A one-dimensional model for localized and distributed failure

G. del Piero and L. Truskinovsky*

Facoltà di Ingegneria, via Saragat 1, 44100 Ferrara, Italy
** Department of Aerospace Engineering and Mechanics, University of Minnesota, 110 Union Street, Minneapolis, MN 55455, U.S.A.*

Abstract: For an elastic body with limited strength, the equilibrium configurations can be obtained by minimization of an energy functional containing two contributions, bulk and cohesive: the bulk energy is a function of strain and the cohesive energy is a function of the relative displacement on a surface of discontinuity. In the present communication we consider the simplest one-dimensional problem for a bar with this type of energy in a hard device. We assume that the bulk energy is convex, and we vary the concavity propertiess of the cohesive energy, obtaining thereby three distinct modes of failure. If the cohesive energy is concave for all admissible displacements, failure occurs with the formation of a single crack, and the opening of the crack may be either abrupt or gradual, depending on the length of the bar. If the cohesive energy is concave at large displacements but convex at the origin, the deformation may progress at constant stress (yielding), through formation of an infinite numer of infinitesimal cracks (structured deformation). Finally, when the cohesive energy is characterized by two domains of concavity, (in the vicinity, and far away from the origin), separated by a domain of convexity, fracture procedes through a successive formation of a finite number of cracks of small but finite size. We conjecture that the different modes of fracture, produced by this simple model, may be associated with various experimentally well-documented regimes of localized and distributed damage.

1. INTRODUCTION

A standard model in fracture mechanics is based on the assumption that the total energy of a body is a sum of a bulk term, representing the strain energy, and of a surface term, representing the energy associated with the displacement discontinuity. This assumption was introduced by Griffith, and modified later by Barenblatt to account for the cohesive forces which oppose fracture opening.

A one-dimensional model of this type with convex bulk energy and concave cohesive energy is capable of reproducing a phenomenon of *localized fracture*: for a bar subject to a prescribed elongation, the minimum of the total energy corresponds to configurations with a single crack [2, 4, 5]. The overall response in this case may be either discontinous, with an abrupt drop of stress, or gradual, with a continuous decrease of stress, depending on the length of the bar. These two regimes, which one can loosely associate with *brittle* and *ductile* fracture, can also be obtained from a model of a chain made of nonlinear springs with a Lennard-Jones potential: for a sufficiently large number of springs, this discrete system can be approximated by a continuum model with a convex bulk and a concave cohesive energy [8].

This type of behavior changes drastically if the cohesive energy is convex at the origin and concave away from the origin [5]. In this case, for elongations belonging to a certain interval, there are no piecewise continuous minimizers, and the energy minimum is attained at a configuration with an infinite number of infinitesimal cracks. This situation can be described in the context of the theory of *structured deformations* [6]. The resulting expression of the energy minimum in the one-dimensional model agrees with the three-dimensional relaxation in the class of functions with this level of regularity [3].

In the present communication we report some preliminary results for a one-dimensional model with a cohesive energy which is convex on a finite segment separated from the origin and concave outside. As we show, this model predicts the formation of a finite number of cracks, one after another, as the total elongation increases [7]. This regime of distributed cracking can be viewed as a "quantized" propagation of *damage*. When the concave region near the origin shrinks to zero the model recovers structured deformations, and when the convex region disappears a localized fracture appears as another limiting case.

Article published online by EDP Sciences and available at http://dx.doi.org/10.1051/jp4:1998812

The three models of interfacial energy which we compare in this paper may be related to three different types of interatomic interactions, with range not necessarily limited to nearest neighbours [7]. Within this interpretation, macroscopically significant differences in material behavior are attributed to structural changes at the microscopic level.

2. EQUILIBRIUM AND METASTABILITY

Consider a bar of length l, with strain energy of the form

$$E(f) = \int_0^l w(f'(x))\, dx + \sum_{x\in\Gamma(f)} \theta([f](x)) , \tag{2.1}$$

where f is the axial deformation of the bar, f' is the deformation gradient, $[f](x)$ is the jump of f at the point x, and $\Gamma(f)$ is the set of all jump points of f. The bar is subject to a prescribed elongation β, so that

$$\beta l = \int_0^l f'(x)\, dx + \sum_{x\in\Gamma(f)} [f](x) \tag{2.2}$$

is the length of the bar after deformation. To exclude interpenetration of matter, we make the assumptions

$$f'(x) > 0 \quad \forall x\in(0, l) , \qquad [f](x) > 0 \quad \forall x\in\Gamma(f) , \tag{2.3}$$

which ensure the injectivity of f. The functions w and θ are the *bulk* and the *cohesive (interfacial)* energy; they are assumed to be twice differentiable and to satisfy the conditions $w(1) = w'(1) = 0$ and $\theta(0+) = 0$, $\theta'(0+) > 0$. We suppose that w is strictly convex, while for θ different possibilities will be taken into consideration:

(i) θ is strictly concave,
(ii) θ is convex in a right neighborhood of the origin and strictly concave outside,
(iii) θ is convex in an interval away from the origin and strictly concave outside.

The graphs of w, θ, and of their derivatives w' and θ' are shown in Fig.1.

Equilibrium configurations are characterized by the non-negativeness of the first variation of E when the unilateral constraint $(2.3)_2$ holds with the equality sign, and by the vanishing of the first variation otherwise. A necessary condition for equilibrium is that the deformation gradient f' be constant across the bar. Other necessary conditions are [4, 7]:

$$w'(f') = \theta'([f](x)) \qquad \forall x\in\Gamma(f) , \tag{2.4}$$

$$w'(f') \leq \theta'(0+) . \tag{2.5}$$

In particular, the deformation $f'_m := (w')^{-1}(\theta'(0+))$ is an upper bound for f', and $\theta'(0+)$ is an upper bound for θ'. With reference to Fig.1b, the latter condition rules out as unstable all jumps in the interval $(0, c)$ for energies of type (ii), and in the interval (b, c) for energies of type (iii).

Among all equilibrium configurations, of interest are those which are *metastable*, i.e., which are local minimizers of the energy. The analysis of the second variation provides the inequality

$$w''(f') + l\, \theta''([f](x)) \geq 0 \qquad \forall x\in\Gamma(f) , \tag{2.6}$$

and the condition that there may be no more than one jump point with $\theta'' \leq 0$ [4, 5, 7]. If N is the number of jumps, we conclude that N≤1 in cases (i) and (ii), in which only negative values of θ'' are allowed. In case (iii), there may be any number of jumps, but at least (N-1) of them must belong to the ascending branch of the $(\theta', [f])$ curve.

Let us focus on case (iii) and consider an equilibrium configuration with (N-1) jumps on the ascending branch and one in either of the descending branches. Denote by θ''^+, θ''^- the corresponding values of $\theta''(f')$; clearly, $\theta''^- < 0 < \theta''^+$. One can prove that the condition

$$l\,\theta''^-\theta''^+ + w''(f')\,((N\text{-}1)\theta''^- + \theta''^+) \geq 0\,, \tag{2.7}$$

is necessary for metastability. It can also be shown that Equation (2.4) and the strict inequalities (2.5)-(2.7) form a set of sufficient conditions, relative to a class of perturbations in which both variations of existing discontinuities and magnitudes of newly created discontinuities are small. In a more general class of perturbations, which allow for splitting of the existing discontinuities, the necessary condition (2.5) must be replaced by the stronger condition

$$\theta([f](x)) \;\leq\; \theta(\lambda[f](x)) + \theta((1\text{-}\lambda)[f](x)) \qquad \forall\lambda\in(0,1)\,, \tag{2.8}$$

and the sufficient conditions must be relaxed accordingly [7].

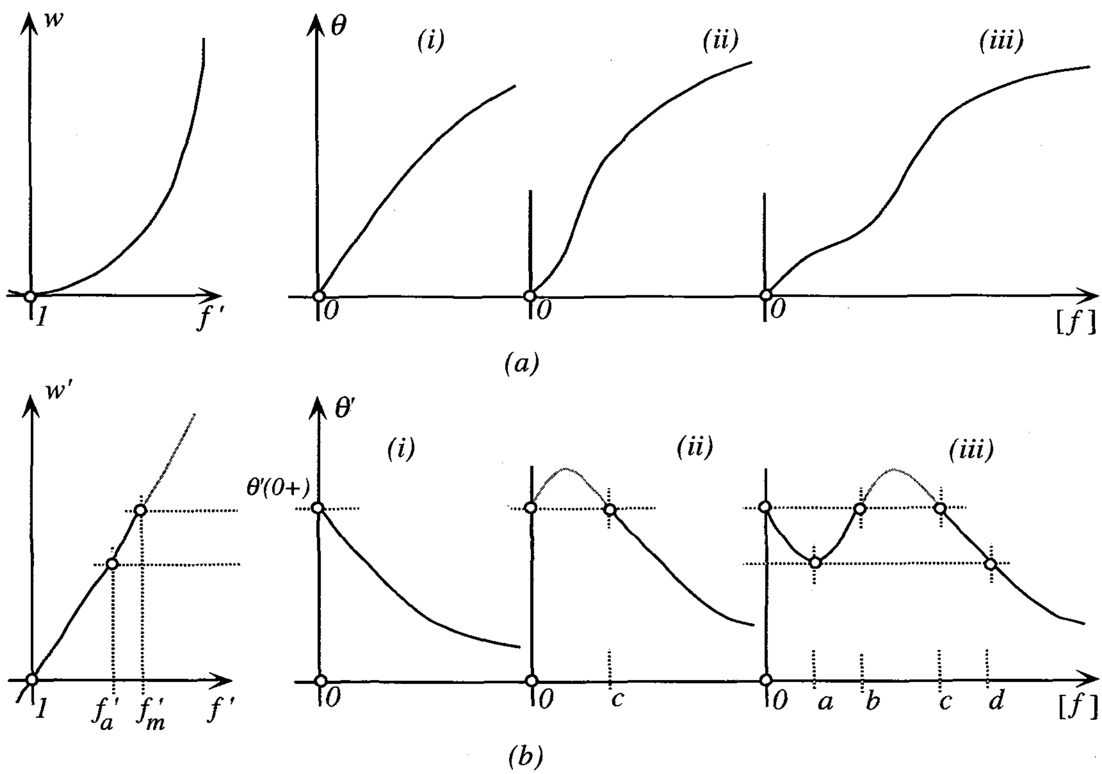

Figure 1: Bulk energy w and interfacial energies θ, (a), and corresponding stresses, (b).

To illustrate the inequality (2.7), consider the jumps a, d and the deformation f'_a shown in Fig.1b and satisfying $w'(f'_a) = \theta'(a) = \theta'(d)$. In this case, $\theta''^+ = \theta''(a) = 0$ and $\theta''^- = \theta''(d) < 0$, so that the inequality (2.7) is violated; by continuity, it is also violated at all jumps in a left neighborhood of d, when coupled with the corresponding jumps in a right neighborhood of a.

The inequality (2.7) has a simple interpretation in terms of the slope of the overall stress-strain relation. Assume that, for a given overall strain β, there is an equilibrium configuration with N-1 jumps on the ascending branch and one in either of the descending branches of the $(\theta', [f])$ curve, characterized by the deformation gradient $f'(\beta)$ and by the values $\theta''^{-}(\beta)$, $\theta''^{+}(\beta)$ of θ''. Denote by $\hat{E}(\beta)$ and $\hat{\sigma}(\beta)$ the corresponding values of energy and stress. It may be shown [7] that

$$\hat{\sigma}(\beta) = l^{-1}\frac{d}{d\beta}(\hat{E}(\beta)) = w'(f'(\beta)) \ , \tag{2.9}$$

and that

$$l\frac{d}{d\beta}(\hat{\sigma}(\beta)) = \frac{w''(f'(\beta))\,\theta''^{-}(\beta)\,\theta''^{+}(\beta)}{l\,\theta''^{-}(\beta)\,\theta''^{+}(\beta) + w''(f'(\beta))\,((N-1)\theta''^{-}(\beta)+\theta''^{+}(\beta))} \ . \tag{2.10}$$

In the right-hand side of the last equation, the denominator is non-negative by condition (2.7), while the numerator is negative because $\theta''^{-}(\beta)$ is negative. Therefore, among all equilibrium configurations of the considered type, only those located on a descending branch of the overall stress-strain curve (σ, β) can be metastable. This conclusion remains true for configurations, present in all cases (i) to (iii), with only one jump, located on the descending branch of the $(\theta', [f])$ curve [4, 5, 7].

Finally we notice that, for all three cases, equilibrium configurations with no jumps and with $f' < f'_m$ are metastable. This is also true for equilibrium configurations in case (iii), with all jumps on the ascending branch of the $(\theta', [f])$ curve [7].

3. GLOBAL ENERGY MINIMA AND STRESS-STRAIN CURVES

In this section we study the relation between the total energy E and the overall strain β, and we demonstrate that this relation may be of a very different nature, depending on the form of the cohesive energy. For each of the three cases listed in the preceding section, explicit investigation of a complete set of equilibrium configurations at a given load β identifies the configurations corresponding to the absolute minimum of the energy functional. As a by-product of this analysis, in each case we recover the overall stress-strain relation corresponding to the path of global minimization.

3.1 The cases (i) and (ii)

For a cohesive energy of type (i), both functions w' and θ' are monotonic. Equilibrium configurations may be of two types: with no jumps (bar without cracks) and with one jump (bar with a crack).

In the first case, Equations (2.1) and (2.2) reduce to

$$E(f) = l\,w(f') \ , \qquad \beta = f' \ , \tag{3.1}$$

and the energy can be expressed as a function of the parameter β

$$\hat{E}(\beta) = l\,w(\beta) \ , \tag{3.2}$$

with β in $(1, f'_m)$, as required by the equilibrium condition (2.5). The stress-strain relationship

$$\hat{\sigma}(\beta) = w'(\beta) \tag{3.3}$$

then follows from Equation (2.9); it is one-to-one and monotonic.

In the second case, the energy $\hat{E}(\beta)$ is given parametrically by the conditions

$$E(f) = l\,w(f') + \theta([f]) \ , \tag{3.4}$$

$$\beta = f' + l^{-1}[f] \ , \tag{3.5}$$

$$w'(f') = \theta'([f]) \ , \tag{3.6}$$

and an elementary analysis shows that the corresponding stress-strain relation is neither monotonic nor one-to-one. The two branches with N=0 and N=1are represented in Fig.2.i.b; the N=0 branch is an ascending curve, which becomes a segment of a straight line in the special case, shown in the figure, of a quadratic bulk energy w; it is metastable for $\beta < f'_m$ and unstable for $\beta > f'_m$. The N=1 branch is a curve which for large l may have an unstable ascending part. The two branches meet at the point A', where $\beta = f'_m$. For $\beta > f'_m$ a decrease of the stress, and therefore of the carrying capacity of the bar, occurs. The point A' can then be regarded as the onset of fracture; hence, in this model fracture is described as a bifurcational phenomenon [8].

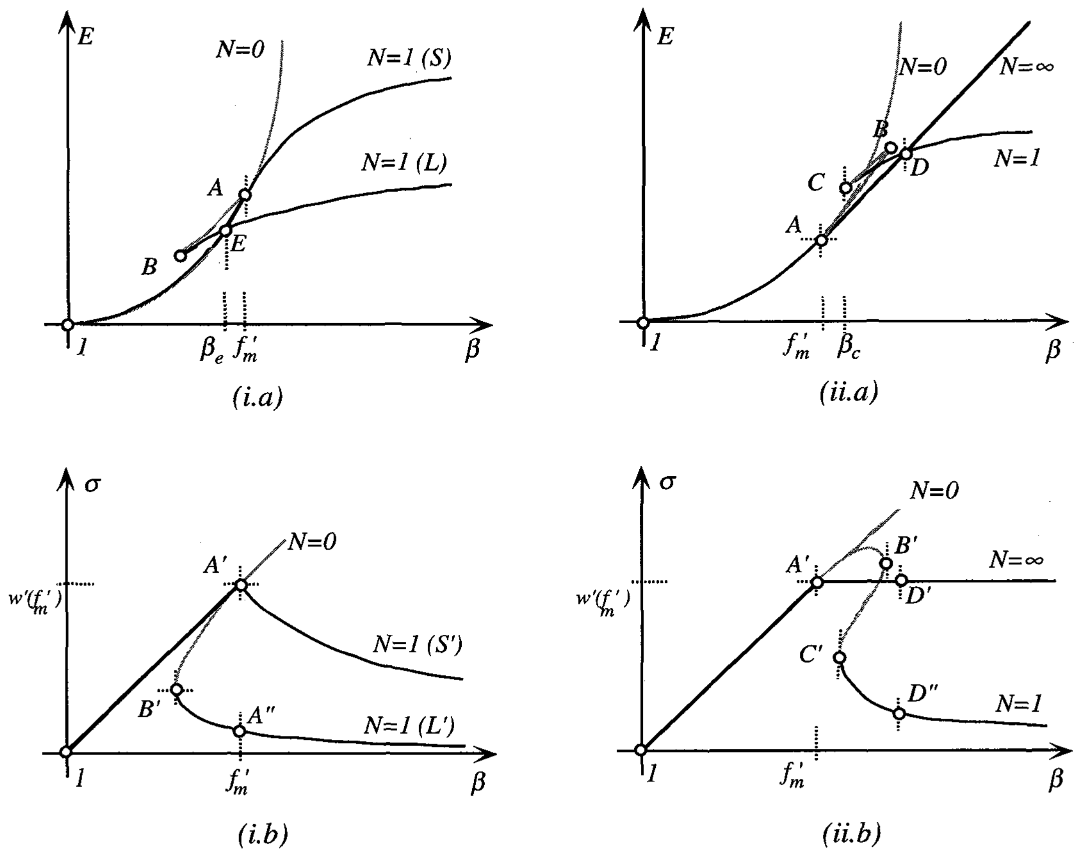

Figure 2: Energy-strain curves (i.a), (ii.a) and stress-strain curves (i.b), (ii.b), for large (L) and small (S) values of l, for cohesive energies of type (i) and (ii): unstable -------- and metastable ——— equilibrium configurations.

The curves L', S' in Fig.2.i.b show that there are two different fracture modes, depending on the value of l. For large l, curve L', there is a sudden drop in the stress, whereas for small l, curve S', the stress decreases gradually and without jumps. This influence of the length of the bar on the fracture mode reflects the *size effect* [2].

In both modes, the decrease of stress is accompanied by a decrease in bulk deformation, so that the further elongation of the bar is totally supported by the increase of the jump. In this way, the model reproduces *strain localization.* Since a one-dimensional model cannot distinguish between rupture by separation and rupture by shear, the localized deformation may represent both *necking* and *sliding* across an inclined plane.

The overall energy-strain curve is shown in Fig.2.i.a. There is a bifurcation point A, corresponding to the bifurcation point A' of the stress-strain curve. For the ductile mode, curve S, the energy is a single-valued function of β, whereas for the brittle mode, curve L, it is multi-valued for all β in the interval between the points A, B, where there are three equilibrium configurations for each value of the total elongation. In the brittle mode, the energy minimum is attained at a configuration with N=0 for all β smaller than a critical value $\beta_e < f'_m$, and at a configuration with N=1 for all $\beta > \beta_e$. A crack then has to form at some β in the interval (β_e, f'_m).

For cohesive energies of type (ii), the function θ' is not monotonic; however, the inequality (2.5) eliminates the ascending branch of the θ' curve, and the analysis reduces to the previous one. Equilibrium configurations may be again of two types, with no jumps and with one jump only. In both cases, the energy $\hat{E}(\beta)$ and the overall stress $\hat{\sigma}(\beta)$ are given by the same equations as before.

A substantial difference between this case and the previous one emerges from Fig.2.ii.a, which shows that here may be a range (f'_m, β_c) of elongations with no metastable configurations of the type considered above. The analysis shows that the minimum of the energy in this range is attained for N=+∞, which means a simultaneous opening of an infinite number of infinitesimal cracks, and that the corresponding stress $\hat{\sigma}(\beta)$ stays constant at the value $w'(f'_m)$ [5]. This particular regime of deformation can be described precisely within the theory of *structured deformations* [6], and the whole response can be associated with the phenomena of *yielding* or of *infinitesimal damage propagation.*

At point D in Fig.2.ii.a, the global minimum path returns to a branch of metastable states with N=1, with a corresponding drop in the stress, from point D' to D" of Fig.2.ii.b. This may be interpreted as a transition from yielding to brittle fracture, with an ultimate strain localization. We notice that the yielding regime is absent at small l.

3.2 The case (iii)

For cohesive energies of type (iii) there is no one-to-one correspondence between deformation gradients and jumps, and the emerging picture is considerably more complex. Here, to emphasize the ideas, we consider the function θ' represented in Fig.1, case (iii); an alternative possibility, of $\theta'(0+)$ being the absolute maximum of θ', is discussed in [7].

As shown in the preceding section, the condition (2.7) implies that, for f' in a right neighborhood of f'_a, a jump in the decreasing branch $(c,+\infty)$ of θ' cannot coexist with a jump in the increasing branch. For simplicity, we assume that this is true for all f' in (f'_a, f'_m). In this case, metastable configurations may be of only four types:

- with no jumps,
- with one jump on one of the descending branches $(c,+\infty)$, $(0, a)$,
- with one jump on the descending branch $(0, a)$ and N-1 jumps on the ascending branch (a,b),
- with N jumps on the ascending branch (a,b).

For configurations of the first type the energy is given by Equation (3.2), and for configurations of the second type, which prevail at large elongations, the energy is again determined by Equations (3.4). For configurations of the third type, the energy $\hat{E}(\beta)$ is given parametrically by the conditions

$$E(f) = l\,w(f') + \theta([f]^-) + (N-1)\theta([f]^+)\ , \tag{3.7}$$

$$\beta = f' + l^{-1}[f]^- + l^{-1}(N-1)\,[f]^+\ , \tag{3.8}$$

$$w'(f') = \theta'([f]^-) = \theta'([f]^+)\ , \tag{3.9}$$

where $[f]^-$ and $[f]^+$ are the jumps in the branches $(0, a)$ and (a, b) corresponding to f'. The overall stress $\hat{\sigma}(\beta)$ is represented by an infinity of branches, one for each N. On these branches, the parts with a positive slope are unstable by the condition (2.7).

Each of these branches is connected to a branch of the fourth type, whose energy $\hat{E}(\beta)$ is determined by the conditions

$$E(f) = l\,w(f') + N\theta([f]^+)\ , \tag{3.10}$$

$$\beta = f' + l^{-1}N\,[f]^+\ , \tag{3.11}$$

$$w'(f') = \theta'([f]^+)\ . \tag{3.12}$$

The overall stress $\hat{\sigma}(\beta)$ is again obtained from Equation (2.9).

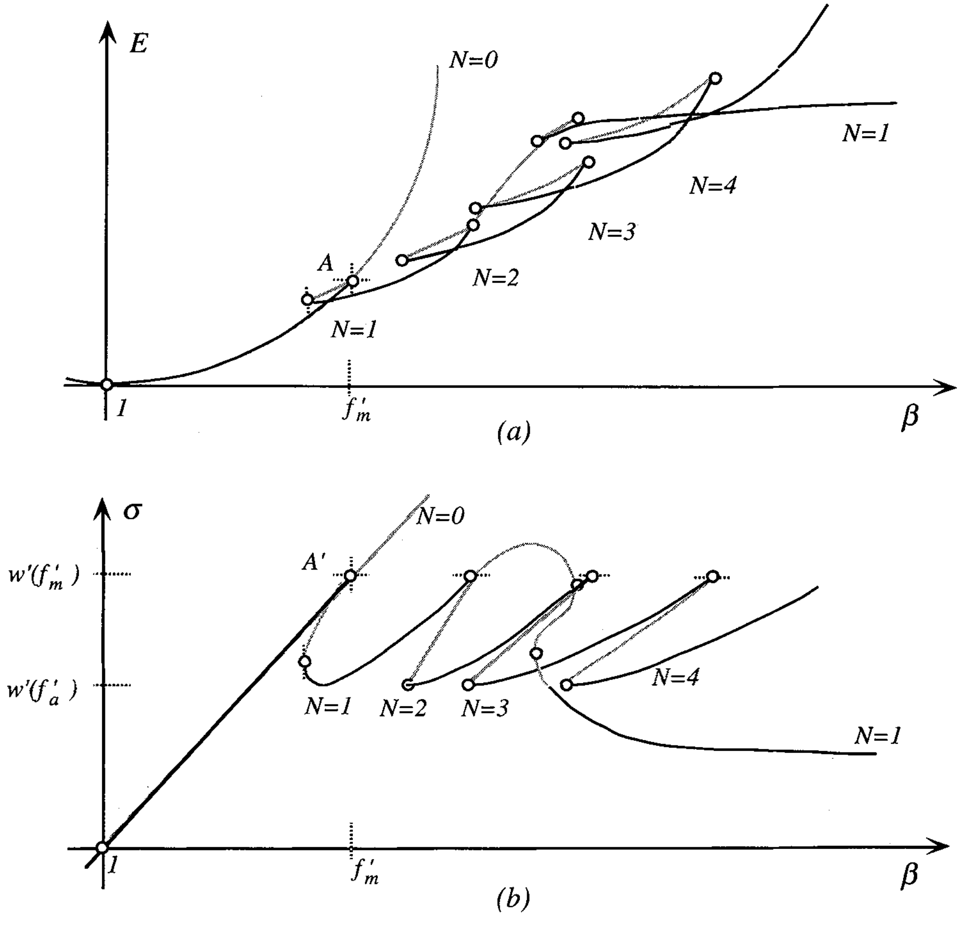

Figure 3:
Energy-strain curve (a) and stress-strain curve (b) for cohesive energies of type (iii):
unstable ---------- and metastable ——— equilibrium configurations.

The dependence of energy and stress upon the total elongation is shown qualitatively in Fig.3, where, for simplicity, only some of the unstable branches are drawn. One can show that most of the metastable configurations are of the fourth type. Energy minima are attained at increasing values of N, starting from N=1 at $\beta = f'_m$, and with more and more cracks forming until the mode N=1 prevails once again. After the formation of the first crack, the stress oscillates in the interval $(w'(f'_a), w'(f'_m))$, and each new crack determines a drop in the stress, recovered during subsequent deformation. The total number of cracks formed depends on the length of the bar.

It is interesting that failure propagates in a discrete manner, although the model contains no element of discreteness. This quantization is similar to the one observed in [9] for the Ericksen bar on elastic foundation.

From the experimental viewpoint, the oscillations in the stress-strain curve predicted by the model are reminiscent of the oscillations observed by Elam in very slow tensile tests at the hard testing machine [1, Sect. 4.24]. Moreover, the successive formation of cracks may provide an interpretation of the inhomogeneity of deformation along the specimen, a phenomenon known since the early tests of Savart [1, Sect. 4.31]; perhaps, it may also explain the propagation of discontinuous deformation along the bar, discovered by McReynolds [1, ibid.].

Acknowledgement

This research has been supported by a CNR grant within the Progetto Strategico *Problemi della progettazione complessa* (G.D.), and by the NSF grant DMS 9501433 (L.T.).

References

[1] Bell J.F., "The experimental foundations of Solid Mechanics", *Handbuch der Physik,* **VIa/1**, (Springer, Berlin, 1973).
[2] Carpinteri A., "Cusp catastrophe interpretation of fracture instability", *J. Mech. Phys. Solids,* **37**, (1989), 567-582.
[3] Choksi R., Fonseca I., "Bulk and interfacial energy densities for structured deformations of continua", *Arch. Rat. Mech. Analysis,* **138**, (1997) 37-103.
[4] Choksi R., Del Piero G., Fonseca I., Owen D.R., "Structured deformations as energy minimizers in some models of fracture and hysteresis", in preparation.
[5] Del Piero G., "One-dimensional ductile-brittle transition, yielding, and structured deformations", Proc. IUTAM Symposium "Variations de domaines et frontières libres en mécanique des solides", Paris, April 1997, forthcoming;
[6] Del Piero G., Owen D.R., "Structured deformations of continua", *Arch. Rat. Mech. Analysis,* **124**, (1993) 99-155.
[7] Del Piero G., Truskinovsky L., in preparation.
[8] Truskinovsky L., "Fracture as a phase transition", Contemporary research in the mechanics and mathematics of materials, R.C.Batra and M.F.Beatty Eds. (CIMNE, Barcelona, 1996), 322-332.
[9] Truskinovsky L., Zanzotto G., " Ericksen's bar revisited: energy wiggles", *J. Mech. Phys. Solids,* **44**, (1996) 1371-1408.