Soft nucleation of an elastic crease
(supplementary online material)

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Internal asymptotics. We first recall that the internal asymptotics is represented by the mapping $x_1 = R/\sqrt{2}\sin(2\Theta)$, $x_2 = R/\sqrt{2}\cos(2\Theta)$ where $R = \sqrt{(X_1)^2 + (X_2)^2}$ and $\Theta = \tan^{-1}(X_1/X_2)$. Then the components of the deformation gradient at the free surface $X_2 = 0$ are: $F_{11} = F_{22} = 0$, $F_{21} = -1/\sqrt{2}$, and $F_{12} = \sqrt{2}$; The corresponding components of the Piola-Kirchhoff stress are:

$$P_{11} = P_{22} = 0 \qquad P_{21} = 2\sqrt{2}w^{'}(5/2) - p_{in}/\sqrt{2}, \qquad P_{12} = -\sqrt{2}(w^{'}(5/2) - p_{in}). \tag{S1}$$

Intermediate asymptotics. The Green's function in the intermediate problem $G(X_1, X_2)$ satisfies the equation:

$$\tilde{\nabla}\bar{\nabla}G(X_1, X_2) = \delta(X_1, 0). \tag{S2}$$

Here $\delta(\mathbf{X})$ is the 2D Dirac function, $\tilde{\nabla} = (\partial_{\tilde{x_1}\tilde{x_1}} + \partial_{\tilde{x_2}\tilde{x_2}})$ and $\bar{\nabla} = (\partial_{\bar{x}_1\bar{x}_1} + \partial_{\bar{x}_2\bar{x}_2})$ are Laplace operators with respect to the stretched variables: $\tilde{x}_1 = \lambda\ X_1/s,\ \tilde{x}_2 = s\ X_2/\lambda$ and $\bar{x}_1 = s\ X_1,\ \bar{x}_2 = X_2/s$.

The scaling factor s and the parameter α are given in the main text. To write down the boundary conditions for Eq. (S2) we introduce the components of the incremental stress δp_{ij}. In the portion of the free surface $X_2 = 0$ that is free of tractions, we must have $\delta p_{12}(X_1, 0) = \delta p_{22}(X_1, 0) = 0$. The applied singular force must be balanced only by reactions at $X_2 = \bar{L}$:

$$-1 = \int_{-\bar{L}}^{\bar{L}} \delta p_{22}|_{X_2=\bar{L}}\ dX_1\ + 2\int_0^{\bar{L}} \delta p_{12}|_{X_1=\bar{L}}\ dX_2. \tag{S3}$$

The solution of the resulting boundary value problem for zero prestress $\lambda = 1$ is known [1] and we can easily adopt it to the case $\lambda \neq 1$. To this end we generalize the classical ansatz for the Green's function (for semi-infinite domain)

$$G(X_1, X_2) = \tilde{a}\ \underline{G}(\tilde{x}_1, \tilde{x}_2) + \bar{a}\ \underline{G}(\bar{x}_1, \bar{x}_2),$$

where $\underline{G}(x, y) = x\log\left(x^2 + y^2\right) + 2y\arctan\left(x/y\right)$. We then apply the boundary conditions on the free surface to find:

$$\tilde{a} = -\bar{a}\frac{s^2\left(s^4 + 1\right)\lambda}{s^4 + \lambda^4}. \tag{S4}$$

Using the trigonometric identity $\arctan(y) + \arctan(y^{-1}) = \pi/2$ we can then rewrite Eq.(S3) in the form

$$-1 = \bar{a}\frac{4\pi\left(s^8-\lambda^4\right)\left(2\left(\lambda^2-1\right)^3\left(\lambda^2+1\right)^2s^4\alpha+\lambda^2\left(\lambda^4+s^8+\left(\lambda^6+3\lambda^2-2\right)s^4\right)\right)w^{'}(I_\lambda)}{\lambda^3s^3(\lambda^2+s^4)(\lambda^4+s^4)} \tag{S5}$$

which finally gives:

$$\bar{a} = -\frac{\lambda^3s^3\left(\lambda^2+s^4\right)\left(\lambda^4+s^4\right)}{4\pi(s^8-\lambda^4)\left(2(\lambda^2-1)^3(\lambda^2+1)^2s^4\alpha+\lambda^2(\lambda^4+s^8+(\lambda^6+3\lambda^2-2)s^4)\right)w^{'}(I_\lambda)}. \tag{S6}$$

In the case when the point force has the magnitude δf we obtain $\hat{a} = \delta f\ \bar{a}$ which after straightforward simplifications gives:

$$\hat{a} = \delta\ell\ \frac{\lambda s^3\left(\lambda^4 - 1\right)\left(\lambda^4 + s^4\right)}{\pi\left(s^4 - \lambda^2\right)\left(\lambda^4 + s^8 + s^4\left(\lambda^6 + \left(\lambda^2 - 1\right)\alpha + 3\lambda^2 - 2\right)\right)}. \tag{S7}$$

The stream function in the incremental problem can be then written as $\delta\phi(X_1, X_2) = G(X_1, X_2)\delta f$, which produces the following incremental stress components:

$$
\begin{aligned}
\delta p_{11}(x_1,x_2) &= \hat{a}\,\frac{4\left(s^4+1\right)x_1^2 x_2\left(s^8-\lambda^4\right)w(I_\lambda)}{\lambda^3 s\left(s^4x_2^2+x_1^2\right)\left(s^4x_1^2+\lambda^4x_2^2\right)};\\
\delta p_{12}(x_1,x_2) &= \hat{a}\,\frac{4x_1\left(s^8-\lambda^4\right)w(I_\lambda)\left(\lambda^4s^8x_2^2+s^4\left(\left(\lambda^4-1\right)x_1^2+2\lambda^4x_2^2\right)+\lambda^8x_2^2\right)}{\lambda^3 s(\lambda^4+s^4)\left(s^4x_2^2+x_1^2\right)\left(s^4x_1^2+\lambda^4x_2^2\right)};\\
\delta p_{21}(x_1,x_2) &= \hat{a}\,\frac{4\left(s^4+1\right)x_1x_2^2\left(s^8-\lambda^4\right)w(I_\lambda)}{\lambda s\left(s^4x_2^2+x_1^2\right)\left(s^4x_1^2+\lambda^4x_2^2\right)};\\
\delta p_{22}(x_1,x_2) &= \hat{a}\,\frac{4x_2\left(s^8-\lambda^4\right)w(I_\lambda)\left(\lambda^2s^8x_1^2+s^4\left(\lambda^4x_2^2\left(2\lambda^8\alpha+\lambda^6-4\lambda^4\alpha+3\lambda^2+2\alpha\right)-2x_1^2\left(\lambda^8\alpha-2\lambda^4\alpha+\lambda^2+\alpha\right)\right)+\lambda^6x_1^2\right)}{\lambda^3 s(\lambda^4+s^4)\left(s^4x_2^2+x_1^2\right)\left(s^4x_1^2+\lambda^4x_2^2\right)}.
\end{aligned}
\tag{S8}
$$

The corresponding pressure increment δp is

$$
\delta p(x_1,x_2) = -\hat{a}\,\frac{4x_2\left(s^8-\lambda^4\right)w^{'}(I_\lambda)\left(s^8x_1^2+\lambda^2s^4\left(x_2^2\left(2\lambda^8\alpha+\lambda^6-2\lambda^4\alpha+\lambda^2\right)-2\left(\lambda^4-1\right)x_1^2\alpha\right)+\lambda^4x_1^2\right)}{\lambda^2 s\left(\lambda^4+s^4\right)\left(s^4x_2^2+x_1^2\right)\left(s^4x_1^2+\lambda^4x_2^2\right)}.
\tag{S9}
$$

In the main paper we also use the first corrections to this asymptotic field. The expression for the corresponding stream function is given in the main paper and here we provide the explicit formulas for the displacement field at the free surface:

$$
\delta u_1|_{X_2=0} = \frac{\pi\hat{a}\left(-\lambda^4+2s^2-1\right)\left(\left|\lambda\left(\delta\ell+X_1\right)\right|-\left|\lambda\left(\delta\ell-X_1\right)\right|\right)}{4\delta\ell\lambda s};
\tag{S10}
$$

$$
\delta u_2|_{X_2=0} = \frac{\hat{a}\left(\left(\lambda^4+1\right)s^2-2\lambda^2\right)\left(\left(\delta\ell+X_1\right)\log\left(1+X_1/\delta\ell\right)+\left(\delta\ell-X_1\right)\log\left(1-X_1/\delta\ell\right)\right)}{4\delta\ell s\lambda^2}
\tag{S11}
$$

Observe that at $|X_1| \geq \delta\ell$, the continuous function $\delta u_1(X_1, 0)$ in (S10) coincides with the discontinuous one obtained under the assumption of infinitely localized load: the regularizing correction appears only at $|X_1| < \delta\ell$. The function $\delta u_2(X_1, 0)$ in (S10) matches its analog for the infinitely localized load at $|X_1| >> \delta\ell$, however, the singularity at $X_1 = 0$ is now smoothed.

Path independent integral. Using the straight path connecting the point A' to A and applying Eqs.(S1) we obtain :

$$
\Delta\mathcal{W}^+(\delta\ell)/\delta\ell = J|_{A'A} = \int_{AA'}(\mathsf{T}\mathbf{n}\cdot\mathbf{m})ds = \int_{-\delta\ell}^{\delta\ell}(w(5/2)-P_{21}F_{12})dX_1 = 2\delta\ell(w(5/2)-2w^{'}(I_\lambda)(\lambda^2-\lambda^{-2})),
\tag{S12}
$$

where we preserved only the linear term in $\delta\ell$. Then, using the fact that along the path BD normal tractions vanish, $\delta p_{12} = \delta p_{22} = 0$, we can write $J|_{BD} = \int_{\bar{L}}^{L} w|_{X_2=0}dX_1$, and expand the energy w up to the first order in ϵ we obtain:

$$
\begin{aligned}
\int_{\bar{L}}^{L} w|_{X_2=0}dX_1 = \quad & \int_{\bar{L}}^{L}[(w(I_\lambda) - 2w^{'}(I_\lambda)(\lambda\delta u_{1,1} + \lambda^{-1}\delta u_{2,2}) + o(\epsilon)]dX_1 = \\
& \int_{\bar{L}}^{L}[(w(I_\lambda) - [(2w^{'}(I_\lambda)(\lambda - \lambda^{-3})\delta u_{1,1}]dX_1 + o(\epsilon) = \\
& w(I_\lambda)(L - \bar{L}) - 2P_{11}(\lambda)\delta u_1(\bar{L}, 0) + o(\epsilon)
\end{aligned}
\tag{S13}
$$

where we used the incompressibility constraint for the intermediate asymptotics and the fact that in the external asymptotics $\delta u_1(L, 0) = 0$.

References

[1] A Flamant. Sur la répartition des pressions dans un solide rectangulaire chargé transversalement. *CR Acad. Sci. Paris*, 114:1465–1468, 1892.