Supplemental Material for the paper "Functionality of disorder in muscle mechanics"

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Mapping to the Random-Field Ising Model

We start with the energy function (1) in the main text and assume that the internal variable y is eliminated using the condition $\partial E/\partial y = 0$. Then,

$$y = \frac{\lambda_f z}{1+\lambda_f} + \frac{1}{N(1+\lambda_f)} \sum_i x_i.$$

and the relaxed energy reads

$$E(x_i, z) = -\frac{1}{2N(1+\lambda_f)} \left(\sum_i x_i \right)^2 + \sum_i (1+x_i) v_i - \frac{\lambda_f z}{1+\lambda_f} \sum_i x_i + \sum_i \frac{x_i^2}{2} + \frac{N\lambda_f z^2}{2(1+\lambda_f)}$$

Since x_i is either 0 or -1, we may write $\sum_i x_i^2 = -\sum_i x_i$ and $\left(\sum_i x_i\right)^2 = \sum_i \sum_j x_i x_j = \sum_{i,j} x_i x_j$. In terms of spin variables, $2x_i = s_i - 1$, with $s_i = \pm 1$ the relaxed energy can be written as,

$$\begin{aligned} E(s_i, z) = &-\frac{1}{8N(1+\lambda_f)} \sum_{i,j} s_i s_j \\ &- \sum_i \left(\frac{2\lambda_f z - 1}{4(1+\lambda_f)} + \frac{1}{4} - \frac{v_i}{2} \right) s_i \\ &+ \sum_i \left(\frac{\lambda_f z(1+z)}{2(1+\lambda_f)} + \frac{1}{4} + \frac{v_i}{2} - \frac{1}{8(1+\lambda_f)} \right) \\ &= -\frac{J}{2N} \sum_{i,j} s_i s_j - \sum_i h_i s_i + f(z). \end{aligned} \tag{1}$$

where $J = \frac{1}{4(1+\lambda_f)}$, $h_i = \frac{2\lambda_f z - 1}{4(1+\lambda_f)} + \frac{1}{4} - \frac{v_i}{2}$ and $f(z) = \sum_i \frac{\lambda_f z(1+z)}{2(1+\lambda_f)} + \frac{1}{4} + \frac{v_i}{2} - \frac{1}{8(1+\lambda_f)}$.

Computation of the free energy

Using the self-averaging property of the free energy in the thermodynamic limit, we write

$$\mathcal{F}(\beta, z) = -\lim_{N\to\infty} (N\beta)^{-1} \langle \log \mathcal{Z}(\beta, z; \{v\}) \rangle_v,$$

where the averaging $\langle\cdot\rangle_v$ is over the disorder, $\beta = \kappa_0 a^2/(k_B T)$, and

$$\mathcal{Z} = \int dy \sum_{x \in \{0,-1\}^N} \exp(-\beta E(\boldsymbol{x}, y, z; \{v\}).$$

The mean field nature of the model allows one to rewrite this expression in the form

$$\mathcal{Z} = \int dy \exp(-\beta N [\frac{\lambda_f}{2}(z-y)^2 - \frac{1}{\beta N} \sum_{i=1}^N \log \tilde{\mathcal{Z}}]),$$

where $\tilde{\mathcal{Z}} = e^{-\frac{\beta}{2}(y+1)^2} + e^{-\beta(y^2/2 + v_i)}$ is the partition function of a single Huxley-Simmons element [1, 2]. In the thermodynamic limit, we can use the saddle-point approximation to obtain $\mathcal{F}(\beta, z) = \tilde{\mathcal{F}}(y_0, \beta, z)$, where $\tilde{\mathcal{F}}(y, \beta, z) = \beta \frac{\lambda_f}{2}(z-y)^2 - \langle \log \tilde{\mathcal{Z}} \rangle_v$ and $y_0(\beta, z)$ is the minimum of $\tilde{\mathcal{F}}$. More explicitly,

$$\begin{aligned} \mathcal{F}(\beta, z) = &\frac{\lambda_f}{2}(z-y_0)^2 + \frac{1}{4}(y_0+1)^2 + \frac{1}{2}(\frac{y_0^2}{2} + v_0) \\ &- \frac{1}{\beta} \int dv\, p(v) \log \left[2\cosh[\frac{\beta}{4}(1 + 2y_0 - 2v)] \right], \end{aligned} \tag{2}$$

where y_0 solves the self-consistency equation,

$$y_0 = \frac{2\lambda_f z - 1}{2(\lambda_f + 1)} + \int dv\, \frac{p(v)}{2(\lambda_f+1)} \tanh \left[\frac{\beta}{4}(1 - 2v + 2y_0) \right].$$

Boundary between phases II and III

Using the expression for the partial free energy,

$$\begin{aligned} \tilde{\mathcal{F}}(\beta, z, y) = &\frac{\lambda_f}{2}(z-y)^2 + \frac{1}{4}(y+1)^2 + \frac{1}{2}(y^2/2 + v_0) \\ &- \frac{1}{\beta} \int dv\, p(v) \log \left[2\cosh[\frac{\beta}{4}(1 + 2y - 2v)] \right] \end{aligned}$$

we can write the condition $\partial^2 \tilde{\mathcal{F}}(\beta, z, y)/\partial y^2 = 0$ in the form

$$\lambda_f + 1 - \frac{\beta}{4} \int dv\, p(v) \operatorname{sech}^2 \frac{\beta}{4}(1 - 2v + 2y_0) = 0.$$

If we use the Gaussian distribution of disorder introduced in the main text and use new variables $\eta = \beta(1 + 2y_0)/2$

and $\bar{v} = \beta v$ we can rewrite this equation in the form

$$\lambda_f + 1 - \frac{\beta}{4}\int d\bar{v}\frac{e^{-\frac{(\bar{v}-\beta v_0)^2}{2\sigma^2\beta^2}}}{\sqrt{2\pi\sigma^2\beta^2}}\,\mathrm{sech}^2\,\frac{1}{2}(\eta-\bar{v}) = 0.$$

Note that the variance of disorder appears in this formula only in the combination $\sigma^2\beta^2$. This means that, modulo some obvious adjustments, the small disorder $\sigma \to 0$ and large temperature $\beta \to 0$ limits are complimentary. The same can be said about the small temperature $\beta \to \infty$ and the large disorder $\sigma \to \infty$ limits.

Zero disorder limit. In the limit $\sigma \to 0$ we have $p(v) \to \delta(v - v_0)$ and the boundary between phase II and III is defined by the equation

$$\lambda_f + 1 = \frac{\beta}{4}\,\mathrm{sech}^2\,\frac{\beta}{4}(1 - 2v_0 + 2y_0).$$

Since $\mathrm{sech}^2\, x \in [0, 1]$, this equation does not have solutions y_0 for $\beta > 4(\lambda_f + 1)$ and therefore the point r is defined by the condition $\beta = 4(\lambda_f + 1)$.

To get the next term of the asymptotic expansion we introduce the new variable $\xi = (1 - 2v + 2y_0)/4$ and assume that the temperature is large $\beta \to 0$. Then we can expand $\log \mathrm{sech}^2\,\beta\xi \approx -\beta^2\xi^2 + O(\beta^4)$, which implies that $\mathrm{sech}^2\,\beta\xi \approx e^{-\beta^2\xi^2}$. Using this approximation we can compute the integral and represent the boundary between phase II and III in the form

$$\lambda_f + 1 = \frac{e^{\frac{-(y_0-v_0+1/2)^2}{2(2T^2+\sigma^2)}}}{2\sqrt{2(2T^2+\sigma^2)}}.$$

where $T = 1/\beta$. Since $e^{-x^2} \in (0, 1]$ the criticality condition is

$$(\lambda_f + 1)2\sqrt{2(2T^2 + \sigma^2)} = 1$$

The equivalent quenched disorder is then defined by the condition $\sigma_{eq}^2 = 2T^2 + \sigma^2$.

Zero temperature limit. In the zero temperature limit $\beta \to \infty$ we use the fact that $\lim_{k\to\infty} \frac{k}{2}\,\mathrm{sech}^2\, kx \to \delta(x)$. to rewrite the equation defining the boundary between phase II and III in the form

$$(\lambda_f + 1)\sqrt{2\pi\sigma^2} = e^{-\frac{(y_0+1/2-v_0)^2}{2\sigma^2}}.$$

Here the *r.h.s* is defined in the interval $(0, 1]$ and therefore there are no solutions y_0 if $(\lambda_f + 1)\sqrt{2\pi\sigma^2} > 1$ where we used the fact that $\sigma, \lambda_f > 0$. The point q is then defined by the condition $(\lambda_f + 1)\sqrt{2\pi\sigma^2} = 1$.

To obtain the next term of the asymptotic expansion we assume that disorder is large $\sigma \to \infty$. In this case we can still approximate the function $\mathrm{sech}^2(x)$ by the Gaussian distribution but now the approximation should be good not at $x = 0$ but globally. To this end we need to require that the two functions are equally normalized

$$\frac{1}{4T}\int dv\,\mathrm{sech}^2\frac{1-2v-y_0}{4T} = \frac{1}{\sqrt{4\pi T^2}}\int dv e^{-\frac{(v-y_0-1/2)^2}{4T^2}} = 1,$$

where again $T = 1/\beta$. With this normalization the integral can be again computed and we obtain the condition

$$(\lambda_f + 1)\sqrt{2\pi} = \frac{e^{\frac{-(y-v_0+1/2)^2}{2(2T^2+\sigma^2)}}}{\sqrt{2T^2+\sigma^2}}.$$

The criticality criterion is then

$$(\lambda_f + 1)\sqrt{2\pi(2T^2 + \sigma^2)} = 1,$$

which allows us to introduce the effective disorder by the condition $\sigma_e^2 = 2T^2 + \sigma^2$.

Gibbs free energy

In the case of soft device the relevant potential is,

$$G = \sum_{i=1}^{N}\left[(1 + x_i)v_i + \frac{1}{2}(y - x_i)^2\right] - ty. \tag{3}$$

Following the approach used in the case of hard device, we obtain the expression for the Gibbs free energy

$$\begin{aligned}\mathcal{G}(\beta, t) = &-ty_0 + \frac{1}{4}(y_0 + 1)^2 + \frac{1}{2}(\frac{y_0^2}{2} + v_0)\\ &- \frac{1}{\beta}\int dv\, p(v)\log\left[2\cosh[\frac{\beta}{4}(1 + 2y_0 - 2v)]\right]\end{aligned} \tag{4}$$

where now y_0 solves the equation

$$t = y_0 + \frac{1}{2} - \frac{1}{2}\int dv\, p(v)\tanh\left[\frac{\beta}{4}(1 - 2v + 2y_0)\right]. \tag{5}$$

The tension elongation relation is then a solution of $y = -\partial\mathcal{G}/\partial t$.

Edwards-Anderson order parameter

In the absence of disorder, a natural order parameter is

$$\phi = \frac{1}{N}\sum_{i=1}^{N}\langle s_i\rangle_T\,,$$

where $s_i = 2x_i + 1$. To find $\phi(z, \beta)$ we notice that since all cross-bridges are the same we can write $\phi = 2\langle x_i\rangle_T + 1$

where

$$\langle x_i\rangle_T = -Z(\beta,z)^{-1}e^{-\beta E(x_i=-1,y_0,z)}$$

with

$$Z(\beta,z) = e^{-\beta N\left[\frac{\lambda_f}{2}(z-y_0)^2-\frac{1}{\beta}\log(e^{-\frac{\beta}{2}(y_0+1)^2}+e^{-\beta(y_0^2/2+v)})\right]}.$$

By combining these expressions we obtain

$$\langle x_i\rangle_T = -\frac{1}{1+e^{\beta(y_0-v+1/2)}}.$$

In the presence of disorder, the average values $\langle x_i\rangle_T$ are different for different cross-bridges and the macroscopic parameter $\phi(z,\beta)$ is no longer sufficient to differentiate between microscopic configurations. To this end we can introduce an analogue of the Edwards-Anderson parameter from the theory of spin glasses

$$q_{EA} = \frac{1}{N}\sum_{i=1}^{N}\left\langle \langle s_i\rangle_T^2\right\rangle_v.$$

where we distinguish between the thermal average $\langle\cdot\rangle_T$ and the ensemble average $\langle A\rangle_v = \int dv p(v)A(v)$. If the parameter ϕ characterizes the average occupancy of the pre-power stroke state, the nonzero value of q_{EA} means that individual cross bridges are 'frozen' either in pre- or post-power-stroke states even if in average, both states appear to be equally occupied. The knowledge of this parameter is needed, for instance, if one is interested in computing the effect of the random field on mechanical susceptibility (stiffness) [3]

In terms of the variables x_i the definition of q_{EA} reads

$$q_{EA} = \frac{1}{N}\sum_{i=1}^{N}\left[4\left\langle \langle x_i\rangle_T^2\right\rangle_v + 4\left\langle \langle x_i\rangle_T\right\rangle_v + 1\right],$$

where

$$\left\langle \langle x_i\rangle_T^2\right\rangle_v = \int dv\frac{p(v)}{(1+e^{\beta(y_0-v+1/2)})^2},$$

and

$$\left\langle \langle x_i\rangle_T\right\rangle_v = -\int dv\frac{p(v)}{1+e^{\beta(y_0-v+1/2)}}.$$

Boundary between phases I and II

Note first that $\frac{\partial t}{\partial z} = \lambda_f(1-\frac{\partial y_0}{\partial z})$, and therefore to get zero stiffness we must have $\partial y_0/\partial z = 1$, Here y_0 is found from the self-consistency condition given by Eq. 5 in the main text and therefore

$$\frac{\partial y_0}{\partial z} = \frac{\lambda_f}{\lambda_f+1} + \frac{\beta}{4(1+\lambda_f)}\int dv\, p(v)\,\mathrm{sech}^2\left[\frac{\beta}{4}(1-2v+2y_0)\right]\frac{\partial y_0}{\partial z}, \tag{6}$$

which is equivalent to

$$1 = \frac{\beta}{4}\int dv\, p(v)\,\mathrm{sech}^2\left[\frac{\beta}{4}(1-2v+2y_0)\right].$$

The condition that this equation has a root y_0 does not contain λ_f and therefore the boundary between phases I and II is λ_f independent.

Zero disorder limit. In the limit $\sigma\to 0$ we can again assume that the probability density $p(v)$ is infinitely localized and compute the integral explicitly. We obtain

$$\frac{4}{\beta} = \mathrm{sech}^2\frac{\beta}{4}(1-2v+2y_0).$$

Since $\mathrm{sech}^2 x\in[0,1]$, this equation does not have solutions y_0 if $\beta<4$, hence $\beta_c=4$, which is the coordinate of our point s. The higher order asymptotic expansion can be obtained following the same procedure as in the case of the boundary between phases II and III.

Zero temperature limit. In the limit $\beta\to\infty$, we can again use the fact that the function $\frac{k}{2}\mathrm{sech}^2 kx$ converges to the delta function as $k\to\infty$. Therefore, assuming that the probability distribution $p(v)$ is Gaussian we obtain,

$$1 = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y_0+1/2-v_0)^2}{2\sigma^2}}.$$

Using the same arguments as in the zero disorder limit and noticing that $e^{-x^2}\in(0,1]$, we conclude that this equation has solution only if $\sigma\geq 1/\sqrt{2\pi}$. Therefore, the critical value of the disorder in this limit is $\sigma_c = 1/\sqrt{2\pi}$, which corresponds to our point r. The expansion around this point can be obtained as in the case of the boundary between phases II and III considered above.

Axial offset

Experimental studies using electron microscopy (EM) and x-ray diffraction have shown that the biding of cross-bridges is restricted to limited segments of the actin filament known as target zones [4, 5]. These zones are represented by two to three actin monomers, see Fig. 1. Moreover, it was found [6] that the probability distribution of axial offsets from the target zone center is approximately Gaussian and that at least 60% of the attached cross-bridges are displaced within half of the spacing be-

tween actin monomers which corresponds to the offset of 2.76nm.

The offset can be represented by the reference elongation $y_0 = v_0 - 1/2$ which marks the boundary between pre and post-power stroke states. Because the parameters v_0 and y_0 differ by a constant, the variance of δv_0 is equal to the variance δy_0. Hence, placing disorder in the energetic bias v_0 is equivalent to introducing variable axial offset.

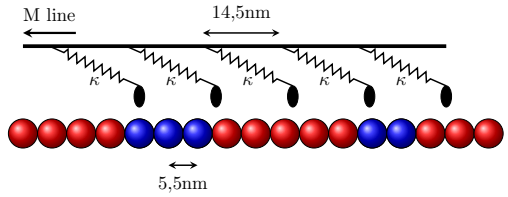

Figure 1. Schematic representation of the attachment sites. Each sphere represents an actin monomer; blue color delineate target zones.

Gaussian distribution of offsets. If we suppose that the distribution of axial offsets between the myosin head and the actin biding site is Gaussian we can estimate its standard deviation by noting that the probability that the variable deviation lies in the range $\pm k\sigma$ is given by,

$$\Pr(\mu - k\sigma \leq X \leq \mu + k\sigma) = \mathrm{erf}(\frac{k}{\sqrt{2}}), \qquad (7)$$

we then use the fact that 60% is in the range ± 2.76nm to find $k = 0.842$ and $\sigma = 3.3$nm.

Critical response in soft and hard ensembles

In Fig. 2 we illustrate the mechanical responses in the adjacent critical regimes marked as A and B in Fig. 2 of the main text. In the associated critical points, indicated here by small circles and intended to represent the physiological regime of isometric contractions, the susceptibilities diverge. The closeness of these two regimes in the parameter space allows the system to exhibit the whole repertoire of behaviors from zero to infinite rigidity.

Two half-sarcomeres in series

Here we present an elementary illustration of the fact that the equilibrium response of a bundle of contractile units connected in series and placed in a hard device, cannot be described by local equilibrium constitutive relations obtained in either soft or hard device ensembles. Instead, the system exhibits an intermediate behavior.

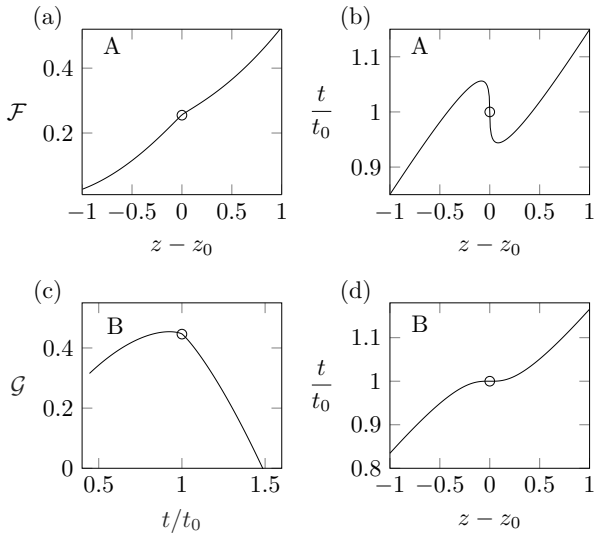

Figure 2. The response of the system in the critical regimes A and B shown in Fig. 2 of the main text : (a) and (b) are the Helmholtz free energy and the tension-elongation curve in the hard device ensemble; (c) and (d) are the Gibbs free energy and the associated tension-elongation curve in the soft device ensemble. Critical points are marked by the small circles.

Consider two elementary contractile units in series, see [7] for the analysis of M such elements. Each of the two elements represents a parallel connection of N cross-bridges. The total energy per cross bridge in dimensionless form for a system placed in a hard device reads

$$E_2 = \frac{1}{2}\left\{\frac{1}{N}\sum_i^N[(1+x_{i1})v_{i1} + \frac{1}{2}(y_1 - x_{i1})^2 + \frac{\lambda_f}{2}(z_1 - y_1)^2] + \frac{1}{N}\sum_i^N[(1+x_{i2})v_{i2} + \frac{1}{2}(y_2 - x_{i2})^2 + \frac{\lambda_f}{2}(z_2 - y_2)^2]\right\} \qquad (8)$$

The equilibrium response of the system is obtained by computing the partition function

$$\mathcal{Z}_2(z,\beta) = \int \exp[-2\beta N E_2]\delta(z_1 + z_2 - 2z)d\boldsymbol{x}$$

where $d\boldsymbol{x} = \prod_i^N dx_{i1}dy_1 \prod_j^N dx_{j2}dy_2$ and z is the (average) elongation imposed on the system. We can rewrite the expression for $\mathcal{Z}_2$ in the form

$$\mathcal{Z}_2(z,\beta) = \int dy_1 dy_2 \exp\left\{-\beta N[-\frac{\lambda_f}{2}(z - y_1 - y_2)^2 - \frac{1}{\beta}\int dv p(v)\log\tilde{\mathcal{Z}}_1(y_1,v)\tilde{\mathcal{Z}}_2(y_2,v)]\right\} \qquad (9)$$

where $\tilde{\mathcal{Z}}_i(y_i, v) = e^{-\frac{\beta}{2}(y_i+1)^2} + e^{-\beta(y_i^2/2+v)}$. The free energy per cross-bridge is then $\mathcal{F}_2(z,\beta) = -\frac{1}{2N}\log \mathcal{Z}_2(z,\beta)$. The equilibrium tension-elongation relation for this system, obtained from the relation $t(z,\beta) = \partial\mathcal{F}_2(z,\beta)/\partial z$, is shown by the thick line in Fig. 3(a). Similar thick line in Fig. 3(b) shows the equilibrium response of a single contractile element placed in the hard device.

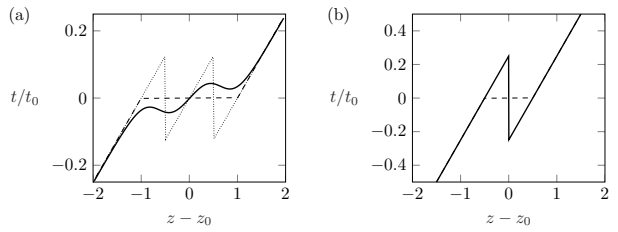

Figure 3. (a) Tension elongation relations for a system containing two half-sarcomeres in series placed in a hard device. Thick line: equilibrium response. Dotted (dashed) line: the response of two contractile elements in series, each one endowed with its own equilibrium the hard (soft) device constitutive law. (b) Response of a single half-sarcomere. Thick line: hard device; dashed line: soft device. Parameters are: $\beta = 30$, $\sigma = 0$, $v_0 = 0$, $\lambda_f = 1$.

We now compare this behavior with the one obtained under the assumption that the two elements in series are characterized by their equilibrium free energies computed either in a hard or in a soft ensembles.

For instance, using the hard device ensemble we can write the total (Helmholtz) free energy of the two element system in the form $E_2^{hd} = \mathcal{F}(z_1,\beta) + \mathcal{F}(z - z_1, \beta)$, where $\mathcal{F}$ is the free energy of a half-sarcomere given by Eq. 2. The extra variable z_1 can be eliminated using the equilibrium condition $\partial\mathcal{F}(z_1,\beta)/\partial z_1 = \partial\mathcal{F}(z - z_1,\beta)/\partial z_1$. The resulting tension elongation curve is shown in Fig. 3 (a) by a dotted line.

Similar analysis can be performed based on the response functions for the elements loaded in a soft device. Here we need to use equilibrium (Gibbs) free energies of the elements (Eq. 5 in the main text) and since the elements in series share the value of tension we obtain $G_2^{SD} = 2\mathcal{G}(t,\beta)$. The ensuing response of the series bundle is shown in Fig. 3(a) by a dashed line. In Fig. 3(b), the dashed line show the equilibrium response of a single contractile element loaded in a soft device.

Observe, first, that the equilibrium response predicted by the two 'constitutive models' contains discontinuities, while the response of the actually equilibrated system (two half-sarcomeres in series) is smooth. Note also that the actual response curves do not coincide with either of the two 'constitutive models' and exhibit some intermediate behavior with features mimicking both models simultaneously. The observed discrepancy is due to the fact that in a fully equilibrated system none of the contractile elements is loaded in either soft or hard device and that the overal response of the system is fundamentally non-affine, see also [7, 8].

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