Continuum Mech. Thermodyn. (2008) 20: 21–62
DOI 10.1007/s00161-008-0072-2

ORIGINAL ARTICLE

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Asymptotic expansions by Γ-convergence

Received: 20 July 2007 / Accepted: 1 February 2008 / Published online: 2 April 2008

Abstract Our starting point is a parameterized family of functionals (a 'theory') for which we are interested in approximating the global minima of the energy when one of these parameters goes to zero. The goal is to develop a set of increasingly accurate asymptotic variational models allowing one to deal with the cases when this parameter is 'small' but finite. Since Γ-convergence may be non-uniform within the 'theory', we pose a problem of finding a uniform approximation. To achieve this goal we propose a method based on rectifying the singular points in the parameter space by using a blow-up argument and then asymptotically matching the approximations around such points with the regular approximation away from them. We illustrate the main ideas with physically meaningful examples covering a broad set of subjects from homogenization and dimension reduction to fracture and phase transitions. In particular, we give considerable attention to the problem of transition from discrete to continuum when the internal and external scales are not well separated, and one has to deal with the so-called 'size' or 'scale' effects.

Keywords Asymptotic expansions · Gamma-convergence · Uniformity · Fracture · Phase transitions · Homogenization · Multi-scale modelling

PACS 02:30.Xx · 46.50.+a · 62.20.mm

0 Introduction

Most of the models in continuum mechanics describing equilibrium configurations are based on the minimization of functionals which contain a small parameter of either constitutive or geometrical nature. It is natural to try to use the smallness of the parameter to replace the original model by a simpler one. The well-known examples of such simplifications are the low-dimensional theories of thin-walled structural elements (e.g., [26]), the homogenized models of composite materials (e.g., [48]) and the continuum models of crystal lattices (e.g., [11]). In all those cases the approximate models are more tractable than their prototypes because they do not contain the small parameter and enjoy the advantages of reduced dimensionality, homogeneity or continuity. Often, the simplified model can be constructed by the more or less straightforward dropping of the 'small' terms. While such pointwise limits can be rigorously justified in some situations (e.g., [7,9]), there are other cases when the limit is non-trivial due to only weak convergence of the minimizers (e.g., [67]). In those cases in order to derive the limiting theory, one has to use more sophisticated methods, in particular, the methods of Γ-convergence [14,28,32].

Communicated by L. Evans

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In some cases the limiting models derived by the methods of Γ-convergence are degenerate, and it is clear that a physically meaningful model must contain the original small parameter. Typical examples include the plate theories combining membrane deformations with bending [37], the theories of phase transitions with surface tension [49], and the theories of fracture assuming a nonzero toughness [39]. The small parameter must also be preserved when the ratio of the internal to external scales is small but not too small, for instance, when the main modeled phenomenon is associated with the 'size' or 'scale' effects [33,62]. Heuristic models of this type have been suggested in applications and it is not uncommon that several such approximations have overlapping domains of application (e.g., Bernoulli and Timoshenko theories for rods [7,43,52], Kirchhoff and Mindlin–Reissner models for plates [3,7,43], gradient and nonlocal continuum models of crystal elasticity [41,49,59], various regularizations in the models of phase transitions and strain localization [6,44]). The present study has been originally motivated by the desire of the authors to rigorously distinguish between the theories of Griffith [39] and Barenblatt [5] in fracture mechanics under the assumption that they represent various asymptotic limits of a lattice model with Lennard–Jones interactions [19,64].

To justify the higher-order corrections rigorously, it is necessary to formally define Γ-asymptotic expansion. In the current mathematical literature, the issue is usually addressed by constructing a Γ-limit in the proper limiting space and then improving it inside the same space through increasingly more accurate approximations of the minimal value (Γ-*development*, see [2]). The obvious problem with this approach is that the class of minimizers is decided already in the first step, and after this class is exhausted, the process of the improvement of the minimizer terminates (*locking*). In addition, as we show in the paper, the higher-order Γ-limits may simply ceases to exist (*choking*). Another type of problems concerns parameterized families of functionals, since at certain value of the parameter the Γ-convergence may cease to be uniform. In these cases we encounter physically interesting singular phenomena, associated, for instance, with nucleation, buckling or material failure, and it is disastrous that around these singular points the precision of the conventional Γ-development drops dramatically.

In this paper we propose a new methodology aimed at overcoming the drawbacks of the existing formal approach and reconciliating the ad hoc approaches with Γ-convergence. Our starting point is the concept of Γ-asymptotically equivalent functionals of given order, which generalizes the corresponding concept in the classical perturbation methods [36,40,54]. We show that, outside the zero-order Γ-limit, the class of Γ-equivalent variational approximations of a given minimization problem may be rather large and raise the question of the additional criteria securing the uniqueness of the asymptotic expansion. In search of such criteria, we move from single functionals to 'theories' interpreting the latter as parameterized sets of functionals, where parameters may characterize the geometry of the domain, the boundary conditions, the bulk 'loading', or the constitutive behavior (see examples in [27,37,46,60]). To deal with the parameter values where the Γ-limit is singular (for instance, discontinuous), we propose a way of rectifying the singular behavior by local blow up of the functional and constructing the 'table' of Γ-limits, which fully characterizes the crossover between different singular regimes. We then show that if the 'tables' in all singular points are known the uniform expansion can be constructed by matching the boundary layer type Γ-expansions near the singular points with the regular Γ-expansions outside these points.

Despite its mainly theoretical focus, the paper contains a series of illuminating examples of equivalent theories and discusses the multiplicity of ways of generating uniform approximations with respect to classes of boundary conditions. In our selection of examples, we intentionally resisted the temptation of dealing with the most general cases and instead limited ourselves almost exclusively to the problems where a one-dimensional, scalar version of a particular model could substitute its multi-dimensional, tensorial analog. Some of our examples use arguments that can be derived from the known theories, and in these cases the details are not included or only sketched. The full proofs are given only for the cases that are not present in the literature.

The paper is organized as follows. To motivate the subsequent definitions, in Sect. 1 we review the existing asymptotic procedures based on Γ-convergence. In Sect. 2 we introduce 'theories' and then show in Sect. 3 that within a 'theory' the standard Γ-development can be nonuniform. In Sect. 4 we introduce the concept of Γ-equivalence and study the main properties of the asymptotic factorization of the set of functionals. Some systematic methods of generating Γ-equivalent functionals are proposed and discussed in Sect. 5. In Sect. 6 we extend the definition of Γ-equivalence to 'theories'. The regular points within the 'theories' are studied in Sect. 7. The structure of the singular points constitute the subject of Sect. 8 where we also formally define a 'table' of Γ-limits and present several detailed computations of various 'tables' in the problems of physical interest. In Sect. 9 we pose the problem of rectifying singular points and produce a rather general recipe for constructing uniformly equivalent theories. Our conclusions are summarized in Sect. 10.

1 Background

Consider a minimization problem involving a small parameter ε. Suppose that one would like to approximate the original problem with a new one where the dependence on this parameter has been eliminated. In the situation where the object of interest is the global minimum of the energy, the adequate language has been developed by De Giorgi [30,32]. His approach is based on the notion of Γ-convergence which we briefly review below to make our presentation self-contained (see [14,15,28] for more details).

The first requirement to Γ-convergence is that

$$F_\varepsilon \xrightarrow{\Gamma} F^{(0)} \tag{1}$$

implies

$$\min F_\varepsilon \to \min F^{(0)}. \tag{2}$$

The second requirement is that (almost) minimizers of F_ε converge to minimizers of $F^{(0)}$

$$u_\varepsilon \longrightarrow u^{(0)}, \tag{3}$$

even though the meaning of convergence may be very weak. The third requirement concerns the stability of the Γ-limit with respect to the addition of continuous perturbations

$$(F_\varepsilon + G) \xrightarrow{\Gamma} (F^{(0)} + G). \tag{4}$$

If the condition (4) is satisfied, then, once the Γ-limit $F^{(0)}$ is computed, the result can be used to describe a whole class of problems. This implies that Γ-convergence can deal with some simple 'theories'.

We now proceed with the formal definition:

Definition 1.1 Let X be a first-countable space (e.g., a metric space) and let $F_\varepsilon : X \to [-\infty, +\infty]$. Then F_ε Γ-*converges* to F_0 as $\varepsilon \to 0$ (and F_0 is the Γ-*limit* of F_ε) if the following two conditions are satisfied for all $x \in X$:

(i) (lim inf inequality) for all $x_\varepsilon \to x$ $F_0(x) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon)$.
(ii) (existence of a recovery sequence) there exists $\overline{x}_\varepsilon \to x$ such that $F_0(x) = \lim_{\varepsilon\to 0} F_\varepsilon(\overline{x}_\varepsilon)$.

From this definition one can see that implication (1) ⇒ (2) is valid if some *equi-coerciveness* assumptions on F_ε are satisfied (i.e., if we may find converging minimizing sequences) and throughout this paper, we suppose that such assumptions indeed hold.

We illustrate the notion of Γ-equivalence by following examples. They deal with the derivation of a continuum elasticity theory as the asymptotic limit of different lattice models.

Example 1.2 For ε such that $N = \frac{1}{\varepsilon} \in \mathbb{N}$ consider the functional

$$F_\varepsilon(u) = \varepsilon \sum_{i=1}^{N} W(u_i) \tag{5}$$

where $u : \{1, \ldots, N\} \to \mathbb{R}$ and

$$\varepsilon \sum_{i=1}^{N} u_i = 0. \tag{6}$$

The energy (5) describes a chain of particles with nearest neighbor (NN) interactions. The system can also be interpreted as a set of springs connected in series and therefore not interacting modulo the 'mean field' interaction with a hard loading device. Suppose that W is strictly convex. Then one can show that (see [21])

$$F^{(0)}(u) = \int_0^1 W(u)\mathrm{d}t, \tag{7}$$

with constraint

$$\int_0^1 u(s)\,\mathrm{d}s = 0. \tag{8}$$

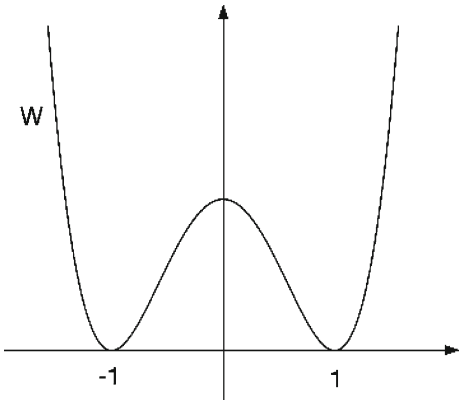

Fig. 1 A double-well potential (Example 1.4)

Example 1.3 To show that the previous result is stable with respect to adding long-range interactions, we introduce a constant $J > 0$ and consider the following 'elastic' Ising model

$$F_\varepsilon(u) = \sum_{i=1}^{N} \varepsilon \left(W(u_i) - J u_i u_{i+1} \right), \tag{9}$$

where we set $u_{N+1} = u_1$ to avoid boundary effects. We again impose the constraint (6). If we now rewrite the energy in the form

$$F_\varepsilon(u) = \sum_{i=1}^{N} \varepsilon \left(\widetilde{W}(u_i) - 2J \left| \frac{u_i + u_{i+1}}{2} \right|^2 \right),$$

it can be interpreted as the model of a chain with the nonlinear interaction of nearest neighbors characterized by the convex potential

$$\widetilde{W}(z) = W(z) + J|z|^2,$$

and an additional linear interaction of the next to nearest neighbors (NNN model). Suppose that $W(z) \geq C(|z|^2 - 1)$ with $C > J$, so that the energies F_ε are equi-coercive. Then, if the function

$$\overline{W}(z) = \widetilde{W}(z) - 2J|z|^2 = W(z) - J|z|^2.$$

is convex one can show that (see [21] for more details)

$$F^{(0)}(u) = \int_0^1 \overline{W}(u)\, dt.$$

The two examples above represent the simplest cases of periodic convex homogenization where the result could also be obtained by pointwise convergence. The next example shows the simplest case where the notion of Γ-limit is essential.

Example 1.4 Consider again Example 1.2 and suppose now that W is a double-well potential as shown in Fig. 1. We obtain (see [14] Theorem 4.3)

$$F^{(0)}(u) = \int_0^1 W^{**}(u)\, \mathrm{d}t, \qquad \int_0^1 u\, \mathrm{d}t = 0.$$

Here and often in the sequel, we denote by W^{**} the convex envelope of W, whose appearance highlights the formation of microstructure.

1.1 Γ-development

If the description given by $F^{(0)}$ is too coarse, further information can be obtained by iteration of the Γ-limit procedure. More specifically if some $\alpha > 0$ exists such that

$$F_\varepsilon^{(\alpha)} := \frac{F_\varepsilon - \min F^{(0)}}{\varepsilon^\alpha} \xrightarrow{\Gamma} F^\alpha, \tag{10}$$

then, using again the fundamental property (2) of Γ-convergence, we obtain

$$\min F_\varepsilon^{(\alpha)} \left(= \frac{\min F_\varepsilon - \min F^{(0)}}{\varepsilon^\alpha} \right) \to \min F^{(\alpha)}. \tag{11}$$

In other words, one can write the more accurate development

$$\min F_\varepsilon = \min F^{(0)} + \varepsilon^\alpha \min F^{(\alpha)} + o(\varepsilon^\alpha). \tag{12}$$

Remark 1.5 The convergence of minima takes place if there exist a precompact sequence of minimizers of F_ε, or, more in general, if there exists a precompact ε^α-minimizing sequence; i.e., x_ε such that $F_\varepsilon(x_\varepsilon) = \inf F_\varepsilon + o(\varepsilon^\alpha)$, which implies the equi-coerciveness of $F_\varepsilon^{(\alpha)}$.

The process of Γ-*development* [2] (or development by Γ-convergence) is formally resumed in the equality

$$F_\varepsilon \overset{\Gamma}{=} F^{(0)} + \varepsilon^\alpha F^{(\alpha)} + o(\varepsilon^\alpha). \tag{13}$$

The general equality (13) is only formal since the domains of the functionals may be different, and even when they are equal the energy $F^{(0)} + \varepsilon^\alpha F^{(\alpha)}$ is equal to $+\infty$ outside the set of minimizers of $F^{(0)}$.

We say that a Γ-development is *complete* if for all $0 < \gamma < \alpha$ we have

$$F^{(\gamma)}(u) := \Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon(u) - \min F^{(0)}}{\varepsilon^\gamma} = \begin{cases} 0 & \text{if } u \text{ is a minimizer of } F^{(0)} \\ +\infty & \text{otherwise} \end{cases}$$

Note that if the Γ-development is not complete, i.e. $F^{(\gamma)}$ does not have the form above for some γ, but $\min F^{(\gamma)} = 0$, then such $F^{(\gamma)}$ plays no role in (12).

As an illustration we may compute the scaled Γ-limit of the energies in Example 1.2 at various orders. The computation shows that all higher-order limits are trivial; therefore, the development is complete already after the first step. This explains the 'stiffness' of the classical elasticity theory which works in a remarkably broad range of scales from meters to nanometers.

Our first example of a nontrivial development illustrates the 'size' effect in classical elasticity due to lattice incompatibility with the 'shape' of the macroscopic boundary.

Example 1.6 To capture the incompatibility effect in the one-dimensional setting we consider a functional almost equivalent to the one in the Example 1.2,

$$F_\varepsilon(u) = \sum_{\{i:\varepsilon i\in(0,1)\}} \varepsilon\, W(u_i), \tag{14}$$

again with constraint (6). While the zero-order Γ-limit is the same as the one in Example 1.2, to proceed further a correction must be made at order ε. If in place of $F_\varepsilon(u)$, we consider

$$\tilde{F}_\varepsilon(u) = \sum_{\{i:\varepsilon i\in(0,1)\}} \varepsilon\, W(u_i) - c_\varepsilon, \tag{15}$$

with constraint (8), and

$$c_\varepsilon = \varepsilon \left(\left[\frac{1}{\varepsilon} - 1 \right] - \frac{1}{\varepsilon} \right) W(0)$$

(here $[t]$ is the integer part of t), then the development is the same as in Example 1.2. The constant c_ε is due to the approximate identification of a discrete function u with its piecewise-constant counterpart in $(0, 1)$. This can be easily dealt with in dimension 1, but may be a delicate problem in higher dimension. In the rest of the paper we will mainly stick to the case $\varepsilon = 1/N$ with $N \in \mathbb{N}$ and avoid inaccuracy problems. In the general case some additive constant c_ε must be taken into account as in the present example.

Remark 1.7 In the computation of the higher-order Γ-limits some non-trivial scale analysis must be performed to understand what is the relevant scaling ε^α (or, more general, $f(\varepsilon)$). Note however that, up to rescaling, we can always suppose that $F^{(0)}$ is non-trivial and, if needed, that the next relevant scale is ε.

Once the first development is computed, the analysis at successively lower scales

$$1 >> f_1(\varepsilon) >> \cdots >> f_m(\varepsilon)$$

can be performed by iteration (in these notations, in the development above we have taken $f_1(\varepsilon) = f(\varepsilon)$, or ε^α). Then we obtain a development

$$F_\varepsilon \overset{\Gamma}{=} F^{(0)} + f_1(\varepsilon)F^{(1)} + \cdots + f_m(\varepsilon)F^{(m)} + o(f_m(\varepsilon)), \tag{16}$$

where (with a little abuse of notation with respect to (13)), we have set

$$F^{(j)} = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon^{(j)}, \text{ where } F_\varepsilon^{(j)}(u) := \frac{F_\varepsilon(u) - \sum_{i<j} f_i(\varepsilon)\, m^{(i)}}{f_j(\varepsilon)}, \tag{17}$$

and $m^{(i)} = \min F^{(i)}$.

We now discuss some limitations of the straightforward Γ-development.

1.2 'Locking' of minimizers

A rather unfortunate consequence of the definition (11) is that the approximate energy is infinite outside the set of minimizers of $F^{(0)}$. The latter may reduce to a trivial set (e.g., a single point) that cannot be refined by the successive Γ-limits. In what follows we shall be referring to this property of the straightforward Γ-development as the *locking* of the minimizers.

Remark 1.8 If $F^{(0)}$ has a unique minimum point u_0 then the computation of $F^{(\alpha)}$ reduces to that of $F^{(\alpha)}(u_0)$. If F_ε are equi-coercive, then this amounts to computing

$$F^{(\alpha)}(u_0) \ \left(= \min F^{(\alpha)}\right) = \lim_{\varepsilon\to 0} \frac{\min F_\varepsilon - \min F^{(0)}}{\varepsilon^\alpha}.$$

This observation will be frequently used in the sequel.

Example 1.9 To illustrate the locking phenomenon, consider

$$F_\varepsilon(u) = \int_0^1 (|u'|^2 + \varepsilon|u|^2)\mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1. \tag{18}$$

We can compute the Γ-development with respect to the strong L^2 topology (or equivalently with respect to the weak H^1 topology), and obtain $I_\varepsilon \overset{\Gamma}{=} F^{(0)} + \varepsilon F^{(1)} + o(\varepsilon)$, where

$$F^{(0)}(u) = \int_0^1 |u'|^2\mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1. \tag{19}$$

and

$$F^{(1)}(u) = \begin{cases} \frac{1}{3} & \text{if } u = t \\ +\infty & \text{otherwise.} \end{cases} \tag{20}$$

The computation of the last Γ-limit is trivial since the problem (19) admits the only minimum point

$$u(t) = t.$$

This expression should be compared to the actual minimizer u_ε of the original problem, whose formal asymptotic expansion goes as follows

$$u_\varepsilon(t) = t + \varepsilon\frac{1}{6}(t^3 - t) - \varepsilon^3\frac{1}{18}(t - t^3) + o(\varepsilon^2).$$

One can see that the successive Γ-development locks the minimizer and does not allow to improve its quality beyond what have been found in the first step even though finer and finer information about the minimizer is needed to compute the higher-order Γ-limits.

The next example shows that the locking of the minimizer may not happen at the level of the first approximation but may be delayed and occur during the subsequent higher-order development.

Example 1.10 Let $W : \mathbb{R} \to \mathbb{R}$ be a continuous double-well potential with wells (absolute minima) in ± 1 and more than linear growth at ∞ (e.g., $W(s) = \min\{(s+1)^2, (s-1)^2\}$). Consider

$$F_\varepsilon(u) = \int_\Omega (W(u) + C\varepsilon^2|\nabla u|^2)\,\mathrm{d}x, \qquad u \in H^1(\Omega), \int_\Omega u\mathrm{d}x = 0.$$

If we use the weak L^1-topology, suggested by the superlinear growth conditions of W at infinity, then the first Γ-limit is (see, e.g., [14,15])

$$F^{(0)}(u) = \int_\Omega W^{**}(u)\,\mathrm{d}x, \quad u \in L^1(\Omega), \int_\Omega u\mathrm{d}x = 0. \tag{21}$$

Note that even though we may find piecewise-constant minimizers with $u = \pm 1$, due to the degeneracy of the energy (21) they are not unique. Only the measures of the sets where $u = 1$ and $u = -1$ are known at this stage while the location of the interfaces (internal boundary layers) between the states with $u = 1$ and $u = -1$ and even their number remain unspecified. This information, however, can be recovered in the next step of Γ-development, which locks the minimizer. Indeed, we obtain (see, e.g., [14,15])

$$F^{(1)}(u) = c_W \mathcal{H}^{n-1}(S(u)), \qquad |u| = 1 \tag{22}$$

Here $u \in \{\pm 1\}$ is piecewise constant, $\int_\Omega u\mathrm{d}x = 0$ and

$$c_W = 2\sqrt{C}\int_{-1}^{1}\sqrt{W(s) - \min W}\,\mathrm{d}s,$$

where $S(u)$ denotes the interface between the phases $\{u = \pm 1\}$ and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional (surface) measure.

The minimization of the functional (22) fixes the location of the interface and locks the minimizer. In the subsequent approximations only the minimal value is changing. Interestingly, the relevant scale successive to ε, is of exponential type $\varepsilon e^{-c_1/\varepsilon}$ and not of the form ε^α [23,70]. Indeed, we recall that in the one-dimensional case with $\Omega = (0, 1)$ the set of 'locked' minimizers is $\{u_0, -u_0\}$, where

$$u_0(t) = \begin{cases} -1 & \text{if } t < 1/2, \\ 1 & \text{if } t > 1/2. \end{cases}$$

The development reads as

$$F_\varepsilon \overset{\Gamma}{=} F^{(0)} + \varepsilon F^{(1)} + \varepsilon e^{-1/2\varepsilon} F^{\infty} + o(\varepsilon e^{-1/2\varepsilon}),$$

where

$$F^{\infty}(u) = \begin{cases} C^{\infty} & \text{if } u = \pm u_0 \\ +\infty & \text{otherwise,} \end{cases}$$

and the constant C^{∞} may be computed in terms of the limit of minimum problems as in Remark 1.8. One can see that, the first approximation (at scale 1) locks the phase fractions, the second (at scale ε) fixes the geometry of the interface, and the higher-order approximations describe exponentially weak corrections due to interaction of the interface with the external boundary (size effect). The latter will be the subject of a detailed analysis in Example 8.4, where we consider setting where the size effect is dominant.

Remark 1.11 Since the weak-L^1 equi-coerciveness improves to *strong*-L^1 coerciveness at scale ε, then we may (a posteriori) choose to compute the first Γ-limit $F^{(0)}$ with respect to the strong L^1-topology, obtaining

$$F^{(0)}(u) = \int_\Omega W(u)\,\mathrm{d}x,$$

while $F^{(1)}$ remains unchanged. This shows that sometimes the Γ-limit may look superficially as a pointwise limit even if the argumentation behind its derivation is entirely different.

1.3 'Choking' of Γ-development

Locking of the minimizer is not the only problem which one may encounter while constructing the Γ-development. Thus, one of the higher-order limits may simply cease to exist. We illustrate this phenomenon by the following examples:

Example 1.12 Consider a non-constant strictly positive and bounded one-periodic function $a : \mathbb{R} \to \mathbb{R}$, and the functionals

$$F_\varepsilon(u) = \int_0^1 a\left(\frac{t}{\varepsilon}\right) |u'|^2 \mathrm{d}t \tag{23}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u(1) = 1.$$

A standard argument shows that for all strictly positive and bounded $f : [0, 1] \to \mathbb{R}$, we have

$$\min\left\{ \int_0^1 f(t)|u'|^2 \mathrm{d}t : \ u(0) = 0,\ u(1) = 1 \right\} = \underline{f}, \ \text{where } \frac{1}{\underline{f}} = \int_0^1 \frac{1}{f(s)}\,\mathrm{d}s. \tag{24}$$

We can now apply this result to $f(s) = a_\varepsilon(s) = a(s/\varepsilon)$, and recalling that $a_\varepsilon^{-1} \rightharpoonup (\underline{a})^{-1}$, obtain the well-know fact that the Γ-limit of F_ε is (see , for instance, [20])

$$F^{(0)}(u) = \underline{a} \int_0^1 |u'|^2 \mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1. \tag{25}$$

The unique minimum point of $F^{(0)}$ is $\overline{u}(t) = t$, and in order to compute any further development it suffices to compute it at this function (locking). The next meaningful order is ε. The Γ-limit makes sense for sequences $\varepsilon_j \to 0$ if there exists the limit (see Remark 1.8)

$$\lim_j \frac{1}{\varepsilon_j}\left(\min\left\{\int_0^1 a\left(\frac{t}{\varepsilon_j}\right)|u'|^2 \mathrm{d}t : \ u(0)=0,\ u(1)=1\right\} - \underline{a}\right)$$
$$= \lim_j \frac{1}{\varepsilon_j}\left(\left(\int_0^1 \frac{1}{a(s/\varepsilon_j)}\,\mathrm{d}s\right)^{-1} - \underline{a}\right) = \lim_j \underline{a}^2 \int_{[1/\varepsilon_j]}^{1/\varepsilon_j}\left(\frac{1}{\underline{a}} - \frac{1}{a(s)}\right)\mathrm{d}s. \tag{26}$$

The value of the limit (26) (equal to $F^{(1)}(\overline{u})$) depends on the sequence (ε_j). This means that the development at order ε does not exist, which terminates the process and prevents one from improving upon the first approximation.

Here we encounter another example of the interaction between the boundary of a body and the homogenization procedure. In the case when the scales are well separated $\varepsilon \ll 1$ (i.e., when we consider only the development at order 1) the energy associated with the boundary layers is negligible. As the external (1) and internal (ε) scales get closer to each other, one enters the domain of the size effect where the oscillatory structure of the corrections to the homogenized theory, preventing the minimal value of the approximate functional from converging, becomes more and more noticeable.

The next example shows that the Γ-development may not terminate before a sufficiently high order of the approximation.

Example 1.13 Consider again Example 1.4 where the functional F_ε (see (5)) can be thought as the discretization of the continuum energy $\int_0^1 W(u)\,dt$. We assume that the discretization is perfectly compatible with the 'shape' of the body, meaning that we set $\varepsilon = 1/N$. In Example 1.4 we have obtained the following result:

$$F^{(0)}(u) = \int_0^1 W^{**}(u)\,\mathrm{d}t, \qquad \int_0^1 u\,\mathrm{d}t = 0.$$

The computation of the Γ-limit at scale ε gives the same trivial result, as in the convex case. On the contrary, the Γ-limit related to the scale ε^2 , which is trivial in the convex case, does not exist in the non-convex case. To justify this result we refer to the study of the parameterized minimum problems

$$m_\varepsilon(\lambda) = \min\{F_\varepsilon(u) : \ \sum_i \varepsilon u_i = \lambda\}.$$

In the case $W(s) = (s-1)^2 \wedge (s+1)^2$, an analysis of the exact solution of this discrete problem (e.g., [35,57]) pictures the dependence on λ as in Fig. 2 from which we see that $\varepsilon^{-2} m_\varepsilon(0)$ does not converge to $0 = \min F^{(0)}$.

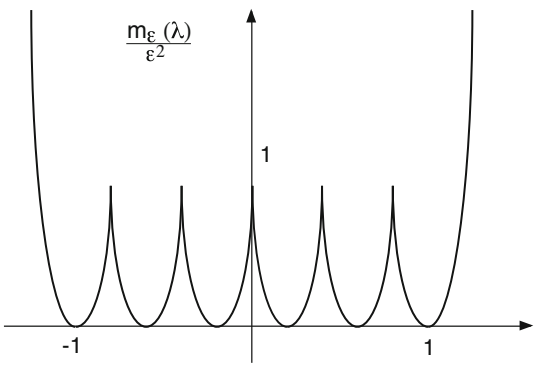

Fig. 2 Minimal energy of a chain of bi-stable springs (Example 1.13)

Indeed, the minimal value at, say, $\lambda = 0$ fluctuates as the small parameter goes to zero. This example shows that for non-convex energies homogenization methods must take into account the scale of the approximation and that different theories may have to be used at different scales (contrary to what one expects in the convex case).

If the parameter λ in Example 1.13 is allowed to vary, the non-existence of the second-order Γ-limit persists for the whole interval $\lambda \in (-1, 1)$. In the next section we discuss more systematically some other typical problems which arise when the Γ-development is applied to a parameterized family of functionals.

2 Theories

We recall that Γ-convergence has been designed to automatically handle functionals parameterized by 'lower-order terms'. In this case the parametrization does not affect the first-order Γ-limit in an essential way, with the corresponding terms being either continuous perturbation or in some way 'compatible' with Γ-convergence. However higher-order Γ-limits do not enjoy the same 'invariance' property with respect to such apparently friendly extensions. Moreover, in the typical problems of interest the parameter enters the functional in a variety of ways that are not at all 'compatible' with Γ-convergence.

In what follows we shall refer to a class of minimization problems originating from a parameterized family of functionals as a 'theory'. This terminology comes from applications where one encounters, for instance, a multiplicity of theories of beams, plates and shells, theories of low or high frequency vibrations, quasi-continuum theories of crystals and cohesive theories of cracks. We adopt the following formal definition:

Definition 2.1 Let $\mathcal{E}$ be a set of positive real numbers with $0 \in \overline{\mathcal{E}}$, and let Λ be a subset of a topological space. Then a family of functionals F_ε^λ is called a *parameterized family* on Λ (the *space of parameters*) or a 'theory'.

We begin the analysis of the 'theories' by listing a series of examples.

1. *Van der Waals's theory of phase transitions* (e.g., [69]). Suppose that W is a double-well potential as in Example 1.10. Take $\Lambda = \mathbb{R}$ or $\Lambda = [-1, 1]$ and define

$$F_\varepsilon^\lambda(u) = \int_0^1 \left(W(u) + \varepsilon^2 |u'|^2\right) \mathrm{d}t, \qquad \int_0^1 u(t)\mathrm{d}t = \lambda. \tag{27}$$

In this case λ represents an imposed integral constraint representing, for instance, average strain in a bar if $v' = u$. A typical problem in this theory is to determine the function

$$m(\lambda, \varepsilon) = m_\varepsilon(\lambda) = \min \left\{ \int_0^1 (W(u) + \varepsilon^2 |u'|^2)\mathrm{d}t : \int_0^1 u\mathrm{d}t = \lambda \right\}, \tag{28}$$

whose first derivative $m'_\varepsilon(\lambda)$ gives the effective stress-strain relation. In the limit $\varepsilon \to 0$, we obtain the famous Maxwell 'common-tangent' construction (e.g., [7]); however, as we show below, the resulting theory does not handle nucleation appropriately, even in the restricted framework of global minimization. To deal with nucleation one needs to develop an approximation which accounts for ε. More broadly, different theories of type (27) have recently been unified under the general title of 'phase-field' models; the main goal of the phase-field model is numerical capturing of sharp discontinuities (e.g., [24]). In this framework it is of interest to construct an intermediate theory which avoids the drawbacks of the sharp interface limit but does not have to resolve the higher derivative terms where it is not absolutely necessary.

2. *1D Lattice theory of fracture* (e.g., ([64])). Let $J : [0, +\infty) \to \mathbb{R}$ be a Lennard–Jones interatomic potential with minimum in 1 and vanishing at $+\infty$ (see Fig. 3), $\varepsilon = 1/N$ with $N \in \mathbb{N}$, and consider the scaled energy

$$F_\varepsilon^\lambda(u) = \sum_{i=1}^{N} \varepsilon\, J\left(\frac{u_i - u_{i-1}}{\varepsilon}\right) \tag{29}$$

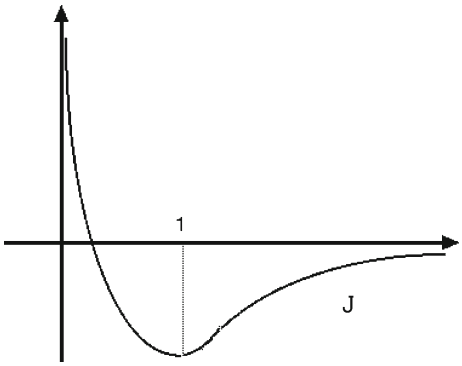

Fig. 3 A Lennard–Jones potential (Example 3.2)

with boundary conditions $u_0 = 0$, $u_N = \lambda$. In this case λ represents an imposed displacement in a hard device. A problem of interest is to compute

$$m(\lambda, \varepsilon) = m_\varepsilon(\lambda) = \min \left\{ \sum_{i=1}^{N} \varepsilon\, J\left(\frac{u_i - u_{i-1}}{\varepsilon}\right), \quad u_0 = 0,\ u_N = \lambda \right\},$$

whose first derivative again defines the effective stress–strain relation. When $\varepsilon = 0$, one obtains a material that does not support tension and breaks at infinitesimal tension ([19,64]). Real cracks, on the contrary, appear only at finite tension. The challenge is then to capture this effect when ε is small but finite. While the theory of fracture in this example looks superficially very different from the theory of phase transitions discussed above, we show in what follows that the two theories are in fact remarkably similar.

3. *Homogenization theory* (e.g., [48]). Take $\Lambda = (0, +\infty)$ and consider a functional

$$F_\varepsilon^\lambda(u) = \frac{1}{\lambda} \int_0^\lambda a\left(\frac{t}{\varepsilon}\right) |u'|^2 \mathrm{d}t \qquad u(0) = 0,\ u(\lambda) = \lambda. \tag{30}$$

In this case λ is a geometrical parameter (the length of a bar). The typical problem in this theory is to find the minimum $m(\lambda, \varepsilon)$ of F_ε^λ and then compute the effective elastic modulus $\bar{a} = 2m(\lambda, \varepsilon)/\lambda^2$. The problem has a classical homogenization solution when $\varepsilon = 0$. The computation of a correction to this result at $\varepsilon \sim \lambda$ constitutes the main task of the theory of 'size effect' in homogenization.

4. *Theory of finite scale micro-structures* (e.g., [58]). This title refers to the broad class of models with competing interactions where certain factors drive the coarsening of the microstructure while the other factors enforce its refinement. Here we consider the simplest model of this type [1,65]. Suppose that $\Lambda = [0, +\infty)$ and define

$$F_\varepsilon^\lambda(u) = \int_0^1 \left(W(u') + \varepsilon^2 |u''|^2 + \lambda u^2\right) \mathrm{d}t, \qquad u(0) = u(1) = 0. \tag{31}$$

The parameter λ represents a combination of material and geometrical parameters and characterizes the 'anti-ferromagnetic' component of the interactions which drives the system towards the refinement of the microstructure. This interaction competes with a 'ferromagnetic' contribution due to the ε term in the energy, which drives the system towards coarsening. The problem is to characterize the scale of the microstructure in the limit when $\varepsilon \to 0$. Due to the approximately periodic arrangement of the optimal microstructure, the adequate parameter is the density of interfaces for the minimizer $N(\lambda, \varepsilon)$. If λ is finite, then $N(\lambda, 0) = \infty$. An interesting question is to predict the value of $N(\lambda, \varepsilon)$ when $\varepsilon \sim \lambda$ and λ is small but finite. Other examples of models with competing interactions can be found in [4,58].

3 Non-uniformity of Γ-developments

In this section we further elaborate on the first two of the above examples in order to illustrate the typical problems encountered by the straightforward Γ-development applied to 'theories'.

Example 3.1 Consider the family of functionals (27). It is well known (see [15,50]) that the volume constraint $\int u\,dt = \lambda$ is compatible with the Γ-limit procedure. We recall that for $|\lambda| \geq 1$ the unique minimizer of F_ε^λ is the constant state $u = \lambda$ for all ε, the Γ-development consist only of one term, and $m_\varepsilon(\lambda) = m^{(0)}(\lambda) = W^{**}(\lambda)$. For all $\lambda \in (-1, 1)$ the development of the minimum values is given by $m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda) + o(\varepsilon)$, where

$$m^{(1)}(\lambda) = c_W \min\left\{\#(S(u)) : u \in \{\pm 1\}, \int_0^1 u\mathrm{d}t = \lambda\right\} = c_W.$$

The plot of the function $m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda)$ illustrating the Γ-development for the minimum values of the functional (27) is given in Fig. 4 together with the corresponding effective stress-strain relation. The horizontal segment on the stress-strain curve between $\lambda = -1$ and $\lambda = 1$ results from Maxwell construction. Observe that nucleation (annihilation) takes place at points 1 and -1 and that the newly forming nucleus has infinitesimal 'size'. This can be compared with the exact solution of the problem at finite ε (e.g., [65,68]) showing that even in the global minimization framework the nucleation starts at a finite distance from the points 1 and -1 and that the first nucleus is finite. To capture this phenomenon our approximate theory of order ε, needs to be corrected near these points at the same order ε.

In a more formal language, we can reformulate the above observations as follows. We first recall that the value $m_\varepsilon(\lambda)$ is continuous with respect to λ, and in particular $\lim_{\lambda\to 1} m_\varepsilon(\lambda) = m^{(0)}(1)$. Then one can write

$$\lim_{\lambda\to 1^-} (m_\varepsilon(\lambda) - m^{(0)}(\lambda) - \varepsilon m^{(1)}(\lambda)) = -\varepsilon c_W,$$

from which we argue that the description given by the Γ-development at scale ε is not accurate close to the point 1. Similar result can be obtained for point -1.

The key point is that near both limits $\lambda = \pm 1$ the external length scale represented here by the distance between the interface and the external boundary and the internal length scale represented by the interface thickness are no longer 'separated': during nucleation both scales become comparable. One encounters this type of non-uniformity also in the multi-dimensional setting when the two interfaces get sufficiently close to each other or when the radius of curvature of one interface becomes comparable to the interface thickness (for instance, during topological transitions).

Despite its rather different formal appearance, the next example is very similar to the previous one. Here instead of the nucleation of a new phase we deal with the nucleation of a crack.

Example 3.2 Consider a lattice theory of fracture with the energy (29). After identifying discrete functions with their piecewise-affine interpolations the Γ-limit at order 1 can be computed as in the other cases involving the passage from discrete to continuous and is simply

$$F_\lambda^0(u) = \int_0^1 J^{**}(u')\mathrm{d}t$$

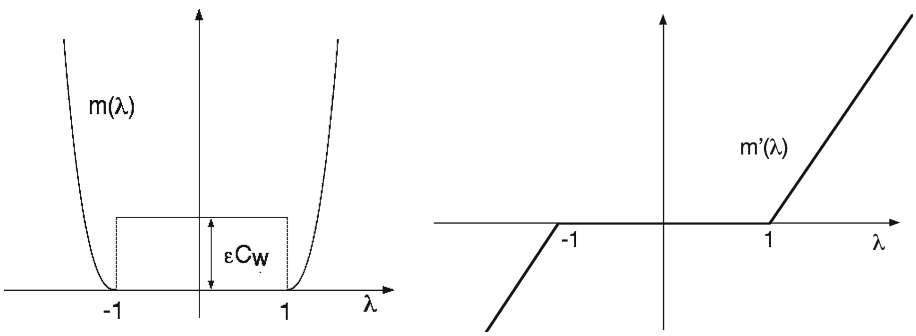

Fig. 4 Approximate minimum values by Γ-development (Example 3.1)

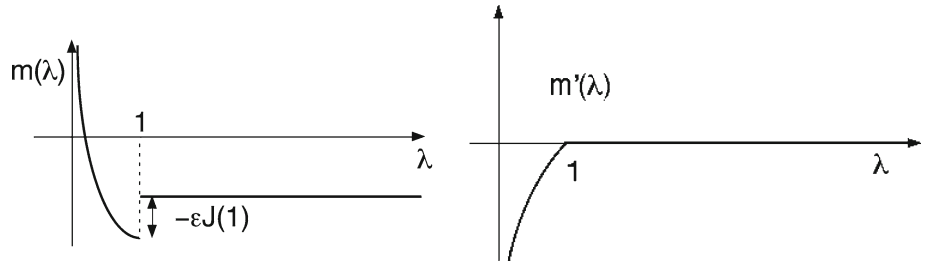

Fig. 5 Approximate minimum values by Γ-development (Example 3.2)

with boundary conditions $u(0) = 0$, $u(1) = \lambda$, defined on all increasing functions $u : [0, 1] \to \mathbb{R}$ such that $u(0) = 0$ and $u(1) = \lambda$ (see e.g., [14,64]; see also Example 6.5). Note that such functions may be discontinuous, but the derivative u' is defined almost everywhere. The integrand J^{**} is constant and equal to $\min J = J(1)$ on $[1, +\infty)$, so that

$$\min F_\lambda^0 = J^{**}(\lambda) = \begin{cases} J(\lambda) & \text{if } \lambda \le 1 \\ J(1) & \text{if } \lambda > 1. \end{cases}$$

To see this we use Jensen's inequality and obtain $\min F_\lambda^0 \ge J^{**}(\lambda)$. If $\lambda \le 1$ then $u_\lambda(t) = \lambda t$ is the only test function for which we have equality. If $\lambda > 1$ then all increasing functions satisfying the boundary conditions and with $u' \ge 1$ are minimizers; in particular the function

$$\widehat{u}_\lambda(t) = \begin{cases} \mathrm{t} & \text{if } 0 \le t < 1 \\ \lambda & \text{if } t = 1, \end{cases}$$

which satisfies the boundary conditions, jumps at $t = 1$, but has $u' = 1$ almost everywhere. One can see that in this approximation the effective material does not support any tension.

The next scale is ε, for which we have

$$F_\lambda^1(u) = \begin{cases} 0 & \text{if } u = u_\lambda \\ +\infty & \text{otherwise} \end{cases}$$

if $\lambda \le 1$, and

$$F_\lambda^1(u) = \begin{cases} -J(1)\#(S(u)) & \text{if } u \text{ is piecewise affine and } u' = 1 \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

for $\lambda > 1$. Here it is understood that u is increasing and satisfies the boundary conditions (see [14] for more details). In particular we see that $\widehat{u}_\lambda$ above is a minimizer for F_λ^1 and $\min F_\lambda^1 = -J(1)$.

Again we can compute the approximation of $m_\varepsilon(\lambda) = \min F_\varepsilon^\lambda$ given by the development by Γ-convergence $m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda)$, and we get

$$m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda) = \begin{cases} J(\lambda) & \text{if } \lambda \le 1 \\ J(1) - \varepsilon J(1) & \text{if } \lambda > 1. \end{cases}$$

The plot of the approximate minimum values obtained by Γ-development is given in Fig. 5 together with the effective stress-strain relation. One can see that the first-order refined theory again delivers the same physically absurd result that the fracture occurs at zero tension. More formally, as in the previous example, we can write

$$\lim_{\lambda \to 1^+} (m_\varepsilon(\lambda) - m^{(0)}(\lambda) - \varepsilon m^{(1)}(\lambda)) = \varepsilon J(1),$$

from which one can argue that the description given by the Γ-development at scale ε is not accurate close to the point 1. Here again, we deal with the phenomenon of nucleation, this time of a crack. For the newly

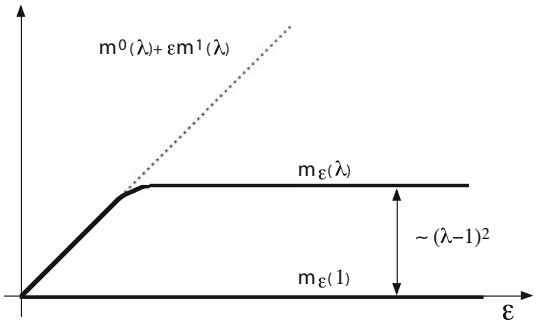

Fig. 6 Non-uniformity of the Γ-development associated with the nucleation phenomenon (Examples 3.1, 3.2)

formed micro-crack the opening is comparable to the small parameter ε, which breaks the scale separation legitimizing the Γ-development.

It is again instructive to compare the approximation of the minimal value delivered by the Γ-development and shown in Fig. 5 with the exact solution of the fracture problem at finite ε (see [19,64] for more details). The local picture near the singular point $\lambda = 1$ is exactly the same as in the case of phase transition problem near the points $\lambda = \pm 1$. Thus, again, at small but finite ε, one can always get sufficiently close to the nucleation point in order to find that the approximation which is supposed to capture the terms of the order ε makes an error at least of the same order. The resulting non-uniformity of the Γ-development is illustrated in Fig. 6, where we show the behavior of the function $m_\varepsilon(\lambda)$ near $\varepsilon \approx 0$ and $\lambda \approx 1$. One can see that no matter how small ε is, it is possible to choose λ sufficiently close to 1 to have the minimal value of the functional approximated by the standard development with an error of order ε. A similar picture can obviously be repeated for both singular points $\lambda = \pm 1$ appearing in the case of a phase transition.

Remark 3.3 Recall that the breakdown of uniform convergence (non-uniformity) in conventional asymptotic expansions for the functions $f_\varepsilon(x)$ is often due to the formation of boundary layers in the x space. As our examples show, in the case of functionals, the non-uniformity of the Γ-development can also present itself through the formation of boundary layers, but now for the function describing the distribution of the minimal values of the functional (function $m_\varepsilon(\lambda)$) in the space of parameters (λ space).

Our main goal in the rest of the paper is to develop an adequate *vocabulary* aimed at overcoming the above drawbacks of the straightforward Γ-convergence and to find the way of reinterpreting rigorously the 'good' approximate theories used in applications.

4 Γ-equivalence

The first observation is that equality of Γ-limits gives an equivalence relation between families of energies; i.e., if $\Gamma\text{-}\lim F_\varepsilon = \Gamma\text{-}\lim G_\varepsilon$ then we may say that F_ε is *equivalent* to G_ε. In this way the concept of Γ-limit can be replaced by that of an equivalence class. Note that the domain of equivalent F_ε and G_ε may be completely different.

Remark 4.1 In order not to make the extraction of a Γ-converging sequence a loss of generality, from now on we will tacitly assume that our Γ-limits are computed with respect to a separable metrizable convergence (which is usually the case in applications).

Definition 4.2 F_ε and G_ε are *equivalent at order* ε^α if there exist translations m_ε such that for all sequences ε_j for which the limits exist we have

$$\Gamma\text{-}\lim_j \frac{F_{\varepsilon_j} - m_{\varepsilon_j}}{\varepsilon_j^\alpha} = \Gamma\text{-}\lim_j \frac{G_{\varepsilon_j} - m_{\varepsilon_j}}{\varepsilon_j^\alpha},$$

and these limits are non-trivial (i.e., they do not take the value $-\infty$ and are not identically $+\infty$).

Observe, that in the spirit of (13), we may write

$$F_\varepsilon \overset{\Gamma}{=} G_\varepsilon + o(\varepsilon^\alpha),$$

even when no Γ-development of either functional exists. If $\alpha = 0$ within this definition, F_ε is equivalent to itself, even when (F_ε) does not converge, and two sequences converging to $+\infty$ are not always equivalent. Note that in the definition above, we may always choose $m_\varepsilon = \min F_\varepsilon$.

While we may also define 'equivalence at order $f(\varepsilon)$', where $f(\varepsilon)$ is any function of ε, we limit ourselves to the scaling ε^α for the sake of simplicity.

Theorem 4.3 *Let H_ε and H'_ε converge continuously to H (i.e, $H_\varepsilon(x_\varepsilon) \to H(x)$ if $x_\varepsilon \to x$; e.g., $H_\varepsilon = H$ a continuous function) and let F_ε and G_ε be equivalent at order ε^α; then $F_\varepsilon + \varepsilon^\alpha H_\varepsilon$ and $G_\varepsilon + \varepsilon^\alpha H'_\varepsilon$ are equivalent at order ε^α*

Proof This follows immediately from the definition, and reduces to the compatibility of the Γ-limit with respect to continuous perturbations if $\alpha = 0$. □

Theorem 4.4 *Let (F_ε) and (G_ε) be equi-coercive and equivalent at order ε^α; then we have*

$$\inf F_\varepsilon = \inf G_\varepsilon + o(\varepsilon^\alpha).$$

Proof The functionals

$$F^\alpha_\varepsilon = \frac{F_\varepsilon - m_\varepsilon}{\varepsilon^\alpha}, \qquad G^\alpha_\varepsilon = \frac{G_\varepsilon - m_\varepsilon}{\varepsilon^\alpha}$$

are equi-coercive. Given (ε_j) converging to 0, upon extraction of a subsequence, by the compactness of Γ-convergence, $F^\alpha_{\varepsilon_j} \to H$ and $G^\alpha_{\varepsilon_j} \to H$ for some coercive H; hence, we have

$$\begin{aligned}\lim_j \frac{\inf F_{\varepsilon_j} - \inf G_{\varepsilon_j}}{\varepsilon^\alpha_j} &= \lim_j \left(\frac{\inf F_{\varepsilon_j} - m_{\varepsilon_j}}{\varepsilon^\alpha_j} - \frac{\inf G_{\varepsilon_j} - m_{\varepsilon_j}}{\varepsilon^\alpha_j} \right) \\ &= \min H - \min H = 0.\end{aligned}$$

The statement of the theorem now follows from the arbitrariness of (ε_j). □

Note that, since we do not require the existence of the Γ-limits, even if minimizers exist, they may not converge. However, arguing by subsequences, we still deduce that the cluster points of ε^α-minimizers of F_ε are the same as those of G_ε. In the particular case when the functional H in the proof above has a unique minimizer, then we may conclude that minimizers of converging subsequences of F_ε and G_ε indexed by the same (ε_j) have the same limits, and in this sense are close.

Below we present several examples of equivalent functionals of different order. The first example illustrates the fact that already in the linear case a multiplicity of equivalent functionals can be easily generated without modifying the structure of the problem.

Example 4.5 An equivalent energy at order ε to F_ε in Example 1.9 can be searched among quadratic functionals of the form

$$G_\varepsilon(u) = \int_0^1 (a_\varepsilon |u'|^2 + b_\varepsilon |u|^2) \mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1, \tag{32}$$

The condition of equivalence at order 1 gives $a_\varepsilon = 1 + o(1)$ and $b_\varepsilon = o(1)$ and the condition of equivalence at order ε gives

$$3a_\varepsilon + b_\varepsilon = 3 + \varepsilon + o(\varepsilon).$$

One can choose for example either

$$G_\varepsilon(u) = \left(1 + \frac{\varepsilon}{3}\right) \int_0^1 |u'|^2 \mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1, \tag{33}$$

or

$$G_\varepsilon(u) = \int_0^1 ((1+\varepsilon)|u'|^2 - 2\varepsilon|u|^2)\mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1. \tag{34}$$

The next example shows that also in the nonlinear case, equivalent problems may have the same general form and differ only in details.

Example 4.6 Let

$$F_\varepsilon(u) = \int_\Omega \left(W(u) + C\varepsilon^2|\nabla u|^2\right) \mathrm{d}x$$

and

$$\widetilde{F}_\varepsilon(u) = \int_\Omega \left(\widetilde{W}(u) + \widetilde{C}\varepsilon^2|\nabla u|^2\right) \mathrm{d}x$$

be two energies as in Example 1.10 with W and $\widetilde{W}$ two double-well potentials with wells in ± 1. Then $\widetilde{F}_\varepsilon$ and F_ε are equivalent at order ε if and only if $\min W = \min \widetilde{W}$ and

$$\sqrt{C}\int_{-1}^1 \sqrt{W(s) - \min W}\,\mathrm{d}s = \sqrt{\widetilde{C}}\int_{-1}^1 \sqrt{\widetilde{W}(s) - \min \widetilde{W}}\,\mathrm{d}s.$$

In this case, by Example 1.10, they are both equivalent to

$$\mathcal{F}(u) = |\Omega| \min W + \varepsilon F^{(1)}(u),$$

with $F^{(1)}$ given by (22). The conditions of equivalence at order 1 are different if we take the weak or the strong L^1-topology. In the first case the condition is $W^{**} = (\widetilde{W})^{**}$; in the second one $W = \widetilde{W}$.

The next two examples illustrate the fact that equivalent theories may also have a rather different structure.

Example 4.7 Consider the discrete model (9) with the double-well energy

$$W(z) = \min\{(z-1)^2, (z+1)^2\},$$

and $J < 1$, so that F_ε are equi-coercive. It is easy to see that $\overline{W}$ from Example 1.3 is itself a double-well potential, with symmetric wells that we denote by $\pm a$. Let $m^0 = \min F^{(0)} = \min \overline{W} = \overline{W}(a)$.

We may now apply to F_ε the first-order analysis of [16] obtaining that the next meaningful scale is ε, and that the next term in the development is

$$F^{(1)}(u) = K_W \#(S(u)), \qquad u \in \{\pm a\},$$

where

$$K_W = \inf\left\{ \sum_{i=-\infty}^{+\infty} \left(\frac{1}{2}\left(W(v_i) + W(v_{i-1}) - 2J v_i v_{i+1}\right) - m^0\right) : \right.$$
$$\left. v : \mathbb{Z} \to \mathbb{R},\ v_i = -a \text{ if } i \le -N,\ v_i = a \text{ if } i \ge N,\ N \in \mathbb{N}\right\}$$

The value K_W represents the energy of an interface which is obtained by means of a 'discrete optimal-profile problem' connecting the two constant (minimal) states $\pm a$. Note that for fixed N the terms in the sum in the minimum problem are 0 for $i \ge N$ and $i < -N-1$; moreover the function $v_i = a \operatorname{sign} i$ is an admissible test function for all $N \ge 1$, from which we obtain $K_W \le 4Ja^2$.

This development gives the equivalent energy at scale ε

$$G_\varepsilon(u) = \begin{cases} m^0 + \varepsilon K_W \#(S(u)) & \text{if } u \in \{\pm a\} \\ +\infty & \text{otherwise.} \end{cases}$$

A comparison with the gradient theory of phase transitions shows that the functional

$$\widetilde{G}_\varepsilon(u) = \int_0^1 (\overline{W}(u) + \varepsilon^2 C|u'|^2)\,\mathrm{d}t, \tag{35}$$

is also equivalent to F_ε at order ε, provided that C is chosen such that

$$K_W = 2\sqrt{C}\int_{-a}^{a} \sqrt{\overline{W}(s) - \min \overline{W}}\,\mathrm{d}s.$$

The minimizers of the approximate energy (35) agree with the exact solution of the original problem which is known in the explicit form for finite ε [66]. It is interesting that another approximation at order ε with the same structure as in (35) but different $\overline{W}(u)$ and C can be formally obtained by a pointwise limit [10,66].

Remark 4.8 In the case of a general W the 'effective' potential $\overline{W}$ is given by the more complex formula

$$\overline{W}(z) = \frac{1}{2}\inf\{W(z_1) + W(z_2) : z_1 + z_2 = 2z\} - 2Jz^2$$

(see [14,21,56]) highlighting oscillations at microscopic scale. Some equivalent energies in this case can be deduced from the analysis in [16].

The next example shows that even the number of variables in equivalent theories with otherwise similar structure may be different.

Example 4.9 Consider a functional which one encounters in the Timoshenko theory of beams

$$F_\varepsilon(u,\phi) = \int_0^l \left(E|\phi'|^2 + \frac{H}{\varepsilon^2}(\phi - u')^2\right)\mathrm{d}t.$$

Below we prove (see also [7]) that the corresponding minimization problem is Γ-equivalent at order 1 to the more conventional Euler–Bernoulli bending problem characterized by the functional

$$G(u) = E\int_0^l |u''|^2\mathrm{d}t.$$

Here we have assumed the identification of G with

$$G(u,\phi) = \begin{cases} E\displaystyle\int_0^l |u''|^2\mathrm{d}t & \text{if } \phi = u' \\ +\infty & \text{otherwise.}\end{cases}$$

To justify the claim we have to show that $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon = G$. It suffices to prove that if $u_\varepsilon, \phi_\varepsilon$ are such that $\sup F_\varepsilon(u_\varepsilon,\phi_\varepsilon) < +\infty$, then, up to subsequences and translations by constants (for ϕ_ε) and affine functions (for u_ε), we have $\phi_\varepsilon \rightharpoonup \phi$ and $u_\varepsilon \rightharpoonup u$ weakly in $H^1(0,l)$, with $u \in H^2(0,l)$ and $u' = \phi$, and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\phi_\varepsilon) \geq E\int_0^l |u''|^2\,\mathrm{d}t.$$

From $\sup \int_0^l |\phi_\varepsilon'|^2\mathrm{d}t < +\infty$ we deduce $\phi_\varepsilon \rightharpoonup \phi$, while from

$$\int_0^l |u_\varepsilon'|^2\mathrm{d}t \leq C\int_0^l (|\phi_\varepsilon|^2 + |u_\varepsilon' - \phi_\varepsilon|^2)\mathrm{d}t \leq C(1+\varepsilon^2)$$

we deduce that $u_\varepsilon \rightharpoonup u$. By the lower semi-continuity of the norm

$$\int_0^l |u' - \phi|^2 \mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^l |u'_\varepsilon - \phi_\varepsilon|^2 \mathrm{d}t = 0,$$

so that $u' = \phi$ and $u \in H^2(0, l)$. Finally,

$$E\int_0^l |u''|^2 \mathrm{d}t = E\int_0^l |\phi'|^2 \mathrm{d}t \le \liminf_{\varepsilon\to 0} E\int_0^l |\phi'|^2 \mathrm{d}t \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \phi_\varepsilon).$$

The obtained result is stable with respect to the addition of the boundary conditions, prescribing, for instance, displacements (hinging) and rotations (clamping) at the endpoints.

5 Systematic methods

Although a sufficiently general method of generating the whole class of Γ-equivalent functionals does not exist, we discuss in this section three rather systematic approaches of producing at least some equivalent functionals.

5.1 'Taylor' expansion

If a Γ-development exists then it is easy to construct an equivalent family as follows.

Theorem 5.1 *Let F_ε and m_ε^α be such that the limit*

$$F^{(\alpha)} = \Gamma\text{-}\lim_{\varepsilon\to 0^+} \frac{F_\varepsilon - m_\varepsilon^\alpha}{\varepsilon^\alpha}$$

exists and is not trivial. Then (F_ε) is equivalent to

$$G_\varepsilon(u) := m_\varepsilon^\alpha + \varepsilon^\alpha F^{(\alpha)}(u)$$

at order ε^α. In particular, if a Γ-development $F^{(0)} + \varepsilon^{\beta_1} F^{(1)} + \cdots + \varepsilon^{\beta_M} F^{(M)} + \varepsilon^\alpha F^{(\alpha)}$ exists, with $0 = \beta_0 < \cdots < \beta_M < \alpha$, then we may take

$$G_\varepsilon(u) := \sum_{k=0}^{M} \varepsilon^{\beta_k} m^{(k)} + \varepsilon^\alpha F^{(\alpha)}(u),$$

with $m^{(k)} = \min F^{(k)}$.

Proof It suffices to apply Definition 4.2 above, with $m_\varepsilon^\alpha = \sum_{k=0}^M \varepsilon^{\beta_k} m^{(k)}$ in the case of a Γ-development. □

Remark 5.2 It must be noted that only the values $m^{(k)} = \min F^{(k)}$ are necessary for the definition of G_ε but not the actual form of $F^{(k)}$. In particular, energies with different developments may be equivalent at scale ε^α.

Example 5.3 An equivalent energy at order ε to F_ε in Example 1.9 can be obtained directly from the development as

$$G_\varepsilon(u) = \begin{cases} 1 + \frac{\varepsilon}{3} & \text{if } u = t \\ +\infty & \text{otherwise.} \end{cases} \tag{36}$$

Example 5.4 An equivalent energy at order 1 and ε in Example 1.12 is given by

$$G_\varepsilon(u) = \left(\underline{a} + \varepsilon \underline{a}^2 b\left(\frac{1}{\varepsilon}\right)\right) \int_0^1 |u'|^2 \mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1, \tag{37}$$

where

$$b(t) = \int_0^t \left(\frac{1}{\underline{a}} - \frac{1}{a(s)}\right) \mathrm{d}s \tag{38}$$

(note that b is 1-periodic, so that we may remove the integer part in the lower extreme of the integral in (26)). Also, note that $\left(\int_0^1 \frac{1}{a(s/\varepsilon_j)}\, ds\right)^{-1}$ can be developed in a power series in terms of ε, $\underline{a}$ and b, obtaining equivalent energies up to order ε^k for all $k \in \mathbb{N}$ of the form

$$G_\varepsilon(u) = \underline{a}\left(1 + \varepsilon \underline{a} b\left(\frac{1}{\varepsilon}\right) + \cdots + \varepsilon^k \underline{a}^k \left(b\left(\frac{1}{\varepsilon}\right)\right)^k\right) \int_0^1 |u'|^2 \mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1.$$

5.2 'Stretched' variables and ansätze

Another widely used approximation scheme is by the computation of the Γ-development of functionals obtained from the original functional through a special change of variables (*ansatz*) which anticipates the structure of the minimizer.

Consider the simplest case where one seeks to construct the lowest order of approximation to a family of functionals $F_\varepsilon(u)$. Suppose that the Γ-limit of $F_\varepsilon(u)$ is either trivial or does not exist. Suppose further that one can find a new variable v, such that

$$u = \frac{1}{\phi(\varepsilon)} \Phi(v) \tag{39}$$

and that a re-scaled functional

$$\Psi_\varepsilon(v) = \psi(\varepsilon) F_\varepsilon\left(\frac{\Phi(v)}{\phi(\varepsilon)}\right) \tag{40}$$

has a nontrivial Γ-limit. Then the functional Ψ_ε expressed in terms of u will deliver the desired equivalent theory. The possibility of nontrivial limits of the type (40) reveals the self-similar structure of the singularity at $\varepsilon = 0$.

A more general formal procedure (here we only deal with the scale 1; i.e., $\alpha = 0$) goes as follows:

1. Find a bijective change of variables $\Phi_\varepsilon : V \to X$ and define $H_\varepsilon(v) = F_\varepsilon(\Phi_\varepsilon(v))$;
2. Compute the Γ-limit $H : V \to X$ of H_ε;
3. Define $G_\varepsilon(u) = H(\Phi_\varepsilon^{-1}(u))$ and prove that G_ε is equivalent to F_ε.

The procedure above can be easily adapted to the general scale ε^α. We also remark that sometimes the passage in Point 3 is not straightforward since the domain of H may be different from that of H_ε. It becomes feasible, however, if other invertible changes of variables exist $\Theta_\varepsilon : V \to V$ carrying the domain of H into the domain of H_ε. In this case we need to put $G_\varepsilon(u) = H(\Theta_\varepsilon^{-1}(\Phi_\varepsilon^{-1}(u)))$.

In many cases, a possible change of variable is of the type $u = u_0 + \varepsilon^\gamma v$ where u_0 is a minimizer for $F^{(0)}$; i.e., $\Phi_\varepsilon(v) = u_0 + \varepsilon^\gamma v$ in the remark above. It is clear that finding the relevant scaling (stretching) requires a deep understanding of the solution to the original problem and can not be fully formalized even though one can of course try to make an exhaustive search through the particular classes of scaling *ansätze*. Some well-know examples of the use of stretched variables can be found in various theories of plates and rods (see, for instance, [7,37,43,55]). Another nontrivial applications of the method can be found in the higher-order approximations for composites [8,61]. Here we illustrate this method by two simple examples.

Example 5.5 Consider again Example 1.9. Our goal is to obtain an equivalent energy at scale ε^2 to F_ε. In this case the locked minimizer is $u_0(t) = t$, so we may choose a change of variables of the form $\Phi_\varepsilon(v) = u_0 + \varepsilon^\gamma v$ with $v(0) = v(1) = 0$, and compute the Γ-limit of

$$H_\varepsilon(v) = \frac{1}{\varepsilon^2}\left(F_\varepsilon(\Phi_\varepsilon(v)) - 1 - \frac{1}{3}\varepsilon\right). \tag{41}$$

If $\gamma < 1$ this limit is identically 0, while if $\gamma > 1$ it is trivial. We then choose $\gamma = 1$ and the change of variables $u = u_0 + \varepsilon v$, for which the Γ-limit of H_ε is

$$H(v) = \int_0^1 (|v'|^2 + 2tv)\mathrm{d}t.$$

By inverting the change of variables $v = (u-t)/\varepsilon$ we get the integral $\frac{1}{\varepsilon^2}\int_0^1 (|1-u'|^2 + 2\varepsilon t(u-t))\mathrm{d}t$, and, after integrating by parts and using the boundary condition, we obtain the following functional which is equivalent to F_ε at scale ε^2

$$G_\varepsilon(u) = \int_0^1 (|u'|^2 + \varepsilon(|u|^2 - |u-t|^2)\mathrm{d}t.$$

Example 5.6 The same scheme can be applied for the linearization of finite elasticity, where the starting energy is of the form

$$F_\varepsilon(u) = \int_\Omega f(\nabla u)\mathrm{d}x, \qquad u(x) = x + \varepsilon\phi(x) \text{ on } \partial\Omega$$

with f a hyperelastic energy density with its minimum on $SO(3)$. The Γ-limits of higher order are locked on the identity $u_0(x) = x$. A change of variables $\Phi(v) = u_0 + \varepsilon v$ allows to express a functional equivalent to F_ε at order ε^2 in terms of the functionals of linearized elasticity. For details we refer to [29].

A nontrivial application of the method of stretched variables will be given in the Example 8.5.

5.3 Matched expansions

Another rather general approximation scheme is based on the assumption that the original functional can be restricted to some part of the domain (say, around the singularities in the original problem), while a functional equivalent to the truncated Γ-development of a finite order is operative in its complement (say, far away from the singularities). In this case the full description corresponding to finite ε is preserved in the domain which is shrinking as $\varepsilon \to 0$, while an approximate description is used in the domain which is enlarging as the small parameter diminishes. This method is used for the fully atomistic resolution of the cores of the defects within continuum elasticity (quasi-continuum method, [63])

The first example deals with the boundary layers in homogenization.

Example 5.7 Consider again Example 1.12. It is not difficult to see that an equivalent theory of arbitrary order can be obtained if we take

$$\widetilde{G}_\varepsilon(u) = \int_0^1 a_\varepsilon(t)|u'|^2\mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1$$

where

$$a_\varepsilon(t) = \begin{cases} \underline{a} & \text{if } t'_\varepsilon \le t \le t''_\varepsilon \\ a\left(\dfrac{t}{\varepsilon}\right) & \text{otherwise,} \end{cases}$$

where $0 \le t'_\varepsilon \le t''_\varepsilon \le 1$ and $t''_\varepsilon - t'_\varepsilon \in \varepsilon\mathbb{N}$. To prove this it is enough to observe that under the imposed conditions the minimum values of the approximate and original functional simply coincide.

In the second example, instead of external boundary layers, we deal with internal boundary layers emerging due to the non-convexity of the energy.

Example 5.8 Consider the energy in Example 4.7 with conditions $\sum_i u_i = 0$ and set for simplicity $u_{N+1} = u_N$. In this case a minimizer of the Γ-limit at order ε has a unique phase transition at the point $1/2$. To resolve the fine features of the transition we can use the discrete formulation close to $1/2$, while using the continuous description 'far' from $1/2$. This can be done by considering the equivalent functional

$$G_\varepsilon(u) = \int_{(0,1)\setminus I_\varepsilon} (\overline{W}(u) + C\varepsilon^2|u'|^2)\mathrm{d}t + \sum_{i=-N_\varepsilon}^{N_\varepsilon-1} \varepsilon(W(u_i) - Ju_iu_{i+1}),$$

(the constant C as in the functional defined in (35)) defined on H^1 functions coinciding in I_ε with their interpolations on the lattice $\varepsilon\mathbb{Z}$. Here we have chosen $N_\varepsilon \in \mathbb{N}$ with $N_\varepsilon \to +\infty$ and $\varepsilon N_\varepsilon \to 0$ and set $I_\varepsilon = \{|t - 1/2| \le \varepsilon N_\varepsilon\}$.

The main difficulty in applying this method is that the exact location of the domain where the full description should be used, is usually unknown a priori. This poses an additional 'localization' problem. An interesting example of the matching method involving first the localization of the fully resolved domain in the case of 1D fracture can be found in [9].

6 Uniform Γ-equivalence

In this section we present a more systematic analysis of parameterized families of functionals ('theories'). We begin with a definition of equivalence for parameterized functionals.

Definition 6.1 Two families of functionals F_ε^λ and G_ε^λ are *equivalent* at order ε^α at $\lambda_0 \in \Lambda$ if $F_\varepsilon^{\lambda_0}$ and $G_\varepsilon^{\lambda_0}$ are *equivalent* at order ε^α.

The definition is illustrated by an example dealing with a size effect in composites.

Example 6.2 Let $\lambda > 0$ and consider the one-dimensional homogenization problem of the form:

$$\min\left\{\frac{1}{\lambda}\int_0^\lambda a\left(\frac{x}{\varepsilon}\right)|u'|^2\mathrm{d}x : u(0) = 0,\ u(\lambda) = \lambda\right\}, \tag{42}$$

where the function a is 1-periodic, bounded and strictly positive. It is convenient to rewrite this problem as a minimum problem for the energy:

$$F_\varepsilon^\lambda(u) = \int_0^1 a\left(\frac{\lambda x}{\varepsilon}\right)|u'|^2\mathrm{d}x, \qquad u(0) = 0,\ u(1) = 1. \tag{43}$$

In Example 1.12 we showed that the Γ-limit of F_ε^λ is

$$F_\lambda^0(u) = \underline{a}\int_0^1 |u'|^2\mathrm{d}x, \qquad u(0) = 0,\ u(1) = 1, \tag{44}$$

independent of λ, and that the Γ-development does not exist at scale ε. An equivalent parameterized functional for all $\lambda > 0$ at scale 1 and ε is

$$G_\varepsilon^\lambda(u) = \left(\underline{a} + \varepsilon\underline{a}^2 b\left(\frac{\lambda}{\varepsilon}\right)\right)\int_0^1 |u'|^2\mathrm{d}t, \qquad u(0) = 0,\ u(1) = 1. \tag{45}$$

After rescaling back to the original variables, we obtain

$$F_\lambda^0(u) = \frac{\underline{a}}{\lambda}\int_0^\lambda |u'|^2 \mathrm{d}x, \qquad u(0) = 0,\ \ u(\lambda) = \lambda, \tag{46}$$

and

$$G_\varepsilon^\lambda(u) = \left(\frac{\underline{a}}{\lambda} + \varepsilon\frac{\underline{a}^2}{\lambda} b\left(\frac{\lambda}{\varepsilon}\right)\right)\int_0^\lambda |u'|^2 \mathrm{d}t, \qquad u(0) = 0,\ \ u(\lambda) = \lambda. \tag{47}$$

Note that for all ε fixed, we have

$$\begin{aligned}\lim_{\lambda\to 0^+} \min F_\varepsilon^\lambda &= \min\left\{\int_0^1 a(0)|u'|^2 \mathrm{d}t : u(0) = 0, u(1) = 1\right\}\\ &= a(0) \neq \underline{a} = \lim_{\lambda\to 0^+} \min F_\lambda^0,\end{aligned}$$

while

$$\lim_{\varepsilon\to 0}\limsup_{\lambda\to 0}\left|\underline{a} - \min G_\varepsilon^\lambda\right| = 0.$$

As we see the equivalence is not uniform because for sufficiently small bodies homogenization may interfere with the boundary. Thus in the limit $\lambda \to 0$, the boundary layers at the extremities of the body start to dominate the effective response. A similar phenomenon for discrete lattices was studied in [22,25].

In view of the previous example it is natural to upgrade Definition 6.1 to a uniform equivalence of parameterized functionals.

Definition 6.3 Two families of functionals F_ε^λ and G_ε^λ are *uniformly equivalent* at order ε^α at $\lambda_0 \in \Lambda$ if there exist translations m_ε^λ such that for all $\varepsilon_j \to 0$ and all $\lambda_j \to \lambda_0$ we have, upon extraction of subsequences,

$$\Gamma\text{-}\lim_{j\to+\infty} \frac{F_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}}{\varepsilon_j^\alpha} = \Gamma\text{-}\lim_{j\to+\infty} \frac{G_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}}{\varepsilon_j^\alpha}, \tag{48}$$

and these Γ-limits are non-trivial.

We say that F_ε^λ and G_ε^λ are uniformly equivalent at order ε^α on Λ if they are uniformly equivalent at all $\lambda_0 \in \Lambda$.

Note again that the space on which uniformly equivalent functionals are defined may be different, and may vary with ε and λ.

Theorem 6.4 *Let Λ be sequentially compact, and let F_ε^λ and G_ε^λ be uniformly coercive and uniformly equivalent at order ε^α on Λ. Then*

$$\sup_{\lambda\in\Lambda}\left|\inf G_\varepsilon^\lambda - \inf F_\varepsilon^\lambda\right| = o(\varepsilon^\alpha). \tag{49}$$

Proof By contradiction, suppose that a sequence (λ_j) exists such that

$$|\inf G_{\varepsilon_j}^{\lambda_j} - \inf F_{\varepsilon_j}^{\lambda_j}| \geq C{\varepsilon_j}^\alpha.$$

By the compactness of Λ, we may suppose that $\lambda_j \to \lambda_0$, and by the definition above that (48) holds. By coerciveness we then obtain that

$$\lim_{j\to+\infty} \frac{\inf F_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}}{\varepsilon_j^\alpha} = \lim_{j\to+\infty} \frac{\inf G_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}}{\varepsilon_j^\alpha},$$

and that the limit is finite. From this we immediately obtain a contradiction.

To give another illustration of non-uniform Γ-equivalence we revisit the problem of discretization for non-convex energies (see our Example 1.2). □

Example 6.5 For ε such that $N = \frac{1}{\varepsilon} \in \mathbb{N}$, and for $\lambda \in \mathbb{R}$ we again consider the parameterized functional

$$F_\varepsilon^\lambda(u) = \sum_{i=1}^{N} \varepsilon\, W(u_i) \qquad \sum_{i=0}^{N} \varepsilon\, u_i = \lambda \tag{50}$$

where $u : \{1, \dots, N\} \to \mathbb{R}$. We recall that, after the identification of u with a piecewise-affine function on $(0, 1)$ and letting $u_i = u(i/N)$, this problem can be viewed as the 'naive' discretization of the continuum problem with the energy $\int_0^1 W(u)\, dt$ and the same boundary conditions.

If W is strictly convex then 'naive' approach works and F_ε^λ is uniformly equivalent at all orders to the parameterized continuous functional

$$G_\varepsilon^\lambda(u) = G^\lambda(u) = \int_0^1 W(u)\mathrm{d}t, \qquad \int_0^1 u(s)\,\mathrm{d}s = \lambda \tag{51}$$

(independent of ε).

If W is not convex then the situation is different and F_ε^λ and G_ε^λ are not equivalent at order ε^2, as we have shown in Example 1.13 for $\lambda = 0$. In fact none of the points in the interval $\lambda \in (-1, 1)$ is a point of uniformity. To construct a uniformly equivalent functional at order ε^2 in the case $W(z) = \min\{(z-1)^2, (z+1)^2\}$ we can, for instance, take

$$W_\varepsilon(z) = \min\left\{ \left(z - 1 + \frac{2i}{N}\right)^2 : \ i = 0, \dots, N \right\}, \tag{52}$$

and define

$$G_\varepsilon^\lambda(u) = \int_0^1 \left(W_\varepsilon(u) + |u'|^2\right) \mathrm{d}t, \qquad \int_0^1 u(s)\,\mathrm{d}s = \lambda. \tag{53}$$

This construction is in some sense trivial, because it presupposes the complete knowledge of the minimizer for the original discrete problem at finite ε. The development of a nontrivial continuum model with the same degree of approximation poses considerable challenge, because in the interval $\lambda \in (-1, 1)$ the system behaves as 'strongly discrete'. Thus in this regime the individual elements transform independently, one after another, so even the weakest forms of the Cauchy–Born rule [34], which is usually the basis of a continuum approximation, can not be expected to work.

7 Regular points

As we have seen in the preceding examples, for parameterized families of minimization problems one can distinguish (in the space of parameters) the regular points, where the approximation is uniform, and the singular points, where it is not. We begin with the formal definition of a regular point in the λ–ε space.

Definition 7.1 Let (F_ε^λ) be a family of parameterized functionals. A point $\lambda_0 \in \Lambda$ is a *regular point* for (F_ε^λ) at scale ε^α if for all $\varepsilon_j \to 0$ and sequences $m_j, \lambda_j \to \lambda_0, \lambda'_j \to \lambda_0$ we have, upon extraction of a subsequence, that

$$\Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j} - m_j}{\varepsilon_j^\alpha} = \Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda'_j} - m_j}{\varepsilon_j^\alpha}. \tag{54}$$

Remark 7.2 We may take $m_j = \inf F_{\varepsilon_j}^{\lambda_j}$ so that for this sequence (m_j) the first limit in (54) is non-trivial.

Note that we may take $\lambda'_j = \lambda_0$ so that for all $\lambda_j \to \lambda_0$

$$\Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j} - m_j}{\varepsilon_j^\alpha} = \Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_0} - m_j}{\varepsilon_j^\alpha}. \tag{55}$$

Theorem 7.3 *Let (F_ε^λ) be an equi-coercive parameterized family such that all λ are regular points at scale ε^α, and let*

$$F_\lambda^\alpha = \Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^\lambda - m_\varepsilon^\lambda}{\varepsilon^\alpha} \tag{56}$$

exist and be non-trivial, where

$$m_\varepsilon^\lambda = \inf F_\varepsilon^\lambda.$$

Then $\lambda \mapsto F_\lambda^\alpha$ is continuous with respect to Γ-convergence.

Proof Since we suppose that our energies are equi-coercive, the topology of Γ-convergence is metrizable and compact (see [28]). Let $\lambda_k \to \lambda_0$ be such that $F_{\lambda_k}^\alpha$ Γ-converge to some F. By a diagonal argument we can find a sequence ε_k such that

$$F = \Gamma\text{-}\lim_k \frac{F_{\varepsilon_k}^{\lambda_k} - m_{\varepsilon_k}^{\lambda_k}}{{\varepsilon_k}^\alpha}. \tag{57}$$

We then have

$$\begin{aligned} F &= \Gamma\text{-}\lim_k \frac{F_{\varepsilon_k}^{\lambda_k} - m_{\varepsilon_k}^{\lambda_k}}{{\varepsilon_k}^\alpha} \\ &= \Gamma\text{-}\lim_k \frac{F_{\varepsilon_k}^{\lambda} - m_{\varepsilon_k}^{\lambda_k}}{{\varepsilon_k}^\alpha} = F_{\lambda_0}^\alpha + \lim_k \frac{m_{\varepsilon_k}^{\lambda} - m_{\varepsilon_k}^{\lambda_k}}{{\varepsilon_k}^\alpha}. \end{aligned}$$

Hence, F and $F_{\lambda_0}^\alpha$ differ by a constant. Note however that by the property of convergence of minima and the renormalization by m_ε^λ we have $\min F = \min F_{\lambda_0}^\alpha = 0$, so that $F = F_{\lambda_0}^\alpha$. □

Theorem 7.4 *Let (F_ε^λ) be a equi-coercive parameterized family such that all λ are regular points at scale ε^α. If for all λ fixed a development of the form*

$$F_\varepsilon^\lambda = \varepsilon^{\beta_0} F_\lambda^{(0)} + \varepsilon^{\beta_1} F_\lambda^{(1)} + \cdots + \varepsilon^{\beta_M} F_\lambda^{(M)} + \varepsilon^\alpha F_\lambda^{(\alpha)} + o(\varepsilon^\alpha) \tag{58}$$

exists with $\beta_0 < \ldots < \beta_M < \alpha$, and we set

$$m_\varepsilon^\alpha(\lambda) = \sum_{k=0}^{M} \varepsilon^{\beta_k} \min F_\lambda^{(k)}, \tag{59}$$

then F_ε^λ is uniformly equivalent to the family

$$G_\varepsilon^\lambda(u) = m_\varepsilon^\alpha(\lambda) + \varepsilon^\alpha F_\lambda^{(\alpha)}(u) \tag{60}$$

at scale ε^α. In particular, if Λ is compact we have

$$\sup_{\lambda\in\Lambda} |\min F_\varepsilon^\lambda - m_\varepsilon^\alpha(\lambda) - \varepsilon^\alpha \min F_\lambda^{(\alpha)}| = o(\varepsilon^\alpha). \tag{61}$$

Proof This immediately follows from Theorem 7.3 above, setting $m_\varepsilon^\lambda = m_\varepsilon^\alpha(\lambda)$, and from Theorem 6.4. □

It is worth noting that in the case $\alpha = 0$ condition (61) reduces to

$$\sup_{\lambda\in\Lambda} |\min F_\varepsilon^\lambda - \min F_\lambda^{(0)}| = o(1); \tag{62}$$

which can be viewed as a *uniform version of the fundamental theorem of* Γ*-convergence* dealing with convergence of minimum problems.

Our next example illustrates the effect of the *regular* interference between the size of the support of a distributed force and the scale of homogenization.

Example 7.5 Consider the one-dimensional homogenization problem

$$\min\left\{\int_{-1}^{1} a\left(\frac{t}{\varepsilon}\right)\mathrm{d}t + \frac{1}{2\lambda}\int_{-\lambda}^{\lambda} u\mathrm{d}t : u(\pm 1) = 0\right\}.$$

Here $\lambda \geq 0$ represents the size of the region where distributed forces are applied. In the singular case $\lambda = 0$, which will be considered in more detail in the next section, the second integral is replaced by $u(0)$. Consider the related family of parameterized functionals

$$F_\varepsilon^\lambda(u) = \int_{-1}^{1} a\left(\frac{t}{\varepsilon}\right)|u'|^2\mathrm{d}t + \frac{1}{2\lambda}\int_{-\lambda}^{\lambda} u\mathrm{d}t, \qquad u(\pm 1) = 0, \tag{63}$$

if $\lambda > 0$, and

$$F_\varepsilon^0(u) = \int_{-1}^{1} a\left(\frac{t}{\varepsilon}\right)|u'|^2\mathrm{d}t + u(0), \qquad u(\pm 1) = 0, \tag{64}$$

if $\lambda = 0$. Since the second term in both cases represents a continuous perturbation, the Γ-limit is simply

$$F_\lambda^0(u) = \underline{a}\int_{-1}^{1} |u'|^2\mathrm{d}t + \frac{1}{2\lambda}\int_{-\lambda}^{\lambda} u\mathrm{d}t, \qquad u(\pm 1) = 0, \tag{65}$$

if $\lambda > 0$, and

$$F_0^0(u) = \underline{a}\int_{-1}^{1} |u'|^2\mathrm{d}t + u(0), \qquad u(\pm 1) = 0, \tag{66}$$

if $\lambda = 0$. Note that F_λ^0 has a unique minimizer for all $\lambda \geq 0$, which we denote by u_0^λ. It gives the following minimum value $m^{(0)}(\lambda)$

$$m^{(0)}(\lambda) = \frac{1}{8\underline{a}}\left(-1 + \frac{2}{3}\lambda\right)$$

Suppose that a is even, so that by a reflection argument also the minimizer u_ε^λ of F_ε^λ is even. In this case the computation of $m^\varepsilon(\lambda) := \min F_\varepsilon^\lambda$ is easily carried over. We have $(u_\varepsilon^\lambda)'(0) = 0$, and u_ε^λ solves the Euler–Lagrange equation

$$\begin{cases} a\left(\dfrac{t}{\varepsilon}\right)u' = \dfrac{t}{4\lambda} & \text{for } 0 < t < \lambda \\[2ex] a\left(\dfrac{t}{\varepsilon}\right)u' = \dfrac{1}{4} & \text{for } \lambda < t < 1, \end{cases}$$

supplemented by the only boundary condition $u_\varepsilon^\lambda(1) = 0$. After computing this solution, we obtain

$$m^\varepsilon(\lambda) = -\frac{1}{8\lambda^2}\int_0^1 \frac{\min\{s^2, \lambda^2\}}{a(s/\varepsilon)}\,\mathrm{d}s.$$

Then the Γ-limit at scale ε is

$$F_\lambda^1(u) = \Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^\lambda(u) - m^{(0)}(\lambda)}{\varepsilon_j} = \begin{cases} K & \text{if } u = u_0^\lambda \\ +\infty & \text{otherwise,} \end{cases}$$

where $K = -\frac{1}{8}\lim_j b(1/\varepsilon_j)$ (b as in (38)). Since we are interested in the dependence of the result on λ we fix a sequence ε_j in such a way that the limit in the definition of K exists. In this case all points $\lambda > 0$ are easily seen to be regular. On the contrary, the point $\lambda = 0$ is not regular and its appropriate neighborhood represents the domain of the size effect. This singular case will be treated in Example 8.7.

Now, we formulate a simple necessary condition of regularity.

Theorem 7.6 *If (F_ε^λ) is a equi-coercive parameterized family which for each λ admits a Γ-development of the form (58), is regular for all scales ε^β with $\beta < \alpha$ and is regular at λ_0 at scale ε^α, then $\lambda \mapsto \min F_\lambda^{(\alpha)}$ is continuous at λ_0.*

Proof The proof can be performed by induction. It suffices to check when $\alpha = 0$, in which case the thesis is that $\lambda \mapsto \min F_\lambda$ is continuous at λ_0, where $F_\lambda = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon^\lambda$. Indeed, if $\lambda_k \to \lambda_0$ then $\min F_{\lambda_k} = \lim_{\varepsilon\to 0^+} \min F_\varepsilon^{\lambda_k}$. By a diagonal argument we may find $\varepsilon_k \to 0$ such that $\lim_k \min F_{\lambda_k} = \lim_k \min F_{\varepsilon_k}^{\lambda_k}$. By the regularity at λ_0 we have that the Γ-limit of $F_{\varepsilon_k}^{\lambda_k}$ is the same as that of $F_{\varepsilon_k}^{\lambda}$; i.e., F_λ. By the convergence of minima we then obtain $\min F_\lambda = \lim_k \min F_{\lambda_k}$. □

In the next example we list several cases studied above where the necessary condition of regularity suggested by Theorem 7.6 fails.

Example 7.7 If F_ε^λ is as in Example 3.1 then all λ different from ± 1 are regular points. From the study of minimum problems for the limit as summarized in Fig. 4, we deduce by Theorem 7.6 that ± 1 are not regular points for F_ε^λ at scale ε. Similarly, for Example 3.2 (see Fig. 5) we deduce that 1 is not a regular point, and for Example 6.2 that 0 is not a regular point.

8 Singular points and 'tables' of Γ-limits

We are now in the position to give the formal definition of a singular point.

Definition 8.1 Let F_ε^λ be a family of parameterized functionals, with $\lambda \in \Lambda$. We say that λ_0 is a *singular point at scale ε^α* if it is not regular; i.e., if there exist m_ε, $\lambda'_\varepsilon \to \lambda_0$ and $\lambda''_\varepsilon \to \lambda_0$ such that (up to subsequences)

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^{\lambda'_\varepsilon} - m_\varepsilon}{\varepsilon^\alpha} \neq \Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^{\lambda''_\varepsilon} - m_\varepsilon}{\varepsilon^\alpha}. \tag{67}$$

Example 8.2 In Example 3.1 the points ± 1 are singular at scale ε; in Example 3.2 the point 1 is singular at scale ε. In both cases (61) does not hold for $\alpha = 1$ even though the limit $F_\lambda^{(1)}$ exists, and we have (taking e.g., Λ a compact set containing a neighbourhood of 1)

$$\sup_\Lambda \left| \min F^{\lambda_\varepsilon} - m^{(0)}(\lambda) - \varepsilon \min F_\lambda^{(1)} \right| \geq C\varepsilon.$$

As we see at a singular point λ_0 the computation of the Γ-limit or Γ-development with fixed λ_0 is not sufficient to accurately describe the behavior of minimum problems. We have then to look at the possibility of different limits along different paths $\lambda_\varepsilon \to \lambda_0$. To simplify the bookkeeping of various distinct limits around the singular point, we introduce the notion of a 'table' of Γ-limits. We limit ourselves to the analysis at scale 1 and ε; the general case requiring only more complex notation.

Definition 8.3 The *table of Γ-limits at scale* 1 for F_ε^λ at λ_0 are all sequences $(\varepsilon_j, \lambda_j)$, and functionals $F^{(0)}_{(\varepsilon_j,\lambda_j)}$ with $\varepsilon_j \to 0$, $\lambda_j \to \lambda_0$, and

$$F^{(0)}_{(\varepsilon_j,\lambda_j)} = \Gamma\text{-}\lim_j F_{\varepsilon_j}^{\lambda_j}.$$

The *table of Γ-limits at scale ε* for F_ε^λ at λ_0 are all sequences $(\varepsilon_j, \lambda_j)$, and functionals $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ with $\varepsilon_j \to 0$, $\lambda_j \to \lambda_0$, and

$$F^{(1)}_{(\varepsilon_j,\lambda_j)} = \Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j} - \min F^{(0)}_{(\varepsilon_j,\lambda_j)}}{\varepsilon_j},$$

etc.

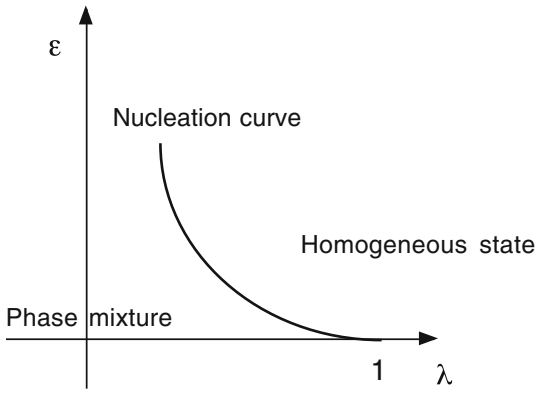

Fig. 7 Crossover boundary describing nucleation threshold in the ε–λ space (Example 8.4)

We recall that if the point λ_0 is regular then

$$F^{(1)}_{(\varepsilon_j,\lambda_j)} = \Gamma\text{-}\lim_j \frac{F^{\lambda_j}_{\varepsilon_j} - \min F^{(0)}_{\lambda_0}}{\varepsilon_j},$$

for all $\varepsilon_j \to 0$ and $\lambda_j \to \lambda_0$. Below we give several examples of singular points.

We begin with the situations when the boundaries between different entries in the table are sharp crossover phenomenon as in the case of nucleation (see below), fracture (see below) and buckling (not discussed here, see [38,42,51]). The first example deals with nucleation (or annihilation) in the case of gradient theory of phase transitions.

Example 8.4 Consider again the energy (27). Suppose that W is of class C^2, with minimum 0 and that $W^{**}(z) = W(z)$ if $|z| \geq 1$. We focus on the singular point $\lambda_0 = 1$.

First note that the functionals $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ are finite only at the constant function $u = 1$, so that it suffices to compute the limit

$$F^{(1)}_{(\varepsilon_j,\lambda_j)}(1) = \lim_j \min \left\{ \int_0^1 \left(\frac{W(v)}{\varepsilon_j} + \varepsilon_j |v'|^2 \right) \mathrm{d}t : \int_0^1 v \mathrm{d}t = \lambda_j \right\} \tag{68}$$

We obtain (see Appendix A.1 for details)

$$F^{(1)}_{(\varepsilon_j,\lambda_j)}(1) = \lim_j \min \left\{ \alpha \frac{(1-\lambda_j)^2}{\varepsilon_j}, \beta \right\}, \tag{69}$$

where

$$\alpha = \frac{1}{2} W''(1), \qquad \beta = c_W. \tag{70}$$

As we see, the existence of the Γ-limit $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ is equivalent to the existence of the limit of the ratio $(1-\lambda_j)^2/\varepsilon_j$. We can summarize our findings in the form of the following table for m_1 (Γ-limit at the constant function $u = 1$) which gives the first-order term in the development of minima.

1. If $(1-\lambda)^2 = C\varepsilon$, where $C \leq \frac{\beta}{\alpha}$, then $m_1 = \alpha C$.
2. If $(1-\lambda)^2 \geq \frac{\beta}{\alpha}$ then $m_1 = \beta$.

The crossover behavior of the system close to the point $\varepsilon = 0$ and $\lambda = 1$ can be pictured in the ε–λ plane, where the line $\varepsilon = \frac{\alpha}{\beta}(1-\lambda)^2$ (for $\lambda < 1$) (*nucleation threshold*) separates the zone with phase mixture from the one where the stable configuration is homogeneous (Fig. 7).

The next example concerns the nucleation of cracks in the discrete lattices with Lennard–Jones interactions.

Example 8.5 We consider the theory (29) and focus as in the Example 3.2 on the singular point $\lambda_0 = 1$. Observe that the functionals $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ are finite only at the affine function $\overline{u}(t) = t$. Then to find the limit along arbitrary sequence in the parameter space it suffices to compute

$$F^{(1)}_{(\varepsilon_j,\lambda_j)}(\overline{u}) = \lim_j \min\left\{\sum_i \left(J\left(\frac{u_i - u_{i-1}}{\varepsilon_j}\right) - J(1)\right) : u_0 = 0,\ u_N = \lambda_j\right\}. \tag{71}$$

To see which sequences produce a nontrivial limit, one can follow the method of Section 5.2 and look for a rectifying change of variables $u = \Phi_\varepsilon(v)$. In this situation, the knowledge of the separation of scales in the discrete solution [64] suggests that one can consider an expansion around $\overline{u}$ of the form $u = \overline{u} + \varepsilon^\gamma v$, which ensures that the 'bulk' energy term (corresponding to springs with $u_i - u_{i-1} \sim \varepsilon$) and the 'surface' energy term (corresponding to springs with $u_i - u_{i-1} \sim 1$) are of the same order. To achieve such parity one must choose $\gamma = 1/2$ because then the surface energy in the stretched variables v_i is of order ε and the bulk energy which scales with the elastic modulus of $\widetilde{J}(z) = J(\sqrt{\varepsilon}z)$ is also of the order ε. A different heuristic argument leading to the same conclusion can be found in [22]. We also remark that very similar reasoning is used in plate theory to justify the approximations containing both membrane and bending terms (e.g., [37]).

Following the above analysis, we set

$$v_i = \frac{u_i - \varepsilon_j i}{\sqrt{\varepsilon}_j}, \qquad \psi_j(z) = \frac{1}{\varepsilon_j}(J(1 + \sqrt{\varepsilon}_j z) - J(1)),$$

and obtain

$$F^{(1)}_{(\varepsilon_j,\lambda_j)}(\overline{u}) = \lim_j \min\left\{\sum_i \varepsilon_j \psi_j\left(\frac{v_i - v_{i-1}}{\varepsilon_j}\right) : v_0 = 0,\ v_N = \frac{\lambda_j - 1}{\sqrt{\varepsilon}_j}\right\}. \tag{72}$$

Now we can characterize the behavior of the original minimum problem via the computation of the Γ-limit for the rescaled functional in the variable v. This Γ-limit has been computed in [22] (see also [21]): it exists whenever the limit of $\min\left\{\alpha(\lambda_j - 1)^2/\varepsilon_j, \beta\right\}$ as $j \to +\infty$ exists, where

$$\alpha = \frac{1}{2}J''(1), \qquad \beta = -J(1), \tag{73}$$

and more precisely

$$\begin{aligned} F^{(1)}_{(\varepsilon_j,\lambda_j)}(\overline{u}) &= \lim_j \min\left\{\alpha \int_0^1 |v'|^2 \mathrm{d}t + \beta \#(S(v)) : v(0) = 0, v(1) = \frac{\lambda_j - 1}{\sqrt{\varepsilon}_j}\right\} \\ &= \lim_j \min\left\{\frac{\alpha(\lambda_j - 1)^2}{\varepsilon_j}, \beta\right\}. \end{aligned}$$

Note the similarity of this case with the previous example even though the methodologies of finding the final result have been (superficially) different. Thus, in the case of phase transitions we used the direct method, while in the case of fracture we used the method of 'stretched' variables.

We can summarize our results in a form of the following 'table', in terms of m_1 (the Γ-limit at $\overline{u}(t) = t$), which also gives the first-order term in the development of minima:

1. If $(1 - \lambda)^2 = C\varepsilon$, and $C \le \frac{\beta}{\alpha}$, then $m_1 = \alpha C$.
2. If $(1 - \lambda)^2 \ge \frac{\beta}{\alpha}\varepsilon$ then $m_1 = \beta$.

The behavior of the system close to the point $\varepsilon = 0$ and $\lambda = 1$ can be again pictured in the ε–λ plane, where the crossover line $\varepsilon = \frac{\alpha}{\beta}(1 - \lambda)^2$ (for $\lambda > 1$) (*fracture threshold*) separates a zone where there is a crack from one where the stable configuration is elastic (Fig. 8).

The next example shows that there may be an infinity of distinct entries characterizing a particular 'table'. This example deals with the theory of finite-scale micro-structures introduced in Sect. 2, which we simplify here to avoid inessential technical difficulties.

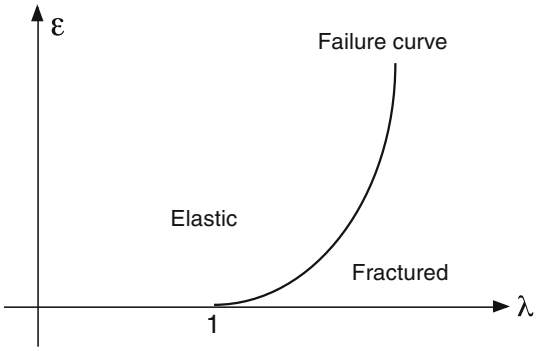

Fig. 8 Crossover boundary describing failure threshold in the ε–λ space (Example 8.5)

Example 8.6 Taking the gradient theory of phase transitions into account, we may consider instead of the energy (31) the following functional:

$$F_\varepsilon^\lambda(u) = 2\varepsilon \#(S(u')) + \lambda \int_0^1 u^2 \mathrm{d}t, \qquad u(0) = u(1) = 0, \ \ |u'| = 1.$$

The equivalence of the two problems in some regimes can be deduced from [1,53,58]. Thus, one can see that the first two terms of the integral in (31) have been replaced by (twice) the number of jump points of the derivative and the constraint that u is piecewise affine with gradient $u' \in \{\pm 1\}$.

The Γ-limit of F_ε^λ is

$$F_\lambda(u) = \lambda \int_0^1 u^2 \mathrm{d}t, \qquad u(0) = u(1) = 0, \ \ |u'| \le 1,$$

with minimum equal to 0. If $\lambda > 0$ the unique minimizer is $u(t) = 0$, while the limit is identically 0 on all admissible functions if $\lambda = 0$.

We may explicitly compute the minimum $m_\varepsilon(\lambda) = \min F_\varepsilon^\lambda$, which is achieved at the function u_N with N creases in $(0, 1)$, $2/N$ =periodic, odd, and equal to $(|2Nt - 1| - 1)/2N$ on $[0, 1/N]$, for which we have

$$F_\varepsilon^\lambda(u_N) = 2\varepsilon N + \frac{\lambda}{12N^2}.$$

The optimal $N = N(\varepsilon, \lambda)$ is obtained by minimizing this quantity for $N \in \mathbb{N}$, $N \ge 1$, obtaining

$$N(\varepsilon, \lambda) \in \left\{ \left[\sqrt[3]{\frac{\lambda}{12\varepsilon}} \right] \vee 1, \left[\sqrt[3]{\frac{\lambda}{12\varepsilon}} \right] + 1 \right\},$$

and

$$\min F_\varepsilon^\lambda = \min \left\{ 2\varepsilon + \frac{\lambda}{12}, 2\varepsilon \left[\sqrt[3]{\frac{\lambda}{12\varepsilon}} \right] + \frac{\lambda}{12 \left[\sqrt[3]{\frac{\lambda}{12\varepsilon}} \right]^2}, 2\varepsilon \left(\left[\sqrt[3]{\frac{\lambda}{12\varepsilon}} \right] + 1 \right) + \frac{\lambda}{12 \left(\left[\sqrt[3]{\frac{\lambda}{12\varepsilon}} \right] + 1 \right)^2} \right\}. \tag{74}$$

For fixed λ we then have

$$N(\varepsilon, \lambda) \sim \sqrt[3]{\frac{\lambda}{12\varepsilon}} \qquad \text{and} \qquad \min F_\varepsilon^\lambda \sim \varepsilon^{2/3} \lambda^{1/3} \sqrt[3]{\frac{2}{3}}.$$

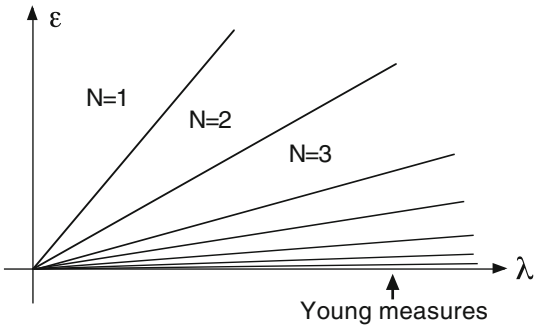

Fig. 9 Crossover boundaries in the parameter space separating microstructures with different number of interfaces (Example 8.6)

We can now examine the behavior at $\lambda = 0$. The first interesting scale here is ε. The scaled energy is then simply

$$\frac{1}{\varepsilon}F_\varepsilon^\lambda(u) = 2\#(S(u')) + \frac{\lambda}{\varepsilon}\int_0^1 u^2 \mathrm{d}t, \qquad u(0) = u(1) = 0, \ \ |u'| = 1, \tag{75}$$

whose Γ-limit depends on the ratio λ/ε. If $\lambda_\varepsilon/\varepsilon \to p \in [0, +\infty)$ then the corresponding Γ-limit is

$$F_p^1(u) = 2\#(S(u')) + p\int_0^1 u^2 \mathrm{d}t, \qquad |u'| = 1.$$

This shows that $\lambda = 0$ is a singular point at scale ε characterized by the following 'table' of Γ-limits:

$$\text{if} C_{k-1}\varepsilon \le \lambda \le C_k\varepsilon, \ \text{then} N = k,$$

Here C_k is an increasing sequence tending to $+\infty$. The values of C_k can be computed from (74), for instance, $C_0 = 0$. The behavior of $F^1_{\lambda/\varepsilon}$ is pictured in Fig. 9 where the number of interfaces of the minimizer is marked explicitly. Note that the behavior of this functional differs from that of functional (31) close to $p = 0$, where the original functional admits homogeneous minimizers (e.g., [65]) that are not allowed by the simplified model.

The next two examples illustrate the cases when the boundary between different entries in a table is diffuse. This means that different asymptotic theories have overlapping domains of application. Both examples deal with size-effects in homogenization.

Example 8.7 We consider F_ε^λ as in (43). Note that the definition can be extended by continuity to $\lambda = 0$ setting

$$F_\varepsilon^0(u) = \int_0^1 a(0)|u'|^2 \mathrm{d}t, \qquad u(0) = 0, \ \ u(1) = 1,$$

independent of ε. The point $\lambda = 0$ is the only singular point at scale 1. The table of Γ-limits at $\lambda = 0$ is obtained by looking at the Γ-limits of functionals

$$F_\varepsilon^{\lambda_\varepsilon}(u) = \int_0^1 a\left(\frac{\lambda_\varepsilon t}{\varepsilon}\right)|u'|^2 \mathrm{d}t, \qquad u(0) = 0, \ \ u(1) = 1.$$

We have two regimes which can be represented as a table:

1) If $\lambda_\varepsilon/\varepsilon \to p \in [0, +\infty)$, then $F^0(u) = \int_0^1 a(pt)|u'|^2\, dt$ with $u(0) = 0, \ u(1) = 1$;
2) If $\lambda_\varepsilon >> \varepsilon$, then $F^0(u) = \underline{a}\int_0^1 |u'|^2\, dt$ with $u(0) = 0, \ u(1) = 1$ (homogenization theory is exact).

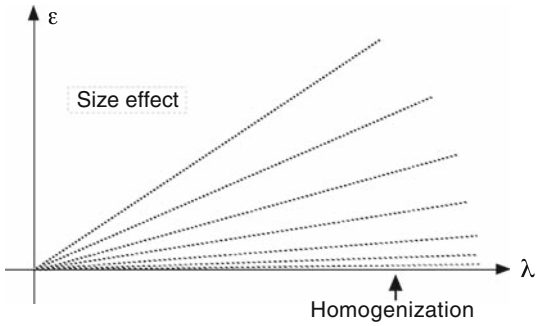

Fig. 10 The domain of size effect in the parameter space (Examples 8.7, 8.8)

Observe that at the next scale ε all $\lambda > 0$ points are singular. The corresponding tables of the Γ-limits depend on the existence (upon subsequences) of the limit

$$K_0 = \lim_{\varepsilon \to 0} b\left(\frac{\lambda_\varepsilon}{\varepsilon}\right)$$

as $\lambda_\varepsilon \to \lambda$. We then obtain

$$F^1_\lambda(u) = \begin{cases} \underline{a}^2 K_0 & \text{if } u(t) = t \\ +\infty & \text{otherwise.} \end{cases} \tag{76}$$

The next example deals with a concentrated body force applied to a composite.

Example 8.8 We consider Example 7.5 and construct the table of Γ-limits at the singular point 0 for the sequence $\varepsilon_j = \frac{1}{j}$, for which $b(1/\varepsilon_j) = 0 = K$. First, we observe that the Γ-limit

$$F^1 = \Gamma\text{-}\lim_j \frac{F^{\lambda_j}_{\varepsilon_j} - m^{(0)}(\lambda_j)}{\varepsilon_j}$$

is finite only at u^0_0. The Γ-limit exists if the limit

$$K_1 = \lim_j \frac{\varepsilon_j^2}{8\lambda_j^2} \int_0^{\lambda_j/\varepsilon_j} \left(t^2 - \left(\frac{\lambda_j}{\varepsilon_j}\right)^2\right)\left(\frac{1}{\underline{a}} - \frac{1}{a(t)}\right) \mathrm{d}t$$

exists, in which case we have the two regimes. They may be presented in the form of a table:

1) If $\lambda_j/\varepsilon_j \to p \in [0, +\infty)$, then

$$F^1(u^0_0) = \begin{cases} 0 & \text{if } p = 0 \\ \dfrac{1}{8p^2} \displaystyle\int_0^p \left(t^2 - p^2\right)\left(\frac{1}{\underline{a}} - \frac{1}{a(t)}\right) \mathrm{d}t & \text{if } p > 0; \end{cases}$$

2) If $\lambda_j >> \varepsilon_j$, then $F^1(u^0_0) = 0$ (homogenization theory is exact).

The division of the parameter space into domains of applicability of different asymptotic theories remains in the present case basically the same as in the previous example (see Fig. 10) with the domain of 'size effect' gradually transforming into the domain of applicability of the homogenized model.

Another interesting set of examples, showing overlapping domains of validity for different asymptotic theories, can be found in the problems involving dimension reduction. In this case the role of λ is played by applied loads which have to be scaled with the slenderness parameter ε if one wants to get a nontrivial Γ-limit. There may be several such limits and in [37,52], one can find some partially filled 'tables', not only providing rigorous justification of the semi-empirical *ansätze* proposed by engineers, but also containing some new entries.

9 Uniform approximations

To construct a uniform approximation for a given theory one has to know the location of the singular points because they have to be treated differently than the regular points. This observation is illustrated by the following example, where the singularity is removed by the quasi-continuum method.

Example 9.1 We may construct equivalent theories at scale 1 to F_ε^λ in (43) in Example 6.2. In Example 8.7 the table of Γ-limits at $\lambda = 0$ was computed noting that 0 is a singular point. This singularity can be removed modifying the Γ-development. A uniformly equivalent functional G_ε^λ to F_ε^λ at $\lambda = 0$ is constructed by setting

$$G_\varepsilon^\lambda(u) = \int_0^1 a_\varepsilon\left(\frac{\lambda}{\varepsilon}x\right)|u'|^2\mathrm{d}x,$$

where

$$a_\varepsilon(y) = \begin{cases} a(y) & \text{if } |y| \le \rho_\varepsilon \\ \underline{a} & \text{if } |y| > \rho_\varepsilon \end{cases}$$

and $\rho_\varepsilon \to +\infty$ are such that $\varepsilon\rho_\varepsilon \to 0$. In this case the homogenized description with the modulus $\underline{a}$ is used only at a sufficiently large scale, while we are resolving all the microscopic details at the small scale.

In most cases, one needs to know not only the location of the singular points but also their structure. We have seen how the behavior of parameterized energies at singular points may be sometimes analyzed by reconstructing directions in the ε–λ space, along which a regular Γ-development exists. Although this is not the general case (recall the oscillatory behavior of the minimal values in Example 1.13 at scale ε^2, and in Example 8.7 at scale ε), it is frequent in applications. Now, if all singular points are rectifiable in this sense, then there exists a specific way to construct a uniform approximation.

To formulate the method we need the following definition of a blown-up (or rectified) functional:

Definition 9.2 Let λ_0 be a singular point for F_ε^λ at scale 1. We say that F_ε^λ admits a *blow up at* λ_0 at order 1 if there exist the energies H_ε^p and a continuous function $p = p(\lambda, \varepsilon)$ such that

(i) H_ε^p Γ-converge to H^p, and all p are regular points;
(ii) $F_\varepsilon^\lambda = H_\varepsilon^{p(\lambda,\varepsilon)}$ for (λ, ε) in a neighbourhood of $(\lambda_0, 0)$.

The definition can be easily extended to the scales ε^α.

We have already seen several examples of singular points where the singular behavior in the original variables could be replaced by a regular behavior of a 'blown-up' functional. It will be convenient to have the rectified functionals ready in our 'twin' cases of phase transition and fracture.

Example 9.3 In Example 8.4 we may 'blow up' functionals F_ε^λ at the point $\lambda_0 = 1$ at order ε. This is equivalent to blowing up the functionals $\frac{1}{\varepsilon}(F_\varepsilon^\lambda - \min F_\lambda^{(0)})$ at order 1. We may then take $p = (1-\lambda)^2/\varepsilon$, and define

$$H_\varepsilon^p(u) = \int_0^1 \left(\frac{W(u)}{\varepsilon} + \varepsilon|u'|^2\right)\mathrm{d}t, \qquad \int_0^1 u\mathrm{d}t = 1 - \sqrt{\varepsilon p}.$$

The rectified functional then takes the form:

$$H^p(u) = \begin{cases} \min\{\alpha p, \beta\} & \text{if } u(t) = 1 \\ +\infty & \text{otherwise} \end{cases}$$

where parameters α and β have been defined in (70).

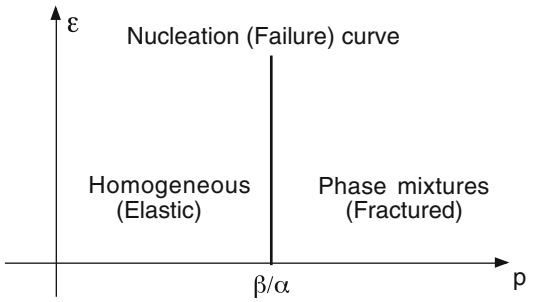

Fig. 11 Rectified parameter space in the cases of phase transition and fracture (Examples 9.3, 9.4)

Example 9.4 As above, in Example 8.5 at $\lambda = 1$, we can take $p = (1-\lambda)^2/\varepsilon$ and define

$$H_\varepsilon^p(u) = \sum_i \left(J\left(\frac{u_i - u_{i-1}}{\varepsilon}\right) - J(1)\right), \qquad u_0 = 0,\ u_N = 1 + \sqrt{\varepsilon p}.$$

The rectified functional can then be written in the form

$$H^p(u) = \begin{cases} \min\{\alpha p, \beta\} & \text{if } u(t) = t \\ +\infty & \text{otherwise} \end{cases}$$

where parameters α and β have been defined in (73).

The following theorem states that for F_ε^λ, which admits a blow up one can construct a uniformly-equivalent family by simply taking H^p computed at $p = p(\lambda, \varepsilon)$.

Theorem 9.5 *Let F_ε^λ admit a blow up at λ_0 by means of energies H_ε^p with $p \in \Pi$ and Π compact; then F_ε^λ is uniformly equivalent to $G_\varepsilon^\lambda = H^{p(\lambda,\varepsilon)}$ at λ_0.*

Proof Let $\lambda_\varepsilon \to \lambda_0$; up to subsequences we may suppose the limits $\lim_{\varepsilon\to 0^+} p(\lambda_\varepsilon, \varepsilon) = p_0$ exist and

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon^{\lambda_\varepsilon} = \Gamma\text{-}\lim_{\varepsilon\to 0^+} H_\varepsilon^{p(\lambda_\varepsilon,\varepsilon)} = H^{p_0} = \Gamma\text{-}\lim_{\varepsilon\to 0^+} H^{p(\lambda_\varepsilon,\varepsilon)}$$

by the regularity of p_0. (Fig. 11) □

Now it is clear that in some cases a uniform approximation can be constructed by asymptotic matching of the rectified structures of the functional around the isolated singular points with the standard Γ-development around the regular points. More precisely, we mean a construction of the energies that are equivalent to the Γ-limit (or Γ-development) far from singular points, and to the 'rectified' energies close to singular points. We may apply this method to a generic theory F_ε^λ by using the following algorithm:

1. Compute the table of Γ-limits of F_ε^λ at every point λ. This actually often subdivides into two steps
   1a) Identify regular points and compute Γ-developments.
   1b) Identify singular points and compute the complete table.
2. Choose the classes of theories which are compatible with the tables of Γ-limits computed; i.e., such that in those classes we may find parameterized energies equivalent to the ones computed in the step above. In many cases these theories will depend on additional parameters;
3. Tune these additional parameters to obtain an equivalent G_ε^λ of the desired form. In practice this is often done separately for regular and singular points. The method is applicable if the corresponding (locally) equivalent theories can be matched.

Of course, the choice of the functionals at Step 2 is not unique, and additional criteria (simplicity, computability, closeness to well-known theories, ability to capture local minimizers, etc) can drive the final selection. In general those functionals will range from 'locked' energies which only bear information about limit minimizers plus a little more detail, to theories as complex as the original functionals F_ε^λ. We may for instance

enlarge the domain of 'locked' equivalent theories or adapt parameterized functionals from a larger class so that the new class either includes or is equivalent to the class of F_ε^λ.

We conclude with two examples showing how the above algorithm can be actually implemented. The first example deals with the lattice model of fracture and brings us back to the original question of the asymptotic relation between the theories of Griffith and Barenblatt.

Example 9.6 In the case of a 1D lattice with Lennard–Jones interactions (see Sect. 2) the interesting interval of boundary conditions is $\lambda > 1$ because for $\lambda \le 1$ the Γ-limit already gives a uniformly-equivalent theory. Indeed at $\lambda \le 1$ one can write a class of uniformly equivalent theories at all orders in the form

$$G_\varepsilon^\lambda(u) = \int_0^1 \psi(u')\mathrm{d}t, \qquad u(0) = 0,\ u(1) = \lambda, \tag{77}$$

provided that $\psi^{**}(z) = J^{**}(z)$ for $z \le 1$.

A singular point is located at $\lambda = 1$ and we know that to rectify the behavior at this point one has to look at the Γ-limits of the scaled functionals in (71) and use the auxiliary variable $v(t) = (u(t) - t)/\sqrt{\varepsilon}$. In the limit we obtain the Mumford–Shah functional

$$\alpha \int_0^1 |v'|^2 \mathrm{d}t + \beta \#(S(v))$$

with the additional constraint of 'increasing jumps' $v^+ > v^-$ on $S(v)$ (see [22]). Now, we follow our algorithm and formally pull back the variable $u = t + \sqrt{\varepsilon} v$ in the limiting functional. We obtain

$$\varepsilon \left(\alpha \int_0^1 |v'|^2 \mathrm{d}t + \beta \#(S(v)) \right) = \alpha \int_0^1 |u' - 1|^2 \mathrm{d}t + \varepsilon\beta \#(S(u)).$$

This approximation of the singular behavior has to be matched with the regular approximation (77) at $\lambda \le 1$ and another regular approximation at $\lambda > 1$.

Having in mind the blown-up energies at $\lambda = 1$ computed above, a class of equivalent theories at $\lambda > 1$ at scale ε (and uniformly equivalent on all compact sets of $(0, +\infty)$) can be chosen in the same general form. Thus, if we consider the class of energies

$$G_\varepsilon^\lambda(u) = \int_0^1 \psi_\varepsilon(u')\mathrm{d}t + \varepsilon \sum_{S(u)} g_\varepsilon(u^+ - u^-), \qquad u(0) = 0,\ u(1) = \lambda,$$

there are two conditions of equivalence: $\psi_\varepsilon = \psi$ which has a unique minimum in 1 with the value $J(1)$ and $\lim_{\varepsilon\to 0} g_\varepsilon(z) = \beta$. In our case these conditions are satisfied by the same energy densities as in the blown-up functional; i.e., $\psi(z) = J(1) + (z-1)^2$ and $g_\varepsilon = \beta$. This gives us a uniformly-equivalent theory for $\lambda \ge 1$ in the form

$$G_\varepsilon^\lambda(u) = J(1) + \alpha \int_0^1 |u' - 1|^2 \mathrm{d}t + \varepsilon\, \beta \#(S(u)), \qquad u(0) = 0,\ u(1) = \lambda \tag{78}$$

with the condition $u^+ > u^-$ on $S(u)$. For $\lambda \le 1$ we may directly use (77) with $\psi(z) = J(z)$.

One can see that the result coincides with the Griffith's theory of brittle fracture with a unilateral condition on the opening. We emphasize, that this theory contains an internal parameter ε and therefore cannot be obtained as a straightforward Γ-limit [64]. The presence in this theory of an internal length scale allows one to distinguish the bulk and surface energies which have different physical dimensions.

The dependence of the minimum values of G_ε^λ on λ and the corresponding stress-strain relation are shown in Fig. 12. We see that fracture in this approximate theory does not take place at infinitesimal tension, as in the straightforward Γ-development (see our Fig. 5), and that the nucleated crack has a finite opening (brutal fracture). These features agree with the solution of the discrete problem at finite ε (see [64]).

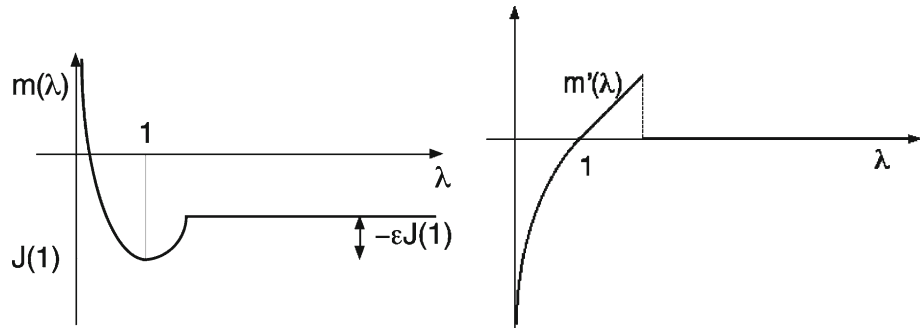

Fig. 12 Approximate minimum values for G_ε^λ in Example 9.6

Once the structure of one uniformly equivalent theory is established, other fracture energies may be constructed that belong to the same equivalence class but may have additional beneficial features. The corresponding development is usually not systematic and requires additional knowledge about the structure of the minimizers in the original problem at finite ε. For instance, one may consider the *cohesive zone* theories of the form

$$G_\varepsilon^\lambda(u) = J(1) + \alpha \int_0^1 |u' - 1|^2 \mathrm{d}t + \varepsilon \sum_{t \in S(u)} g\left(\frac{u^+ - u^-}{\varepsilon}\right), \qquad u(0) = 0,\ u(1) = \lambda.$$

Within the class of functions $g \geq 0$, which are concave and non-decreasing, the conditions of equivalence are:

$$g'(0) > 0 \text{ and } \lim_{z \to +\infty} g(z) = \beta = -J(1).$$

In case (78) we had $g(z) = 0, z = 0$, and $g(z) = \beta, z > 0$. By modifying appropriately the behavior of the Griffith function $g(z)$ near $z = 0$ one can avoid some known limitations of Griffith's theory [45,47]. For example one can take $g(z) = \min\{z, 1\}$ as in the *Dugdale's* theory of fracture and obtain better description of the local minimizers than the Griffith's theory [45]. The approximation can be further improved if $g(z)$ is taken to coincide with a particular rescaling (and translation) of a concave branch of the function $J(z)$, as shown in [64] where the comparison with the discrete theory is also presented. In this case the uniformly equivalent theory coincides with Barenblatt's theory of fracture, which now adequately describes even the fine structure of the bifurcation of a solution with a crack from a homogeneous (Cauchy-Born) solution [19,64].

Our second example concerns the gradient theory of phase transitions (see Sect. 2), and here the goal of a uniformly equivalent theory is to deal adequately with interfaces that are either close to each other or to the exterior boundaries. Such approximation should be able to take into account that an interface adjusts its internal structure while approaching an obstacle. Due to the mentioned similarity between the fracture problem and the phase transition problem, we will use below the insights from Example 9.6.

Example 9.7 Consider again the 'theory' given by (27). To construct the simplest uniformly equivalent theory one can try to modify the straightforward development of F_ε^λ at the singular points $\lambda = \pm 1$. For simplicity we suppose $\min W = 0$ and define the following parameterized family of functionals:

$$G_\varepsilon^\lambda(u) = \begin{cases} \begin{cases} W(\lambda) & \text{if } u = \lambda \\ +\infty & \text{otherwise} \end{cases} & \text{if } |\lambda| \geq 1 \\ \begin{cases} \min\left\{\varepsilon c_W, \frac{1}{2}\left(\left(W''(1)(\lambda-1)^2\right) \wedge \left(W''(-1)(\lambda+1)^2\right)\right)\right\} \#(S(u)) & \text{if } |u| = 1 \text{a.e.} \\ +\infty & \text{otherwise} \end{cases} & \text{if } |\lambda| < 1. \end{cases} \tag{79}$$

Observe that we have matched the regular approximation at $|\lambda| \geq 1$ at order ε given by

$$G_\varepsilon^\lambda(u) = \begin{cases} W(\lambda) & \text{if } u = \lambda \\ +\infty & \text{otherwise} \end{cases}$$

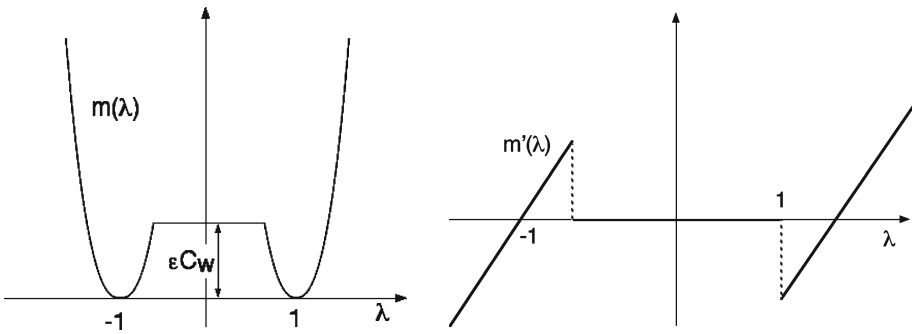

Fig. 13 Approximate minimum values for G_ε^λ in Example 9.7

with the one at $|\lambda| < 1$, given by

$$G_\varepsilon^\lambda(u) = \begin{cases} \varepsilon c_W \#(S(u)) & \text{if } |u| = 1 \text{a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

and, finally, with the rectified singular approximation at $\lambda = 1$ (and similarly at -1) given by

$$G_\varepsilon^\lambda(u) = \begin{cases} \min\left\{\varepsilon c_W, \frac{1}{2}\left(W''(1)(\lambda-1)^2\right)\right\} \#(S(u)) & \text{if } |u| = 1 \text{a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

(see Example 9.3). The uniform equivalence of the resulting theory (79) and the original theory (27) is immediately proven by the reference to Remark 1.8.

The dependence of the minimum values of G_ε^λ on λ and the corresponding stress-strain relation are shown in Fig. 13. We see that the phase transition in this approximate theory takes place at finite stress, contrary to what is predicted by the straightforward Γ-development (see our Fig. 4), and that the nucleus has a finite size. These features agree with the exact solution of the discrete problem at finite ε (see [65,68]).

The main defect of the constructed approximate 'theory' is its rigid structure adopted exclusively to one type of loading. Such theory cannot be easily modified to cover other types of boundary conditions, or generalized to describe higher dimensional case and to deal with local minimizers. The origin of the problem is that the equivalent energy, which we have chosen for the singular point, is excessively simple (minimal). To improve the situation we can choose a broader class of energies with the correct structure of the singularities which would then generate more flexible uniformly equivalent theory.

The choice in this case should be driven by our computation of the energy singularity at ± 1 which implicitly enlarges the functional space to piecewise-constant functions or rather to SBV functions (see [13]). Thus, following the pattern of fracture theory we can take

$$G_\varepsilon^\lambda(u) = \int_0^1 (W(u) + \overline{C}\varepsilon^2|u'|^2)\mathrm{d}x + \varepsilon\beta\#(S(u)), \qquad \int_0^1 u\mathrm{d}x = \lambda \tag{80}$$

defined on $SBV(0,1)$. Here the only role of the gradient terms is to prevent the formation of discontinuities outside the set $S(u)$. Since only equivalence at scale ε is required, the potential W may be replaced by its piecewise-quadratic analog

$$W_0(z) = \frac{1}{2}\min\{W''(-1)(z+1)^2, W''(1)(z-1)^2\},$$

provided that $\overline{C}$ is chosen such that

$$2\sqrt{\overline{C}}\int_{-1}^{1}\sqrt{W_0(s)}\,\mathrm{d}s \geq c_W$$

(see Appendix A.2(c)). The analysis of this theory shows that the convex component of W is now represented by the bulk term in (80), while the concave component (the 'spinodal' region) is described by a (constant) surface energy. We emphasize once again that the resulting theory contains a small parameter ε and therefore cannot be obtained as a straightforward Γ-limit. The problem with the functional (80) is that it is again still too rigid in its affinity to particular boundary conditions and can hardly be used to capture local minimizers [65,68] or to deal with the multi-dimensional case.

To overcome these limitations we may try, as in the case of fracture, to modify the surface energy further. To characterize the classes of eligible surface energies we can deal directly with n-dimensions. First, by enlarging the space from piecewise-constant functions with values ± 1 to all piecewise-constant functions we obtain the class of functionals

$$G_\varepsilon(u) = \int_\Omega W(u)\mathrm{d}x + \varepsilon \int_{S(u)} g(u^+, u^-)\mathrm{d}\mathcal{H}^{n-1}, \tag{81}$$

where $u^\pm$ are the traces on both sides of the set $S(u)$ of discontinuity points of u, and $g \geq 0$ is a subadditive function. Clearly, G_ε is equivalent to F_ε at scale 1. Sufficient conditions on g for G_ε to be equivalent to F_ε at scale ε are

$$g(u, v) \geq 2\sqrt{C}\left|\int_u^v \sqrt{W(s)}\,\mathrm{d}s\right| \quad \text{for all } u, v,$$

and $g(1, -1) = g(-1, 1) = c_W$; for example, $g(u, v) \equiv c_W$ (see Appendix A.2(a)). In an even larger space $SBV(\Omega)$ (see [13,31]), we may take

$$G_\varepsilon(u) = \int_\Omega (W(u) + \overline{C}\varepsilon^2|\nabla u|^2)\mathrm{d}x + \varepsilon \int_{S(u)} g(u^+, u^-)\,\mathrm{d}\mathcal{H}^{n-1} \tag{82}$$

for $u \in SBV(\Omega)$, with g as above and $\overline{C} \geq C$ (see Appendix A.2(b)). If $\overline{C} > C$ then sharp phase transitions are favored. Inside this class a uniformly equivalent theory can be obtained if we take $\{u \in SBV(0, 1) : |u| \geq T\}$ for some $0 < T < 1$ together with the constraint that $u^- \leq -T$ and $u^+ \geq T$ (or the converse) on $S(u)$ and write a functional

$$G^\lambda_\varepsilon(u) = \int_{\Omega_+(u)} \left(\frac{1}{2}W''(1)(u-1)^2 + \varepsilon^2 C|u'|^2\right)\mathrm{d}t + \int_{\Omega_-(u)} \left(\frac{1}{2}W''(-1)(u+1)^2 + \varepsilon^2 C|u'|^2\right)\mathrm{d}t + \varepsilon \sum_{S(u)} g(u^-, u^+),$$

where $\Omega_\pm(u) = \{\pm u \geq T\}$. Note that here the gradient terms are not 'formal' as in (80) and are expected to describe the boundary layers near the internal discontinuities. Therefore, the constants in front of them remain the same as in the original theory.

For the approximation (83) to work, some technical conditions must be imposed on T, C and g, involving the notion of subadditive envelope (see Appendix A.2(d)). As we have already seen, those conditions are satisfied by taking g to be equal to constant c_W, however, the example of fracture shows, that one can do better if the goal is to capture local minimizers. Thus, if we identify g with appropriately re-scaled and translated spinodal component of the function W, we may expect to obtain an approximate theory with a much better capture of the local minimizers.

10 Conclusions

The goal of this paper was twofold. First, we wanted to show that the previous attempts to extend the idea of Γ-convergence beyond the first Γ-limit have not been fully satisfactory. Second, we wanted to find the way of constructing a rigorous asymptotic Γ-expansion which is devoid of the detected flaws by extending to functionals the corresponding machinery developed by Poincaré (1886) in the context of differential equations. This has placed the main focus of the paper on *definitions* rather than theorems.

Thus, we have found it necessary to extend to functionals the concept of an asymptotic equivalence at a particular order. This allowed us to represent the whole set of approximating functionals as a union of the classes of equivalence. While one can propose different criteria making the approximation unique (analytical simplicity, computability, ability to capture local minimizers, etc.), we have selected the one which emphasizes that in application the particular functionals usually appear as representatives of the parameterized families (e.g., Von-Karman theory of plates, lubrication theory, the theory of incompressible elastic solids, etc.). To deal with such 'theories', we had to extend to parameterized families of functionals the concept of Γ-equivalence. Since the so-defined equivalence may not be uniform with respect to parameters, we have been naturally led to the important distinction between the regular and singular values of the parameters. Following the methodology developed for functions, we have shown that even if a conventional Γ-limit in the singular point does not exist, one can often reconstruct the structure of the singularity by the blow-up procedure. We have then showed that the knowledge of the structure of the critical points allows one to construct matched asymptotic Γ-expansions delivering globally uniform approximation of a given order.

We have applied the proposed methodology to problems of practical interest and produced several interesting approximate theories. In some cases our formal development has given a rigorous justification for the existing semi-empirical procedures used by practitioners (e.g., cohesive models of fracture, quasi-continuum models in elasticity, etc.). In the other cases, entirely new approximate theories have been advanced (e.g., a theory of phase transitions with surface energy combined with a possibility of a discontinuity in the gradients).

In conclusion we mention several limitations of our approach. First, the proposed methodology is not fully formalized and its implementation depends on the detailed knowledge of the minimizers of the original functional, which is seldom readily available. Then, even if successful, the method delivers only a particular approximate theory, which may well coexist in applications with other equivalent theories. In those cases one needs to find additional criteria which would justify the use of a given theory in the physical problem of interest. The main problem, however, is that Γ-convergence deals with the global minima while in applications the situations are plentiful when the energy landscape is rather rugged and it is the knowledge of the local minima which is crucial.

Acknowledgments The authors' research was partially supported by the Marie Curie Training Network 'Multi-scale modelling and characterisation for phase transformations in advanced materials' (MRTN-CT-2004-505226). The authors are grateful to C. Evans, A. Schlömerkemper, C. Zanini and L. Scardia for valuable comments on the manuscript.

Appendix

A.1 We prove the second inequality in (69), namely that

$$\lim_j \min\left\{\int_0^1 \left(\frac{W(v)}{\varepsilon_j}+\varepsilon_j|v'|^2\right)\mathrm{d}t : \int_0^1 v\mathrm{d}t=\lambda_j\right\} = \lim_j \min\left\{c_W, \frac{1}{2}W''(1)\frac{(1-\lambda_j)^2}{\varepsilon_j}\right\}.$$

We first note that

$$\begin{aligned} m_j &:= \min\left\{\int_0^1 \left(\frac{W(v)}{\varepsilon_j}+\varepsilon_j|v'|^2\right)\mathrm{d}t : \int_0^1 v\mathrm{d}t=\lambda_j\right\} \\ &\le \frac{W(\lambda_j)}{\varepsilon_j} = \frac{1}{2}W''(1)\frac{(1-\lambda_j)^2}{\varepsilon_j} + o\left(\frac{(1-\lambda_j)^2}{\varepsilon_j}\right) \end{aligned} \tag{83}$$

by testing with $v=\lambda_j$, This shows that if $(1-\lambda_j)^2 << \varepsilon_j$ then $\lim_j m_j = 0$ as desired. Conversely, let

$$(1-\lambda_j)^2 \ge C\varepsilon_j \tag{84}$$

for some $C>0$. Note that in this case we can construct a sequence of the form

$$\widehat{v}_j(t) = \overline{u}\left(\frac{t-t_j}{\varepsilon_j}\right),$$

where $\overline{u}$ is a solution of the optimal profile problem

$$\min\left\{\int_{-\infty}^{+\infty}(W(u)+|u'|^2)\mathrm{d}t : u(\pm\infty)=\pm 1\right\}=c_W,$$

and t_j are suitable translations in order to match the integral constraint, such that $\lim_j \frac{1}{\varepsilon_j}F_{\varepsilon_j}^{\lambda_j}(\widehat{v}_j)=c_W$. By this computation and (83), we then get

$$\lim_j m_j \le \lim_j \min\left\{c_W, \frac{1}{2}W''(1)\frac{(1-\lambda_j)^2}{\varepsilon_j}\right\}.$$

To prove the converse inequality, consider v_j a minimizer for m_j. If $\lim_j \inf v_j \le -1$ then there exit points x_j^-, x_j^+ such that $\lim_j v_j(x_j^\pm)=\pm 1$, and then

$$\lim_j m_j \ge \liminf_j \left|\int_{x_j^-}^{x_j^+}\left(\frac{W(v_j)}{\varepsilon_j}+\varepsilon_j|v_j'|^2\right)\mathrm{d}t\right| \ge \liminf_j \left|\int_{x_j^-}^{x_j^+}\sqrt{W(v_j)}|v_j'|\mathrm{d}t\right| = c_W.$$

If otherwise, $\lim_j \inf v_j \ge C > -1$ then for fixed $\eta > 0$ we have

$$|\{v_j < 1-\eta\}| \le \frac{\varepsilon}{C_\eta}\int_0^1\left(\frac{W(v_j)}{\varepsilon_j}+\varepsilon_j|v_j'|^2\right)\mathrm{d}t \le \varepsilon\frac{c_W}{C_\eta}, \tag{85}$$

where $C_\eta=\min\{W(s) : C\le s\le 1-\eta\}$. Note moreover that it is not restrictive to suppose that $v_j < 1+\eta$. Let $c_\eta = o(1)$ as $\eta\to 0$ be such that $W(z)\ge \frac{1}{2}(W''(1)-c_\eta)(1-z)^2$ on $[1-\eta, 1+\eta]$; then, by Jensen's inequality and (84)

$$\begin{aligned}\lim_j m_j &\ge \lim_j \int_{\{v_j>1-\eta\}}\frac{W(v_j)}{\varepsilon_j}\mathrm{d}t \ge \lim_j \frac{1}{2}(W''(1)-c_\eta)\frac{1}{\varepsilon_j}(1-\lambda_j+O(\varepsilon))^2\\ &\ge \lim_j \frac{1}{2}(W''(1)-c_\eta)\frac{1}{\varepsilon_j}(1-\lambda_j)^2.\end{aligned}$$

Letting $\eta\to 0$ we have the desired inequality.

A.2 We sketch here the proofs for the equivalence statements in Example 9.7.

(a) After noting that the domain of the Γ-limit at order 1 consists of functions in $BV(\Omega;\{-1,1\})$ we remark that

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\frac{G_\varepsilon-\min W}{\varepsilon}\ge \overline{H}, \tag{86}$$

where $\overline{H}$ is the lower-semicontinuous envelope of

$$H(u)=\int_{S(u)} g(u^+,u^-)\mathrm{d}\mathcal{H}^{n-1}$$

defined on piecewise-constant functions. From the relaxation theory for those functionals (see, e.g., [12,17,18]), we deduce that the conditions on g imply that $\overline{H}(u)=H(u)=c_W\mathcal{H}^{n-1}(S(u))$ if $u\in BV(\Omega;\{-1,1\})$, which gives the liminf inequality. Finally, we note that a recovery sequence for such u is simply given by $u_\varepsilon=u$.
(b) As for (a) we can use the same relaxation argument, taking now

$$H(u)=2\sqrt{C}\int_\Omega\sqrt{W(u)}|Du|\mathrm{d}x+\int_{S(u)} g(u^+,u^-)\mathrm{d}\mathcal{H}^{n-1}.$$

(c) We can follow the argument in A.1 above. The condition of equivalence at scale ε is then that

$$c_W = \min\left\{\int_{-\infty}^{+\infty}(W_0(u)+\overline{C}|u'|^2)\mathrm{d}t + \sum_{S(u)} g(u^+,u^-) : u(\pm\infty) = \pm 1 \text{ or } \mp 1\right\},$$

which is implied by the conditions assumed.
(d) As in (c) we may use the argument in A.1, provided that

$$c_W = \min\left\{\int_{R_+(u)}\left(\frac{1}{2}W''(1)(u-1)^2+\varepsilon^2 C|u'|^2\right)\mathrm{d}t \right.$$
$$\left. + \int_{R_-(u)}\left(\frac{1}{2}W''(-1)(u+1)^2+\varepsilon^2 C|u'|^2\right)\mathrm{d}t + \sum_{S(u)} g(u^+,u^-) : u(\pm\infty) = \pm 1 \text{ or } \mp 1\right\},$$

where the infimum is taken over all $u \in SBV(\mathbb{R})$ such that $|u| \geq T$ and $u^- \leq -T$ and $u^+ \geq T$ (or the converse) on $S(u)$, and $R_\pm(u) = \{\pm u \geq T\}$.
Note that if we set

$$g^{T,C}(u,v) = \begin{cases} g(u,v) & \text{if } u \leq -T \text{ and } v \geq T \text{ or the converse} \\ 2\sqrt{C}\left|\displaystyle\int_u^v |s-1|\mathrm{d}s\right| & \text{if } u, v \geq T \\ 2\sqrt{C}\left|\displaystyle\int_u^v |s+1|\mathrm{d}s\right| & \text{if } u, v \leq T \\ +\infty & \text{otherwise,} \end{cases}$$

then the above condition can be equivalently expressed as

$$c_W = \min\left\{\sum_{i=1}^{N} g^{T,C}(z_i, z_{i-1}) : z_0 = \pm 1, \quad z_N = \mp 1, \ \ N \in \mathbb{N}\right\};$$

i.e., that the subadditive envelope of $g^{T,C}$ computed in $(-1, 1)$ and $(1, -1)$ is c_W (see [14]).

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