,\mathrm{d}v + U[\mathbf{e}'](t) \\ &= \frac{1}{2}\overline{W}[\bar{\mathbf{u}}] + \int_\Omega \rho\mathbb{C}[\bar{\mathbf{e}}] \cdot \mathbf{e}'\,\mathrm{d}v + U[\mathbf{e}'](t). \end{aligned}$$

Then, by time-averaging we have

$$\langle U[\mathbf{e}] \rangle = \frac{1}{2}\overline{W}[\bar{\mathbf{u}}] + \int_\Omega \rho\mathbb{C}[\bar{\mathbf{e}}] \cdot \langle \mathbf{e}' \rangle\,\mathrm{d}v + \langle U[\mathbf{e}'] \rangle.$$

However, because $\mathbf{e}'(\mathbf{x}, t) = \mathbf{e}(\mathbf{x}, t) - \bar{\mathbf{e}}(\mathbf{x})$ we see from (2.13) that $\langle \mathbf{e}' \rangle(\mathbf{x}) = \langle \mathbf{e} \rangle(\mathbf{x}) - \bar{\mathbf{e}}(\mathbf{x}) = \mathbf{0}$ and, consequently, we reach

$$\langle U[\mathbf{e}] \rangle = \frac{1}{2}\overline{W}[\bar{\mathbf{u}}] + \langle U[\mathbf{e}'] \rangle. \tag{2.16}$$

Now, to determine $\langle U[\mathbf{e}']\rangle$, it is convenient to observe, using (2.1)–(2.3), (2.15) and the relationships $\mathbf{T}' = \rho\mathbb{C}[\mathbf{e}']$ and $\mathbf{e}' = (\nabla\mathbf{u}' + (\nabla\mathbf{u}')^{\mathrm{T}})/2$, that

$$\begin{aligned} &\operatorname{div}\mathbf{T}' = \rho\ddot{\mathbf{u}}' \quad \text{in } \Omega,\ \forall t > 0, \\ &\mathbf{T}'\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega,\ \forall t > 0; \quad \mathbf{u}'(\mathbf{x}, 0) = -\bar{\mathbf{u}}(\mathbf{x}), \quad \dot{\mathbf{u}}'(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega. \end{aligned} \tag{2.17}$$

Then, with the definition (1.1), the symmetry of $\mathbb{C}$, (2.17) and the aid of the divergence theorem we find that

$$\begin{aligned} U[\mathbf{e}'](t) &= \frac{1}{2}\int_{\Omega} \mathbf{T}' \cdot \nabla\mathbf{u}' \,\mathrm{d}v \\ &= \frac{1}{2}\int_{\partial\Omega} \mathbf{T}'\mathbf{n} \cdot \mathbf{u}' \,\mathrm{d}a - \frac{1}{2}\int_{\Omega} \rho\ddot{\mathbf{u}}' \cdot \mathbf{u}' \,\mathrm{d}v \\ &= -\frac{1}{2}\int_{\Omega} \rho\ddot{\mathbf{u}}' \cdot \mathbf{u}' \,\mathrm{d}v \\ &= -\frac{1}{2}\int_{\Omega} \left(\overset{\cdot}{\overline{\rho\dot{\mathbf{u}} \cdot \mathbf{u}'}} - \rho|\dot{\mathbf{u}}|^2\right) \mathrm{d}v, \end{aligned}$$

the last equation of which uses the fact that (2.15) implies $\dot{\mathbf{u}}'(\mathbf{x}, t) = \dot{\mathbf{u}}(\mathbf{x}, t)$. Now, by time-averaging, recalling that $\dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}(\mathbf{x}, t^*) = \mathbf{0}$ and using (2.6), we obtain

$$\langle U[\mathbf{e}']\rangle = \langle K[\dot{\mathbf{u}}]\rangle, \tag{2.18}$$

which is a well known result concerning the equipartition between kinetic and potential energies. Thus, by substituting (2.18) into (2.16) and then using (2.14) we conclude that

$$\langle U[\mathbf{e}']\rangle = \langle K[\dot{\mathbf{u}}]\rangle = \frac{1}{4}\overline{W}[\bar{\mathbf{u}}] \tag{2.19}$$

and, again using (2.16), we see that

$$\langle U[\mathbf{e}]\rangle = \frac{1}{2}\overline{W}[\bar{\mathbf{u}}] + \frac{1}{4}\overline{W}[\bar{\mathbf{u}}] = \frac{3}{4}\overline{W}[\bar{\mathbf{u}}]. \tag{2.20}$$

To show that this result is independent of the assumption of periodicity, let us introduce the complete set of orthonormal eigenfunctions and eigenvalues, $\{\bar{\mathbf{u}}_i(\mathbf{x}), \omega_i, i = 1, 2, \ldots\}$, which satisfy (1.7) and

$$\begin{aligned} \operatorname{div}(\mathbb{C}[\nabla\bar{\mathbf{u}}_i]) + \rho\omega_i^2\bar{\mathbf{u}}_i = 0 \quad &\text{in } \Omega, \\ (\mathbb{C}[\nabla\bar{\mathbf{u}}_i])\mathbf{n} = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{2.21}$$

and expand the solution $\mathbf{u}(\mathbf{x}, t)$ of (2.1)–(2.3) in the form

$$\mathbf{u}(\mathbf{x}, t) = \sum_{i=1}^{\infty} \bar{\mathbf{u}}_i(\mathbf{x}) g_i(t).$$

Then, we readily find that $g_i(t) = A_i(1 - \cos\omega_i t)$, $i = 1, 2, \ldots$, and it is possible to determine the constants A_i as Fourier coefficients so that this series represents a weak solution of (2.1)–(2.3) in the sense that $\nabla\mathbf{u}(\cdot, t) \in L^2(\Omega)$ for all $t > 0$. Furthermore, it is straightforward to show that the infinite time-average of the displacement field,

$$\langle\mathbf{u}\rangle_\infty(\mathbf{x}) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^{\mathrm{T}} \mathbf{u}(\mathbf{x}, t)\,\mathrm{d}t, \tag{2.22}$$

satisfies $\langle\mathbf{u}\rangle_\infty(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})$ and that conclusions similar to those highlighted in the previous paragraph continue to hold for the relationships between the infinite time-averages of the work, strain energy and kinetic energy, i.e.,

$$\langle\overline{W}[\mathbf{u}]\rangle_\infty = \overline{W}[\langle\mathbf{u}\rangle_\infty] = \overline{W}[\bar{\mathbf{u}}] = \langle U[\mathbf{e}]\rangle_\infty + \langle K[\dot{\mathbf{u}}]\rangle_\infty \tag{2.23a}$$

with

$$\langle U[\mathbf{e}]\rangle_\infty = \frac{3}{4}\overline{W}[\bar{\mathbf{u}}], \qquad \langle K[\dot{\mathbf{u}}]\rangle_\infty = \frac{1}{4}\overline{W}[\bar{\mathbf{u}}]. \tag{2.23b}$$

Based upon the above analyses, we conclude that when an elastic body is set in motion with a 'dead' loading system from an initially undistorted rest state, then, with suitable interpretation, the average work that is supplied to the body by the 'dead' loading is equal to the equilibrium work of CLAPEYRON's theorem. On the average, three quarters of this work appears as strain energy (half due to the equilibrium strain energy as predicted from CLAPEYRON's theorem and a quarter due to the strain energy of the deformation relative to this equilibrium), and the remaining quarter is, on the average, transformed into kinetic energy.

To illustrate the general conclusions reached above, we consider, in Appendix A, a specific one-dimensional elastodynamic problem with 'dead' loading.

2.2. 'SLOW' LOADING

When an ideal elastic body is 'dead' loaded from an undistorted, rest state with an otherwise equilibrium system of loads, the loading is impulsively applied. Consequently, from the dynamical considerations of Section 2.1, the body never reaches equilibrium but, rather, rings by constantly redistributing kinetic and strain energy between its elements. Indeed, the work done to the body at any time $t > 0$ due to the external loading is given by (1.3), but the body is never coincidently at rest and in a state of equilibrium. On the other hand, we expect that if an equilibrium system of loads is achieved sufficiently slowly in time then even an ideal elastic body should distort through a sequence of near equilibrium states and eventually reach a nearly static equilibrium configuration. In this case, the work done to the body at any time t due to the external loading may be calculated using (1.2), but the calculation is no longer trivial because now $\mathbf{t}^*$ and $\mathbf{b}^*$ are not 'dead' but rather depend on time. For dissipationless, ideal elastic bodies it is intuitively clear that the work expended to reach equilibrium should be related to the latter rather than the former calculation.

To gain some general perspective, suppose that the loading system, $\mathbf{t}^*$ on $\partial\Omega$ and $\mathbf{b}^*$ in Ω, is such that

$$\mathbf{t}^* = \mathbf{t}^*(\mathbf{x}, t) = \begin{cases} \dfrac{t}{t_\infty}\bar{\mathbf{t}}(\mathbf{x}), & t \in (0, t_\infty), \\ \bar{\mathbf{t}}(\mathbf{x}), & t \geqslant t_\infty, \end{cases} \tag{2.24}$$

and

$$\mathbf{b}^* = \mathbf{b}^*(\mathbf{x}, t) = \begin{cases} \dfrac{t}{t_\infty}\bar{\mathbf{b}}(\mathbf{x}), & t \in (0, t_\infty), \\ \bar{\mathbf{b}}(\mathbf{x}), & t \geqslant t_\infty, \end{cases} \tag{2.25}$$

where t_∞ is a sufficiently large time constant so that the loads may be considered to be slowly applied. Then, at least for $t \in (0, t_\infty)$, the displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = t\bar{\mathbf{u}}(\mathbf{x})/t_\infty$ and the corresponding strain and stress fields, $\mathbf{e} = \mathbf{e}(\mathbf{x}, t) = t\bar{\mathbf{e}}(\mathbf{x})/t_\infty$ and $\mathbf{T} = \mathbf{T}(\mathbf{x}, t) = t\overline{\mathbf{T}}(\mathbf{x})/t_\infty$, from the strain-displacement and stress–strain relations of Section 1, will satisfy the dynamical equation

$$\operatorname{div}\mathbf{T} + \mathbf{b}^* = \rho\ddot{\mathbf{u}} \quad \text{in } \Omega,\ t \in (0, t_\infty),$$

together with the boundary condition

$$\mathbf{Tn} = \mathbf{t}^* \quad \text{on } \partial\Omega,\ t \in (0, t_\infty)$$

and initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}, \qquad \dot{\mathbf{u}}(\mathbf{x}, 0) = \frac{1}{t_\infty}\bar{\mathbf{u}}(\mathbf{x}) \quad \text{in } \Omega.$$

Clearly, for sufficiently large time constant t_∞ not only is the applied loading 'slow', but the initial state of Ω is undistorted and 'nearly' at rest. Further, at time $t = t_\infty$ the body achieves the equilibrium displacement field $\bar{\mathbf{u}}(\mathbf{x})$ with, again, 'nearly' zero velocity. Moreover, according to (1.2), the work done to Ω up to time $t = t_\infty$ is

$$\begin{aligned} W[\mathbf{u}](t_\infty) &= \int_0^{t_\infty}\left(\int_{\partial\Omega}\frac{t}{t_\infty}\bar{\mathbf{t}}\cdot\frac{1}{t_\infty}\bar{\mathbf{u}}\,\mathrm{d}a + \int_\Omega \frac{t}{t_\infty}\bar{\mathbf{b}}\cdot\frac{1}{t_\infty}\bar{\mathbf{u}}\,\mathrm{d}v\right)\mathrm{d}t \\ &= \frac{1}{2}\left(\int_{\partial\Omega}\bar{\mathbf{t}}\cdot\bar{\mathbf{u}}\,\mathrm{d}a + \int_\Omega \bar{\mathbf{b}}\cdot\bar{\mathbf{u}}\,\mathrm{d}v\right), \end{aligned}$$

so that with $(1.8)_1$ we have

$$W[\mathbf{u}](t_\infty) = \frac{1}{2}\overline{W}[\bar{\mathbf{u}}].$$

In addition, according to (2.6), the kinetic energy of Ω at any time $t \in [0, t_\infty)$ is 'nearly' zero and equal to its initial value because

$$K[\dot{\mathbf{u}}](t) = \frac{1}{2t_\infty^2} \int_\Omega \rho |\bar{\mathbf{u}}|^2 \, \mathrm{d}v, \quad \forall t \in [0, t_\infty).$$

Finally, according to (1.1), the strain energy of Ω at time $t = t_\infty$ is given by

$$U[\mathbf{e}](t_\infty) = U[\bar{\mathbf{e}}],$$

and so with $(1.8)_2$ we may conclude that

$$W[\mathbf{u}](t_\infty) = U[\mathbf{e}](t_\infty).$$

Thus, for sufficiently large time constant t_∞, the body is 'nearly' at rest in equilibrium at time $t = t_\infty$ and the work done to Ω to achieve this 'near' equilibrium state is half that which is supplied, according to LOVE's interpretation of CLAPEYRON's theorem; in fact, this reasoning shows that the work done is equal to the strain energy at time $t = t_\infty$ and, to within a certain degree of approximation the paradox of CLAPEYRON's theorem may be considered resolved. Of course, the body is not exactly at rest in equilibrium at time $t = t_\infty$ and while it was initially undistorted, it was not initially at rest. For large time constant t_∞, according to the given initial conditions there is a small kinetic energy imparted to Ω at time $t = 0$ and this same kinetic energy must be extracted from Ω at time $t = t_\infty$ in order for Ω to strictly remain in equilibrium for all time $t > t_\infty$.

In general, for the loading conditions of (2.24) and (2.25) it readily follows from an application of the power theorem that for all time $t > t_\infty$ we must have

$$K[\dot{\mathbf{u}}](t) + U[\mathbf{e} - \bar{\mathbf{e}}](t) = K[\dot{\mathbf{u}}](t_\infty) = \frac{1}{2t_\infty^2} \int_\Omega \rho |\bar{\mathbf{u}}|^2 \, \mathrm{d}v,$$

where, of course, the right-hand side also represents the kinetic energy that is added to the system at $t = 0$ due to the 'nearly' stationary initial condition. Thus, given an $\epsilon > 0$, for sufficiently large time constant t_∞, both the kinetic energy of Ω and the strain energy of Ω for the difference strain $\mathbf{e}(\mathbf{x}, t) - \bar{\mathbf{e}}(\mathbf{x})$ must remain within an ϵ-neighborhood of zero for all $t \geqslant t_\infty$.

To illustrate these ideas, we consider, in Appendix B, a one-parameter family of one-dimensional elastodynamic problems for a bar of finite length, wherein the applied loading depends on a slowness parameter α. Our aim in this appendix is to exhibit how the retarded nature of the applied loading effects the dynamical behavior and its relationship to the notion of equilibrium.

3. Viscoelasticity

As earlier, we again suppose that the body is initially at rest and undistorted and that it is 'dead' loaded as in the static situation of Section 1. Now, to introduce an elementary form of mechanical dissipation, we consider a viscoelastic body whose constitutive relation is of the KELVIN–VOIGT form

$$\mathbf{T} = \rho\mathbb{C}[\mathbf{e}] + \mathbb{D}[\dot{\mathbf{e}}], \tag{3.1}$$

where $\mathbb{D}$ is a positive definite, completely symmetric (constant) viscosity tensor. The dynamical equation and the boundary and initial conditions are the same as those in (2.1)–(2.3), i.e.,

$$\begin{aligned} &\operatorname{div}\mathbf{T} + \bar{\mathbf{b}} = \rho\ddot{\mathbf{u}} \quad \text{in } \Omega,\ \forall t > 0, \\ &\mathbf{Tn} = \bar{\mathbf{t}} \quad \text{on } \partial\Omega,\ \forall t > 0, \qquad \mathbf{u}(\mathbf{x}, 0) = \dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega. \end{aligned} \tag{3.2}$$

In the usual way, it follows from (3.2) and (2.6) that the classical power theorem holds, i.e.,

$$\int_{\partial\Omega} \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\,\mathrm{d}a + \int_{\Omega} \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\,\mathrm{d}v = \int_{\Omega} \mathbf{T} \cdot \dot{\mathbf{e}}\,\mathrm{d}v + \frac{\mathrm{d}}{\mathrm{d}t} K[\dot{\mathbf{u}}](t), \tag{3.3}$$

which, with the use of (3.1), $(3.2)_3$, (1.1), (1.3) and integration in time, results in

$$\overline{W}[\mathbf{u}](t) = U[\mathbf{e}](t) + K[\dot{\mathbf{u}}](t) + \mathcal{D}(t), \tag{3.4}$$

where $\mathcal{D}(t)$ denotes the dissipation function

$$\mathcal{D}(t) \equiv \int_0^t \left(\int_{\Omega} \mathbb{D}[\dot{\mathbf{e}}] \cdot \dot{\mathbf{e}}\,\mathrm{d}v \right) \mathrm{d}t \geqslant 0, \tag{3.5}$$

for all $t \geqslant 0$.

Because of the dissipative character of viscosity and the special nature of the loading, in that $\bar{\mathbf{t}}(\mathbf{x})$ and $\bar{\mathbf{b}}(\mathbf{x})$ are balanced and correspond with the equilibium displacement field $\bar{\mathbf{u}}(\mathbf{x})$ of Section 1, it is natural to expect that the solution of the problem outlined above will have the 'asymptotic property' $\mathbf{u}(\mathbf{x}, t) \to \bar{\mathbf{u}}(\mathbf{x})$ as $t \to \infty$.* Supposing this is the case, we find from (3.4), (1.1) and (2.6) that in the limit as $t \to \infty$

$$\overline{W}[\bar{\mathbf{u}}] = U[\bar{\mathbf{e}}] + \mathcal{D}_\infty,$$

where

$$\mathcal{D}_\infty \equiv \mathcal{D}(\infty) = \int_0^\infty \left(\int_{\Omega} \mathbb{D}[\dot{\mathbf{e}}] \cdot \dot{\mathbf{e}}\,\mathrm{d}v \right) \mathrm{d}t \geqslant 0. \tag{3.6}$$

---

* Dafermos [5] and Andrews and Ball [1] have studied the questions of existence and asymptotic stability for general one-dimensional KELVIN-VOIGT viscoelasticity theory. With certain smoothness hypotheses, the conclusions in [1, 5] guarantee that the solution to the problem with 'dead' loading and zero initial data asymptotically and strongly approaches the equilibrium state which corresponds to the same 'dead' loads.

Moreover, by using CLAPEYRON's theorem (1.8) we may then conclude that half the work done to reach equilibrium is stored as strain energy and the remaining half is given by

$$\frac{1}{2}\overline{W}[\bar{\mathbf{u}}] = \mathcal{D}_\infty,$$

which is consumed during the dynamical process through viscous dissipation. To summarize, when viscous dissipation is present and the 'asymptotic property' holds then 'dead' loading and equilibrium are, indeed, compatible. Moreover, in practical terms the paradox reached within elasticity theory from LOVE's interpretation of CLAPEYRON's theorem may be resolved by appropriately accounting for the dissipative action of viscoelastic behavior.

4. Thermoelasticity

Within the linear theory of thermoelasticity, when a body is subject to a displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ relative to its undistorted, rest state and coincidently the absolute temperature is changed from its constant reference (room) temperature θ_0 to the field $\theta = \theta(\mathbf{x}, t)$, the HELMHOLTZ free energy per unit mass $\psi = \psi(\mathbf{x}, t)$ is determined by the constitutive equation*

$$\psi = \hat{\psi}(\theta, \mathbf{e}) = \frac{1}{2}\mathbb{C}[\mathbf{e}] \cdot \mathbf{e} - (\theta - \theta_0)\mathbf{M} \cdot \mathbf{e} - c\theta \ln \frac{\theta}{\theta_0}, \tag{4.1}$$

normalized so that $\hat{\psi}(\theta_0, \mathbf{0}) = 0$. Here, $\mathbf{M}$ is the positive definite, symmetric thermal expansion tensor and $c > 0$ is the specific heat at constant deformation, both representing prescribed thermomechanical material properties and herein assumed to be constant. The symmetric stress tensor field $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$ and the entropy field per unit mass $\eta = \eta(\mathbf{x}, t)$ are then determined by the GIBBS relations

$$\mathbf{T} = \widehat{\mathbf{T}}(\theta, \mathbf{e}) = \rho \frac{\partial \hat{\psi}(\theta, \mathbf{e})}{\partial \mathbf{e}} = \rho(\mathbb{C}[\mathbf{e}] - (\theta - \theta_0)\mathbf{M}) \tag{4.2}$$

and

$$\eta = \hat{\eta}(\theta, \mathbf{e}) = -\frac{\partial \hat{\psi}(\theta, \mathbf{e})}{\partial \theta} = \mathbf{M} \cdot \mathbf{e} + c\left(\ln \frac{\theta}{\theta_0} + 1\right), \tag{4.3}$$

respectively. The total HELMHOLTZ free energy of the body is given by

$$\Psi[\theta, \mathbf{e}](t) \equiv \int_\Omega \rho \hat{\psi}(\theta(\mathbf{x}, t), \mathbf{e}(\mathbf{x}, t))\, \mathrm{d}v. \tag{4.4}$$

If we now assume, analogous to Sections 1–3, that the 'dead' loads $\bar{\mathbf{t}}(\mathbf{x})$ on $\partial\Omega$ and $\bar{\mathbf{b}}(\mathbf{x})$ in Ω are balanced in the sense of (1.4) and that $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{x})$ is a corresponding

* In [4] or [8, p. 99], for example, a non-essential quadratic approximation for θ near θ_0 is used in place of the last term in (4.1).

equilibrium displacement field at the uniform temperature $\theta = \theta_0$ then, as in Section 1, we readily see, from an argument totally analogous to that given in (1.8) for CLAPEYRON's theorem and (1.1), that

$$\Psi[\theta_0, \bar{\mathbf{e}}] = \frac{1}{2}\overline{W}[\bar{\mathbf{u}}]; \tag{4.5}$$

i.e., half of the work done to reach equilibrium is stored in the body as HELMHOLTZ free energy. Here, we have used the normalization (1.7) which guarantees uniqueness and eliminates any possible additive infinitesimal rigid field.

Before proceeding with a more detailed continuum thermodynamic analysis for non-isothermal processes, we first give an elementary thermodynamic explanation for the isothermal case $\theta(\mathbf{x}, t) = \theta_0$. Thus, for a finite material body the first law and the second law, in the form of the CLAUSIUS–PLANCK inequality, may be written as

$$\dot{\mathcal{E}}(t) + \dot{\mathcal{K}}(t) = \mathcal{P}(t) + \mathcal{Q}(t), \qquad \dot{\mathcal{H}}(t) \geqslant \frac{\mathcal{Q}(t)}{\theta_0} \quad \forall t > 0, \tag{A}$$

where $\mathcal{E}$, $\mathcal{K}$, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{H}$ denote the internal energy, kinetic energy, mechanical power supply (positive for influx and negative for efflux), heat supply rate (positive for absorbtion and negative for emission) and entropy for the body, respectively. Now, introducing the HELMHOLTZ free energy of the body, $\mathcal{F}(t) \equiv \mathcal{E}(t) - \theta_0\mathcal{H}(t)$, we may write $(\mathrm{A})_1$ in the form

$$\mathcal{P}(t) = \dot{\mathcal{F}}(t) + \theta_0\dot{\mathcal{H}}(t) + \dot{\mathcal{K}}(t) - \mathcal{Q}(t). \tag{B}$$

Then, supposing the body reaches an equilibrium state at some time $t_0 \in (0, \infty]$, we see from (B) that the total work done to the body over the time interval $(0, t_0)$ is given by

$$\mathcal{W} \equiv \int_0^{t_0} \mathcal{P}(t)\,\mathrm{d}t = \Delta\mathcal{F} + \mathcal{D}, \tag{C}$$

where

$$\mathcal{D} \equiv \theta_0\Delta\mathcal{H} - \int_0^{t_0} \mathcal{Q}(t)\,\mathrm{d}t \geqslant 0 \tag{D}$$

represents the total energy dissipated by the body during the (isothermal) process of reaching equilibrium. We are assured that this dissipated energy is non-negative because of $(\mathrm{A})_2$ and the isothermal condition. Now, by naturally interpreting $\mathcal{W}$ as the work $\overline{W}[\bar{\mathbf{u}}]$ to reach equilibrium and $\Delta\mathcal{F}$ as the equilibrium free energy $\Psi[\theta_0, \bar{\mathbf{e}}]$, both noted in (4.5), we see from (4.5), (C) and (D) that

$$\frac{1}{2}\mathcal{W} = \Delta\mathcal{F} \quad \text{and} \quad \frac{1}{2}\mathcal{W} = \mathcal{D}. \tag{E}$$

Clearly, half of the work that is supplied to reach equilibrium is dissipated and the paradox of CLAPEYRON's theorem is resolved.

Now, rather than assume that the temperature field of the body is spatially uniform and constant in time, let us suppose that the body is initially at rest in its undistorted state at the constant temperature θ_0 and that for all time $t > 0$ it is subject to a balanced 'dead' loading system, as is presumed in the equilibrium situation which lead to (4.5) above. In addition, for convenience we suppose that the body is subject to null heat radiation to or from the external environment and that the *boundary temperature* is fixed at θ_0 for all time $t > 0$. Explicitly, the boundary and initial conditions that we consider are

$$\mathbf{Tn} = \bar{\mathbf{t}}, \qquad \theta = \theta_0 \quad \text{on } \partial\Omega,\ \forall t > 0, \tag{4.6}$$

and

$$\mathbf{u}(\mathbf{x}, 0) = \dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{0}, \qquad \theta(\mathbf{x}, 0) = \theta_0 \quad \text{in } \Omega, \tag{4.7}$$

respectively. The dynamical governing equations have the form

$$\operatorname{div} \mathbf{T} + \bar{\mathbf{b}} = \rho \ddot{\mathbf{u}} \quad \text{in } \Omega,\ \forall t > 0, \tag{4.8}$$

and

$$-\operatorname{div} \mathbf{q} + \mathbf{T} \cdot \dot{\mathbf{e}} = \rho \dot{\epsilon} \quad \text{in } \Omega,\ \forall t > 0. \tag{4.9}$$

Here, $\epsilon = \epsilon(\mathbf{x}, t)$ is the internal energy field per unit mass, which is related to the HELMHOLTZ free energy, temperature and entropy through $\epsilon = \psi + \theta\eta$, and $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$ is the heat flux vector field. Also, we note for later reference that with (4.1)–(4.3), we may write (4.9) in the alternative form

$$-\operatorname{div} \mathbf{q} = \rho\theta\dot{\eta} \quad \text{in } \Omega,\ \forall t > 0. \tag{4.10}$$

Now, with the aid of (4.8), (4.6) and (2.6) we again have the power theorem (3.3). Moreover, following a standard line of reasoning which uses (3.3) with (4.9) and an application of the divergence theorem, we recover the global form of the balance of energy:

$$\int_{\partial\Omega} \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \,\mathrm{d}a + \int_{\Omega} \bar{\mathbf{b}} \cdot \dot{\mathbf{u}} \,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} E[\theta, \mathbf{e}](t) + \frac{\mathrm{d}}{\mathrm{d}t} K[\dot{\mathbf{u}}](t) - Q(t). \tag{4.11}$$

Here, $E[\theta, \mathbf{e}](t)$, the total internal energy of the body at time t, may be written conveniently as

$$\begin{aligned} E[\theta, \mathbf{e}](t) &\equiv \int_{\Omega} \rho\hat{\epsilon}(\theta(\mathbf{x}, t), \mathbf{e}(\mathbf{x}, t)) \,\mathrm{d}v \\ &= \Psi_{\theta_0}[\theta, \mathbf{e}](t) + \theta_0 \int_{\Omega} \rho\hat{\eta}(\theta(\mathbf{x}, t), \mathbf{e}(\mathbf{x}, t)) \,\mathrm{d}v, \end{aligned} \tag{4.12}$$

where

$$\Psi_{\theta_0}[\theta, \mathbf{e}](t) \equiv \int_\Omega \rho(\hat{\epsilon}(\theta, \mathbf{e}) - \theta_0 \hat{\eta}(\theta, \mathbf{e}))\mathrm{d}v \tag{4.13}$$

is the total HELMHOLTZ *semi*-free energy* of the body based upon the boundary temperature θ_0 and $Q(t)$ is the total heat rate of the body at time t, which, here, is determined solely by boundary conduction, i.e.,

$$Q(t) \equiv \int_{\partial\Omega} -\mathbf{q} \cdot \mathbf{n}\,\mathrm{d}a. \tag{4.14}$$

Because $\mathbf{n}$ denotes the outer unit normal to $\partial\Omega$, we note that $Q(t) > 0$ (< 0) corresponds to a rate of heat *supply to* (*loss from*) Ω. Thus, by integration of (4.11) in time and use of the initial conditions (4.7), and formulae (4.1), (4.4), (1.2) and (1.3), we arrive at

$$\overline{W}[\mathbf{u}](t) = \Psi_{\theta_0}[\theta, \mathbf{e}](t) + K[\dot{\mathbf{u}}](t) + \mathcal{D}(t), \tag{4.15}$$

where $\mathcal{D}(t)$ denotes the dissipation function

$$\begin{aligned}\mathcal{D}(t) &\equiv \theta_0 \int_\Omega \rho(\hat{\eta}(\theta, \mathbf{e}) - \hat{\eta}(\theta_0, \mathbf{0}))\mathrm{d}v - \int_0^t Q(\tau)\,\mathrm{d}\tau \\ &= \int_0^t \left(\theta_0 \frac{\mathrm{d}}{\mathrm{d}t} H[\theta, \mathbf{e}](t) - Q(t)\right)\mathrm{d}t \geqslant 0,\end{aligned} \tag{4.16}$$

for all $t \geqslant 0$ and where $H[\theta, \mathbf{e}](t)$, the total entropy of the body in the state of temperature $\theta(\mathbf{x}, t)$ and strain $\mathbf{e}(\mathbf{x}, t)$, is defined by

$$H[\theta, \mathbf{e}](t) \equiv \int_\Omega \rho\hat{\eta}(\theta(\mathbf{x}, t), \mathbf{e}(\mathbf{x}, t))\,\mathrm{d}v. \tag{4.17}$$

Observe that the right-hand side of $\mathcal{D}(t)$ in (4.16) contains an expression as integrand which, in the absence of radiation and when the body is emersed in an environment of constant temperature θ_0, is non-negative due to the second law of thermodynamics in the form of the CLAUSIUS–PLANCK inequality. Of course, in this circumstance the CLAUSIUS–PLANCK inequality is implied by the CLAUSIUS–DUHEM inequality.

Because of the dissipative nature of heat conduction and the fact that the mechanical loading $\bar{\mathbf{t}}(\mathbf{x})$ and $\bar{\mathbf{b}}(\mathbf{x})$ and the thermal loading conditions $(4.6)_2$ and $(4.7)_3$,

* See the work on the stability of material phases by Dunn and Fosdick [7, p. 41]. Duhem [6] introduced a similar quantity denoted by him "l'énergie balistique" in his studies on the stability of equilibrium states. Truesdell [15], in his Historical Introit on pp. 39–40, gives a brief account of Duhem's ballistic energy and its first appearances in the more modern researches of the 1960s. Today, the term "ballistic free energy" often is used to denote the sum of the total kinetic energy, the HELMHOLTZ *semi*-free energy and the total potential energy of the applied forces for the body, for certain special processes as, for example, in [2, Section 3.3]. Its main feature is that it is non-negative on these processes and this fact emphasizes its importance in stability analyses.

are associated with the equilibrium state $\mathbf{u} = \bar{\mathbf{u}}(\mathbf{x})$ and $\theta = \theta_0$, it is natural to expect, based on physical considerations, that any possible thermodynamic process, generated according to (4.6)–(4.9), will stabilize in the sense that $\mathbf{u}(\mathbf{x}, t) \to \bar{\mathbf{u}}(\mathbf{x})$ and $\theta(\mathbf{x}, t) \to \theta_0$ as $t \to \infty$.* Provided this asymptotic behavior** is, indeed, the case, we may conclude, from (4.13), (4.15), (4.16) and the fact that $\Psi_{\theta_0}[\theta, \mathbf{e}](t) \to \Psi[\theta_0, \bar{\mathbf{e}}]$, that

$$\overline{W}[\bar{\mathbf{u}}] = \Psi[\theta_0, \bar{\mathbf{e}}] + \mathcal{D}_\infty \tag{4.18}$$

in the limit $t \to \infty$, where

$$\begin{aligned} \mathcal{D}_\infty \equiv \mathcal{D}(\infty) &= \theta_0(H[\theta_0, \bar{\mathbf{e}}] - H[\theta_0, \mathbf{0}]) - \int_0^\infty Q(\tau)\,\mathrm{d}\tau \\ &= \int_0^\infty \left(\theta_0 \frac{\mathrm{d}}{\mathrm{d}t} H[\theta, \mathbf{e}](t) - Q(t)\right)\mathrm{d}t \geqslant 0. \end{aligned} \tag{4.19}$$

Thus, with (4.5) and (4.18) we see that half the work done to reach equilibrium is stored as HELMHOLTZ free energy and the remaining half is given by

* Of course, from an analytical point of view this will depend upon the constitutive structure for the law of heat conduction which, for classical linear theory, may be taken as Fourier's law (4.22).

** In the present context, this problem has yet to be studied. While Dafermos [4] has provided an analysis of the issues of existence and asymptotic stability for the completely linear theory of thermoelasticity, the initial-boundary value problem under consideration here is weakly nonlinear, due to thermal expansion, and slightly different. In its one-dimensional form the fields $u(x, t)$ and $\theta(x, t)$ are sought for $x \in (0, L)$ and for all $t > 0$ such that the dynamical and constitutive equations (4.2), (4.3), (4.8), (4.9) and (4.22) hold subject to null body force and appropriate boundary and initial conditions. Specifically, the governing equations are

$$\begin{aligned} &\sigma_x(x, t) = \rho \ddot{u}(x, t) \quad \forall x \in (0, L),\ \forall t > 0, \\ &\text{with} \quad \sigma(x, t) = E u_x(x, t) - \rho m(\theta(x, t) - \theta_0), \end{aligned}$$

and

$$k\theta_{xx}(x, t) = \rho(m\theta(x, t)\dot{u}_x(x, t) + c\dot{\theta}(x, t)) \quad \forall x \in (0, L),\ \forall t > 0,$$

subject to the following boundary and initial conditions:

$$\begin{aligned} &u(0, t) = 0, \qquad \sigma(L, t) = \bar{\sigma} = \text{const}, \qquad \theta(0, t) = \theta(L, t) = \theta_0 \quad \forall t > 0, \\ &u(x, 0) = \dot{u}(x, 0) = 0, \quad \theta(x, 0) = \theta_0 \quad \forall x \in (0, L). \end{aligned}$$

The material constants ρ, k, m, c and E are positive.

In the completely linear theory, the nonlinear term $\theta\dot{u}_x$ in the third equation above is linearized and replaced by $\theta_0\dot{u}_x$. For the system so linearized and within the more general three-dimensional setting, DAFERMOS has shown that the solution asymptotically and strongly approaches the equilibrium state of uniform temperature in the sense that

$$(u, e \equiv u_x, \sigma)(x, t) \to (\bar{u}, \bar{e}, \bar{\sigma})(x) = \left(\frac{\bar{\sigma}}{E}x, \frac{\bar{\sigma}}{E}, \bar{\sigma}\right), \qquad \theta(x, t) \to \theta_0$$

as $t \to \infty$.

$$\frac{1}{2}\overline{W}[\bar{\mathbf{u}}] = \mathcal{D}_\infty. \tag{4.20}$$

Following classical considerations, we may interpret the first term in the definition $(4.19)_1$ of $\mathcal{D}_\infty$, i.e., the term that involves the total entropy difference, as that part of the change of the total internal energy that is stored in the distorted equilibrium state of the body in the 'primative form of heat' and that is unavailable to do mechanical work at the temperature $\theta = \theta_0$. This is historically referred to as the 'bound' part. Of course, the total HELMHOLTZ free energy $\Psi[\theta_0, \bar{\mathbf{e}}]$ represents the remaining part of the total internal energy, and it is available. According to the definition (4.14), the second term in $\mathcal{D}_\infty$, in $(4.19)_1$, represents the total heat exchange for the body due to the process of conduction (i.e., 'transfer') through its boundary during the thermodynamic process.

Finally, to clearly identify (4.19) as an expression for the dissipated energy due to the internal heat transfer, we first note that with (4.14), (4.17), the divergence theorem, (4.10) and $(4.6)_2$ we may re-write $\mathcal{D}_\infty$ as

$$\begin{aligned}\mathcal{D}_\infty &= \int_0^\infty \left(\int_\Omega (\rho\theta_0\dot{\eta} + \operatorname{div}\mathbf{q})\,\mathrm{d}v \right)\mathrm{d}t \\ &= \int_0^\infty \left(\int_\Omega \left(1 - \frac{\theta_0}{\theta}\right) \operatorname{div}\mathbf{q}\,\mathrm{d}v \right)\mathrm{d}t \\ &= \int_0^\infty \left(\theta_0 \int_\Omega -\frac{\mathbf{q}\cdot\nabla\theta}{\theta^2}\,\mathrm{d}v \right)\mathrm{d}t. \end{aligned} \tag{4.21}$$

Then, as is standard within the linear theory of thermoelasticity, if we assume FOURIER'S law of heat conduction, i.e.,

$$\mathbf{q} = -\mathbf{K}\nabla\theta, \tag{4.22}$$

where $\mathbf{K}$ is the positive definite, symmetric heat conductivity tensor, we see that

$$\mathcal{D}_\infty = \theta_0 \int_0^\infty \left(\int_\Omega \frac{(\mathbf{K}\nabla\theta)\cdot\nabla\theta}{\theta^2}\,\mathrm{d}v \right)\mathrm{d}t \geqslant 0. \tag{4.23}$$

Accordingly, in the case of continuum thermoelasticity the expression (4.23) gives an explicit representation for the total dissipated energy that was identified as $\mathcal{D}$ in our previous more elementary discussion (see (D)). Through (4.20), it accounts for the remaining half of the work that is done to reach equilibrium and provides a thermodynamics based response to the paradox posed in Section 1.

5. Discussion

In this communication we have revisited a well known classical theorem in linear elastostatics due to Emile Clapeyron and offered several interpretations of an apparent paradox associated with the 'mysterious' unaccountability of part of the work done by the loading device to reach equilibrium. Our considerations reveal that this theorem may be viewed in a purely statical framework as a mechanical statement concerning *work* and *elastic strain energy* as did Love [11], and that is where the paradox appears, or it can be viewed more generally as a thermodynamical statement concerning the *work* and the HELMHOLTZ *free energy*, in which case no paradox emerges. We consider the 'thermodynamic' version of CLAPEYRON's theorem, as noted in (4.5), to be the most reasonable one; the issue does not appear to have been addressed previously in the literature.

Within elastostatics, the purely mechanical statement of CLAPEYRON's theorem is ambiguous because only equilibrium ideas are used to deduce it and, therefore, the definition of 'work' is somewhat subjective. In practice, an elastic body adjusts to the application of a loading gradually and part of the associated work is transformed during this *process* into an energy of 'ringing' relative to some average configuration. This 'ringing' may be sizable or negligible depending upon the rate at which the ultimate load is attained. Coincidently, this energy is being removed from the system by the unavoidable action of dissipation and the body tends to an equilibrium state. If, in a particular setting, the process of reaching equilibrium is considered instantaneous relative to the time-scale defined by the physical problem, then the classical theorem applies and the unaccounted work should be considered lost through dissipation. In this case, one can suppose that there is a fast time-scale in the problem and that the associated generation of high frequency vibrations can be considered, from the slower time-scale point of view, to be an effective dissipative action.

We note that circumstances in which some energy may be either 'lost' or 'acquired' are not unknown within the setting of a purely conservative elastic system. For example, when considering *steady state solutions* of linear elastodynamic problems, one characteristically neglects short transient periods in determining the corresponding steady states from prescribed initial conditions. One of the energetic consequences of such a neglect of the transient phase of the process is the necessity to apply so-called radiation conditions in order to determine a unique steady state configuration. Another example originates in nonlinear elastodynamics where the energy is not conserved due to the unavoidable generation of the 'invisible' high frequency vibrations inside the transition layer of *shock waves*.

If the HELMHOLTZ free energy is used instead of the elastic strain energy and the problem is viewed as thermodynamical from the very beginning, the paradox does not surface. The reason is that in this case the system no longer is considered to be energetically closed and the 'macro-mechanical' degrees of freedom are not the only ones present in the system. More specifically, in this case, the adjustment of the body to the applied dead loads involves the activation of the 'micro-mechanical' degrees of freedom not accounted for by the purely mechanical macro-description. The channeling of the macroscopic energy towards these microscopic degrees of freedom is then viewed at the macro-level as the dissipation. The beauty of a continuum thermodynamical description is that these degrees of freedom need not be described explicitly.

Acknowledgements

We wish to thank Chi-Sing Man, J.J. Marigo and W. Warner for helpful comments. L.T. also acknowledges, with appreciation, discussions with I. Müller, P. Podio-Guidugli, J. Rice and K. Wilmanski. We gratefully acknowledge E. Petersen for supplying the calculations and figures related to Appendix B.

Appendix A. 1D Example: 'Dead' Loading

To exemplify the general conclusions reached in Section 2.1 concerning the dynamical implications of 'dead' loading, consider the specific one-dimensional elastodynamic problem of determining the displacement field $u(x,t)$ for $x \in (0, L)$ and for all time $t > 0$ such that

$$Eu_{xx}(x,t) = \rho\ddot{u}(x,t) \quad \forall x \in (0,L),\ \forall t > 0, \tag{A.1}$$

subject to the following boundary and initial conditions:

$$u(0,t) = 0, \qquad \sigma(L,t) = \bar{\sigma} = \text{const} \quad \forall t > 0, \tag{A.2}$$

$$u(x,0) = \dot{u}(x,0) = 0 \qquad\qquad \forall x \in (0,L). \tag{A.3}$$

Here, $E > 0$ is the (constant) Young's modulus and $\sigma(x,t) \equiv Eu_x(x,t)$ denotes the stress.

It is straightforward to show that the solution of (A.1)–(A.3) is periodic in time with period $T = 4L/c$, where $c \equiv \sqrt{E/\rho}$ is the characteristic wave speed, and that in the (x,t)-plane the strain and velocity fields, $e(x,t) \equiv u_x(x,t)$ and $v(x,t) \equiv \dot{u}(x,t)$, are piecewise constant and of the form shown in Figure 1. Moreover, in this one-dimensional setting (2.7) again holds, i.e.,

$$\overline{W}[u](t) = U[e](t) + K[v](t) \quad \forall t \geqslant 0, \tag{A.4}$$

where

$$\begin{aligned} &\overline{W}[u](t) \equiv \bar{\sigma}u(L,t), \qquad U[e](t) \equiv \int_0^L \frac{1}{2}Ee^2\,\mathrm{d}x, \\ &K[v](t) \equiv \int_0^t \frac{1}{2}\rho v^2\,\mathrm{d}x. \end{aligned} \tag{A.5}$$

Thus, from the solution shown in Figure 1 we may readily construct the periodic forms of $\overline{W}[u](t)$, $U[e](t)$ and $K[v](t)$ and they are illustrated in Figure 2.

Now, to analyze these results it is helpful to first note that the unique equilibrium displacement $\bar{u}(x)$, strain $\bar{e}(x)$ and stress $\bar{\sigma}(x)$ fields which correspond to the boundary conditions

$$\bar{u}(0) = 0, \quad \bar{\sigma}(L) = \bar{\sigma}$$

are given by $\bar{u}(x) = (\bar{\sigma}/E)x$, $\bar{e}(x) = \bar{\sigma}/E$ and $\bar{\sigma}(x) = \bar{\sigma}$ for $x \in (0,L)$. In this case, CLAPEYRON's theorem implies that

$$\frac{1}{2}\overline{W}[\bar{u}] = U[\bar{e}] \tag{A.6}$$

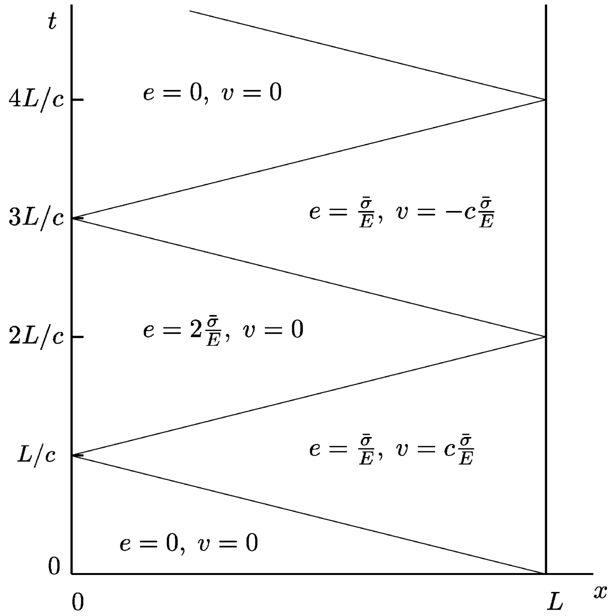

Figure 1. Summary of the solution of (A.1)–(A.3) in the (x, t)-plane.

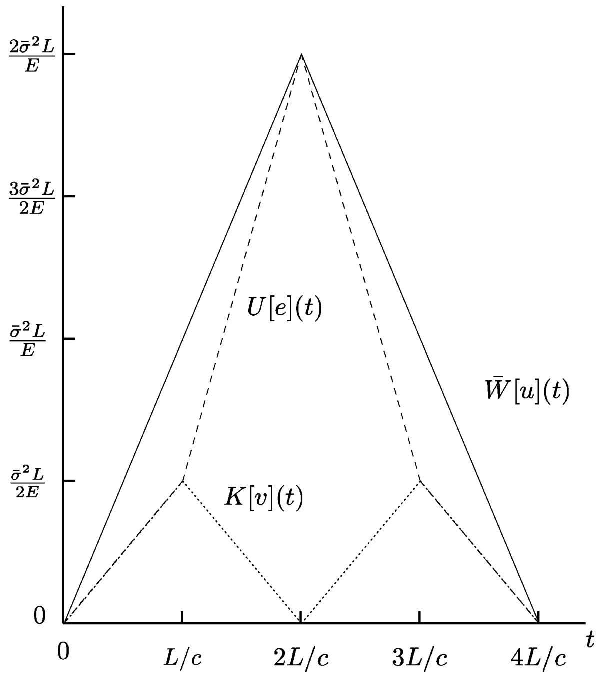

Figure 2. The total work $\overline{W}[u](t)$, strain energy $U[e](t)$ and kinetic energy $K[v](t)$ during one period of motion.

and we easily calculate, using (A.5), that

$$\overline{W}[\bar{u}] = \frac{\bar{\sigma}^2}{E} L, \qquad U[\bar{e}] = \frac{1}{2}\frac{\bar{\sigma}^2}{E} L. \tag{A.7}$$

Notice from Figure 1 that at the discrete times $t = \bar{t} \in \{L/c, 3L/c, \ldots\}$ the displacement and strain fields coincide with those of the equilibrium state, $u(x, \bar{t}) = \bar{u}(x)$ and $e(x, \bar{t}) = \bar{e}(x)$. Thus, from $(\text{A.7})_1$ and Figure 2 we see that

$$\overline{W}[u](\bar{t}) = \overline{W}[\bar{u}], \qquad U[e](\bar{t}) = K[v](\bar{t}) = \frac{1}{2}\overline{W}[\bar{u}]. \tag{A.8}$$

This verifies (2.9) and explicitly shows that at those times when the dynamical displacement field coincides with the equilibrium displacement field, half the work done is stored as strain energy and the remaining half appears as kinetic energy. In passing, we note from Figure 1 that at time $t = 2L/c$ (and periodically thereafter) the body is at rest and it is distorted with a strain field that is *double* what it is in equilibrium. Moreover, from Figure 2 we see that at this time there is a total 'work-energy balance' in the sense that $\overline{W}[u](2L/c) = U[e](2L/c)$. This is a reflection of Poncelet's observation noted earlier in the first footnote of Section 2.1.

Observe, from Figure 1, that $v(x, t^*) = 0\ \forall x \in (0, L)$ and for every $t^* \in \{2L/c, 4L/c, \ldots\}$. Thus, by time-averaging (A.4) over any interval $(0, t^*)$ and using a notation analogous to (2.10) it is clear that

$$\langle \overline{W}[u] \rangle = \overline{W}[\langle u \rangle] = \langle U[e] \rangle + \langle K[v] \rangle, \tag{A.9}$$

where, according to Figure 2 and $(\text{A.7})_1$, we easily calculate

$$\langle \overline{W}[u] \rangle = \overline{W}[\bar{u}], \qquad \langle U[e] \rangle = \frac{3}{4}\overline{W}[\bar{u}], \qquad \langle K[v] \rangle = \frac{1}{4}\overline{W}[\bar{u}], \tag{A.10}$$

in agreement with results more generally obtained in Section 2.1. In addition, from the periodic extension of Figure 2 and the value of $\overline{W}[\bar{u}]$ in (A.7), we readily see that the infinite time-average, constructed analogous to (2.22) for this one-dimensional example, satisfies the general conditions recorded in (2.23), i.e.,

$$\langle \overline{W}[u] \rangle_\infty = \langle U[e] \rangle_\infty + \langle K[v] \rangle_\infty,$$

where

$$\langle \overline{W}[u] \rangle_\infty = \overline{W}[\bar{u}], \qquad \langle U[e] \rangle_\infty = \frac{3}{4}\overline{W}[\bar{u}], \qquad \langle K[v] \rangle_\infty = \frac{1}{4}\overline{W}[\bar{u}].$$

Appendix B. 1D Example: Retarded Loading

In order to exhibit more precisely how the solution of an elastodynamics problem may depend on the slowness of the applied loading, we consider another one-dimensional elastodynamic problem of determining $u(x, t)$ for $x \in (0, L)$ and for all time $t > 0$ such that

$$Eu_{xx}(x, t) = \rho\ddot{u}(x, t) \quad \forall x \in (0, L),\ \forall t > 0, \tag{B.1}$$

subject to the following boundary and initial conditions:

$$u(0, t) = 0, \qquad \sigma(L, t) = (1 - \mathrm{e}^{-\alpha t})\bar{\sigma} \quad \forall t > 0, \tag{B.2}$$

$$u(x, 0) = \dot{u}(x, 0) = 0 \quad \forall x \in (0, L). \tag{B.3}$$

Here, $\alpha > 0$ represents a 'slowness' load parameter which governs the length of time it takes the applied end load to essentially reach the constant value $\bar{\sigma}$. For sufficiently large α, the loading in (B.2) is nearly impulsive and this problem then reduces to that of Appendix A. As α is reduced the loading becomes more retarded and the solution is expected to show less of a dynamic structure. Of course, analogous to (2.7) the mechanical energy balance again holds, so that

$$W[u](t) = U[e](t) + K[v](t), \quad \forall t \geqslant 0, \tag{B.4}$$

where the work done on the body up to time t is now determined by

$$W[u](t) = \int_0^t \sigma(L, \tau)\dot{u}(L, \tau)\,\mathrm{d}\tau, \tag{B.5}$$

and where the corresponding strain energy, $U[e](t)$, and corresponding kinetic energy, $K[v](t)$, are as defined in (A.5).

One of the major questions concerning the solution of the dynamical problem stated above is how the work, strain energy and kinetic energy vary with time relative to the strain energy that would be stored in the same elastic bar in equilibrium under the constant end load $\bar{\sigma}$, i.e., $U[\bar{e}]$ of $(A.7)_2$. In Figures 3–5 we show the normalized work, $W[u](t)/U[\bar{e}]$, normalized strain energy, $U[e](t)/U[\bar{e}]$, and normalized kinetic energy, $K[v](t)/U[\bar{e}]$, as functions of time computed numerically for this problem for a range of slowness load parameters α between $\alpha = 10^4$ sec^{-1} and $\alpha = 10^6$ sec^{-1}. These figures are based on material constants for an aluminum alloy with $E = 76.1 \times 10^9$ Pa and $\rho = 2710$ kg/m^3, for a bar of length $L = 5 \times 10^{-3}$ m, and for a load constant $\bar{\sigma} = 10^7$ Pa. The time axis of these figures is measured in 'time steps' with the final time step of 129760 corresponding to 1200×10^{-6} sec.

One can see that the impulsive-like nature of the loading for large α results in wildly irregular behavior which is sustained over an infinite time. On the contrary, for relatively small α equilibrium appears to be achieved quickly in time with nearly constant limiting values $W[u](t)/U[\bar{e}] \approx 1$, $U[e](t)/U[\bar{e}] \approx 1$ and $K[v](t)/U[\bar{e}] \approx 0$. We conclude that the quantity $\overline{W}[\bar{u}]$ in $(A.7)_1$, while it has units of work and shows up in CLAPEYRON's theorem as exhibited in (A.6), does not represent the work done to reach equilibrium; reasoning based on the computed limiting behavior leads to the conclusion that only half of this value is expended to reach equilibrium and, then, it is manifested totally in the form of strain energy.

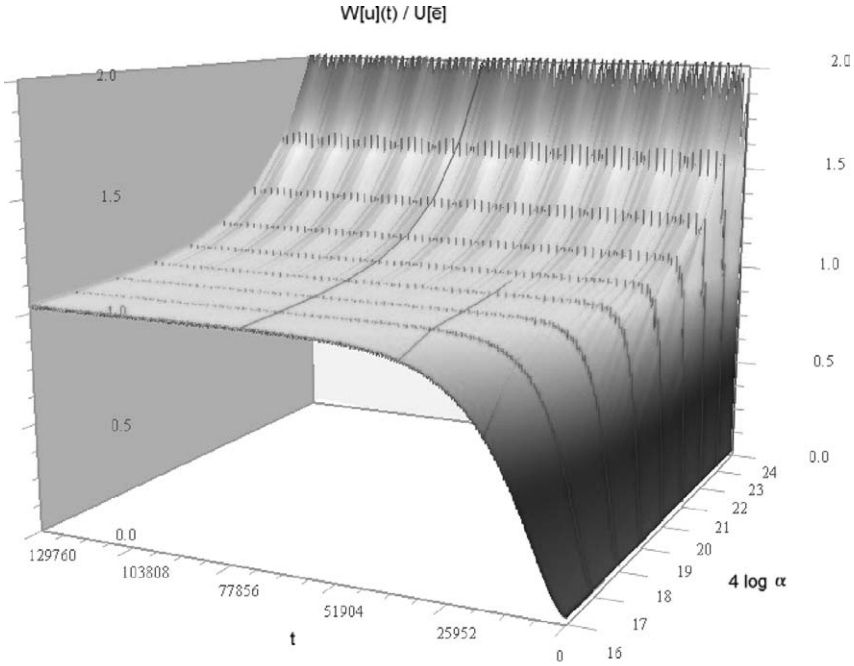

Figure 3. Normalized work $W[u](t)/U[\bar{e}]$ as a function of time for various slowness load values α.

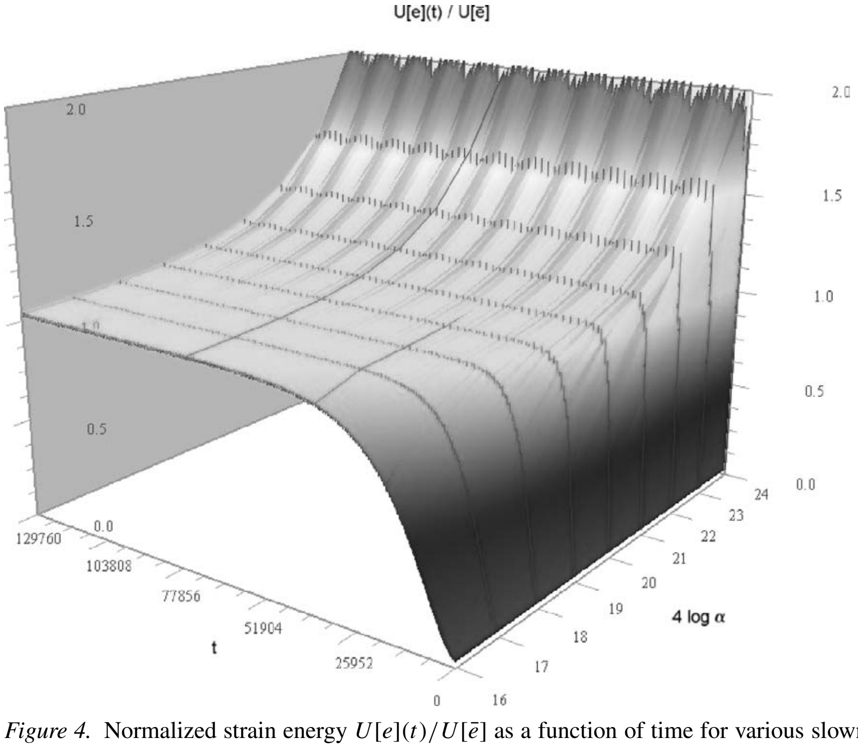

Figure 4. Normalized strain energy $U[e](t)/U[\bar{e}]$ as a function of time for various slowness load values α.

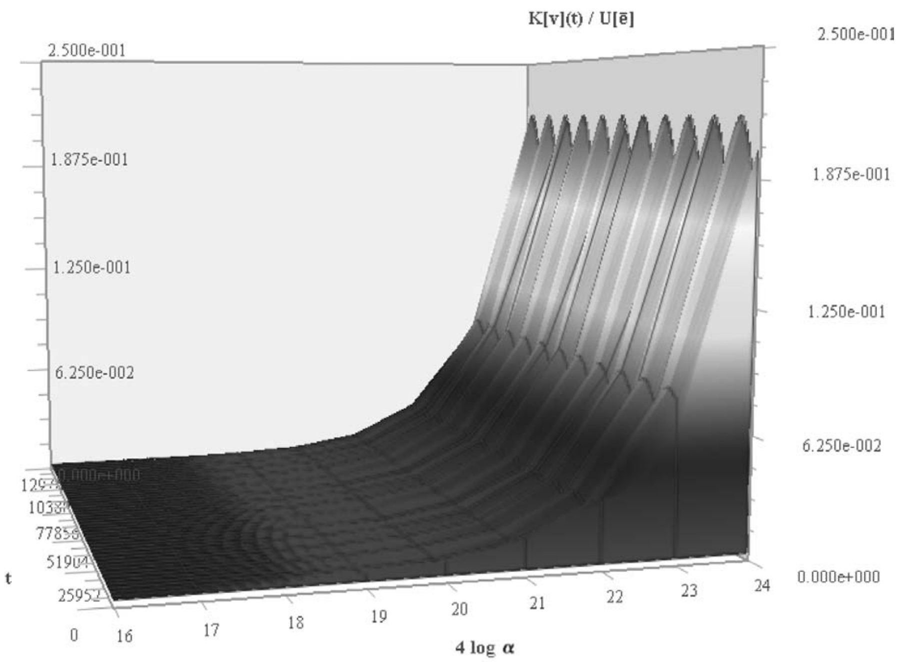

Figure 5. Normalized kinetic energy $K[v](t)/U[\bar{e}]$ as a function of time for various slowness load values α.

Because there are three decades of variation of the slowness load parameter α shown in Figures 3–5, there is much highly oscillatory, rapid time-behavior that is not resolved in these figures. Therefore, in Figures 6–10, we take $\alpha = 10^5$ sec^{-1} and show a more detailed solution of (B.1)–(B.3). The material constants E and ρ, bar length L and load constant $\bar{\sigma}$ are the same as noted above, but the time steps for the time-axis is now such that the final time step of 12800 corresponds to 120×10^{-6} sec. In Figure 6, we see that the strain field $e(x, t)$ is highly irregular in time at the fixed end $x = 0$ where information from the time-dependent loading at the end $x = L$ is reflected back into the bar. The length-axis of this figure is measured in 'length steps' with the final length step of 100 corresponding to 5×10^{-3} m which is the length of the bar. In Figures 7 and 8, we show the normalized total work done $W[u](t)/U[\bar{e}]$ and the normalized kinetic energy $K[v](t)/U[\bar{e}]$ as functions of time. These correspond to the $\alpha = 10^5$ sec^{-1} cross sections of Figures 3 and 5, respectively, for the initial time interval (0, 12800) as noted in these figures. The normalized strain energy $U[e](t)/U[\bar{e}]$ is not shown, but behaves similar to Figure 7. Notice the orders of magnitude reduction of the energy scale used in exhibiting the kinetic energy in Figure 8. In Figures 9 and 10, we show the ratios $U[e](t)/W[u](t)$ and $K[v](t)/W[u](t)$ as functions of time in order to illustrate that it takes only a few 'rings' to almost completely eliminate the total kinetic energy in the bar. Of course, a small motion remains in the bar for all time no matter how small the slowness parameter $\alpha > 0$.

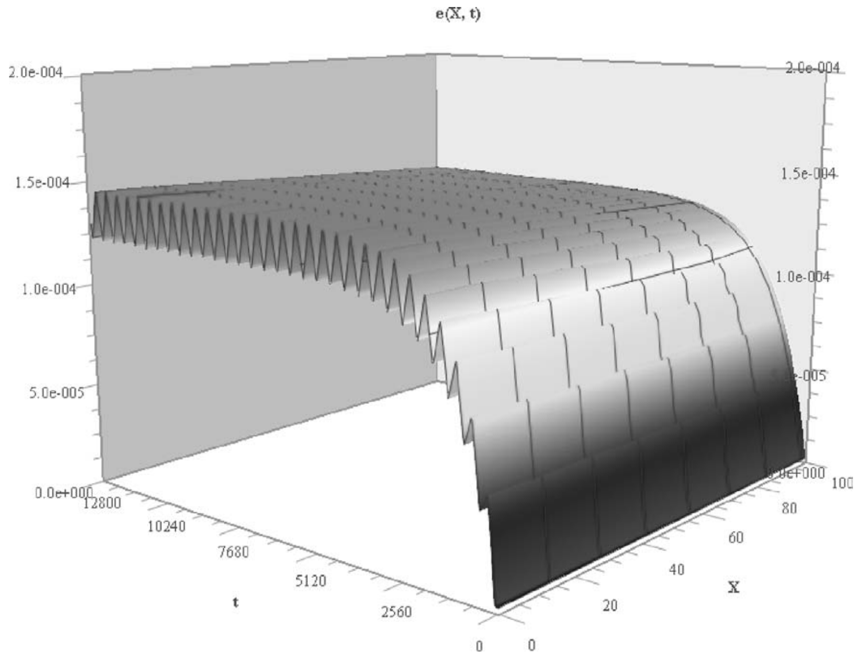

Figure 6. Strain $e(x, t)$ as a function of axial position and time for $\alpha = 10^5$ $\sec^{-1}$.

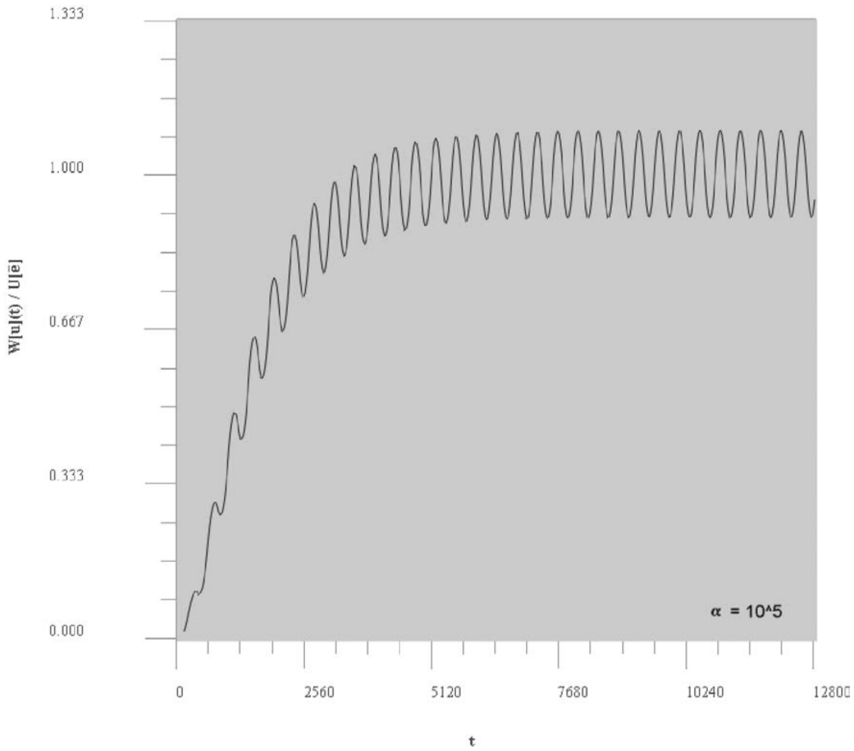

Figure 7. $W[u](t)/U[\bar{e}]$ vs. t: $\alpha = 10^5$ $\sec^{-1}$.

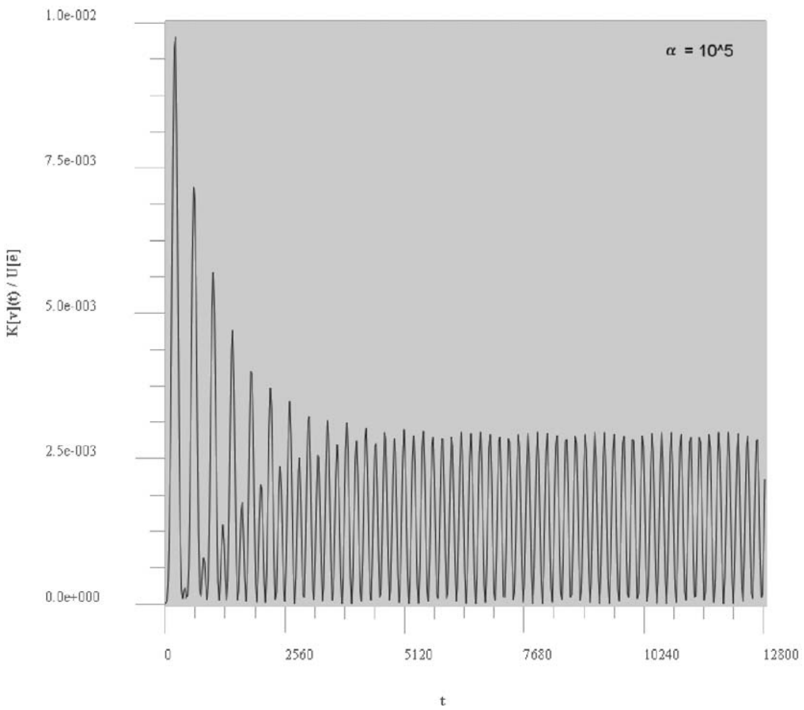

Figure 8. $K[v](t)/U[\bar{e}]$ vs. t: $\alpha = 10^5$ sec^{-1}.

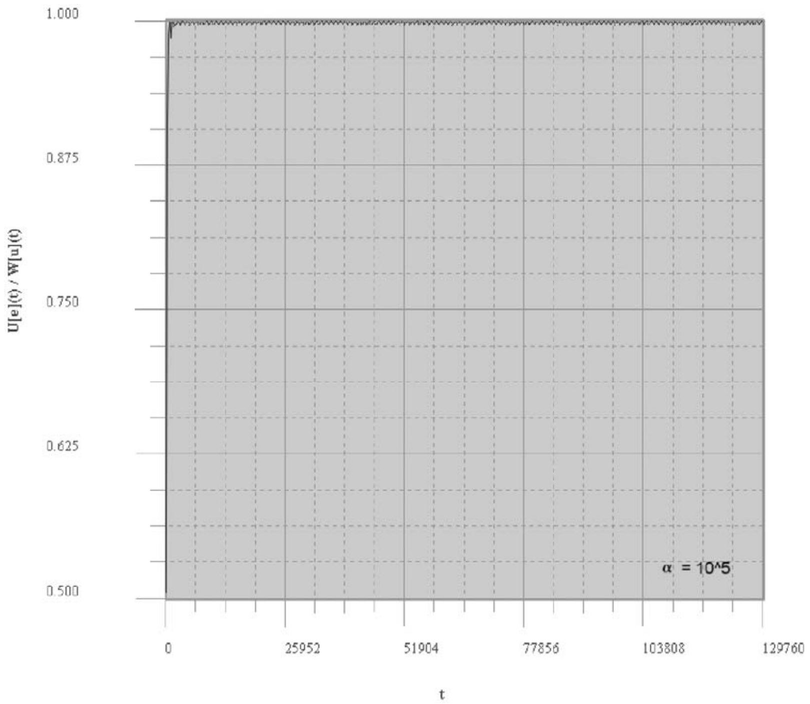

Figure 9. $U[e](t)/W[u](t)$ vs. t: $\alpha = 10^5$ sec^{-1}.

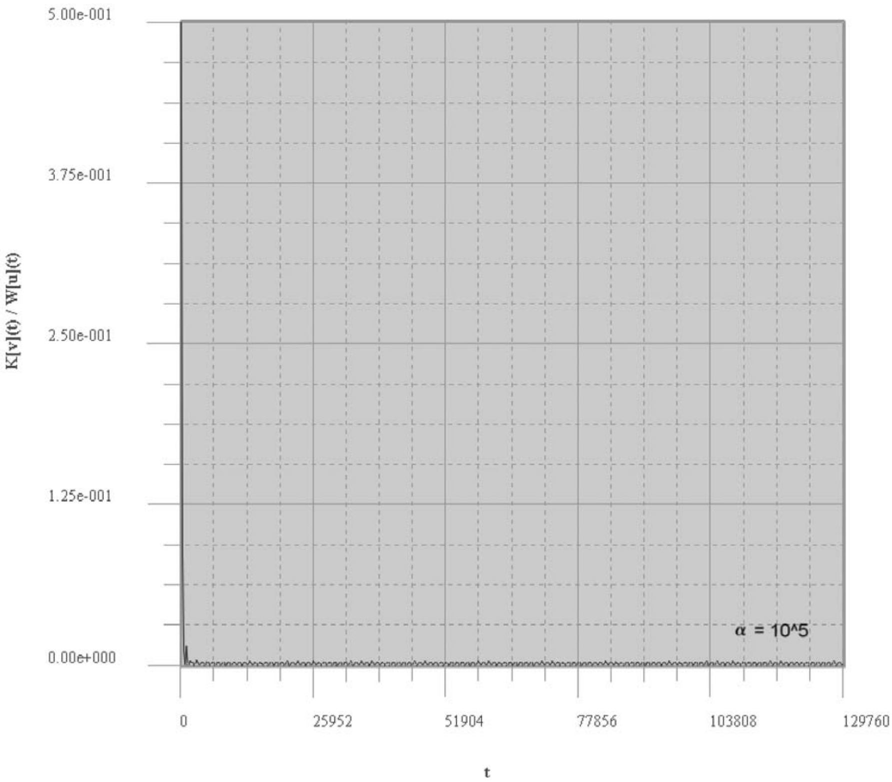

Figure 10. $K[v](t)/W[u](t)$ vs. t: $\alpha = 10^5\ \text{sec}^{-1}$.

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