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Elastic bars with cohesive energy

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Abstract Most quasi-static variational models of fracture are based on the splitting of the energy functional into the sum of two terms: bulk, depending on the displacement gradient, and surface, depending on the displacement discontinuities. In this paper we consider the simplest one-dimensional problem of this type, a bar stretched by a given axial displacement, and systematically compare two alternative interpretations of the surface energy term. In the first interpretation (*elastic model*), the surface energy is viewed as a cohesive energy which is stored and can be recovered. In the second (*inelastic model*), it is irreversibly lost. We show that by assuming an evolution scheme based on local minimization and by varying the convexity-concavity properties of the surface energy the elastic model can reproduce a broad class of macroscopic material responses which have been traditionally treated as unrelated. These responses are associated with monotone loading and range from brittle fracture to rate independent plasticity. However, a realistic description for both loading and unloading is achieved only within the inelastic model.

Keywords Fracture mechanics · Variational methods · Reversible fracture · Rate independent dissipative potential · Subadditivity · Griffith model · Barenblatt model

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1 Introduction

Starting with the seminal work of Griffith [29], theoretical fracture mechanics views a stable equilibrium configuration of a body as a minimizer of the energy functional involving two contributions: a bulk energy in the non-fractured portion of the continuum, and a cohesive energy associated with the fracture surface. In this class of models the bulk energy density depends on the displacement gradient, while the surface energy is a function of the displacement discontinuities, see the reviews [5,53].

In the case of quasi-static loading, two different interpretations of the surface energy have been proposed in the literature. In the first interpretation (our *elastic model*) the surface energy is viewed as a form of elastic energy, which implies that it can be recovered if the decohesion is re-healed (e.g. [28,34]). In the second (our *inelastic model*), the surface term in the energy functional is viewed as irreversible dissipation. The irreversibility assumption is made in most of the engineering studies of fracture (e.g. [10]). To recover a variational

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formulation in this case one has to assume that the dissipation is rate independent (see the general development of these ideas in [26,32,44,48–50]; a review of recent literature can be found in [41,42]. The resulting incremental variational approach to fracture has been very successful, see, e.g., [4,27,37].

In the present paper we take a specific example and offer a parallel analysis of the two models, elastic and inelastic, emphasizing the issue of local versus global minimization. To achieve a full analytical transparency of the results, we simplify the bulk problem by choosing the geometry of the one-dimensional elastic bar, and by assuming that the bar is homogeneous, free of body forces, and loaded by controlled displacements at the ends.

Our analysis shows that the global response of the bar, given by the relation between prescribed elongation and axial force, can be very different depending on the analytical shape of the cohesive energy and, in particular, on its behavior near the origin. Thus, an inelastic model with concave surface energy provides a representation of atomistic fracture (e.g. [9,55]) and distinguishes between brittle and ductile modes determined by the size of the body (e.g [8,19]). In the same inelastic framework, a cohesive energy which is convex near the origin determines a force-elongation curve similar to the response curve of an elastic-plastic material with the macro-fracture surfaces replaced by micro-decohesions diffused over the volume [20,21,24]. An inelastic model assuming a bi-modal cohesive energy which is concave near the origin, convex in a segment away from the origin and concave at large crack openings [23,24] provides a two-step fracture mechanism combining macro-cracking and micro-cracking. This model can reproduce a broad range of damage regimes from crazing rupture in polymers to progressive damage of metals.

In general, our study shows that an evolution scheme based on *local minimization* and an appropriate choice of the *convexity properties* of the cohesive term provide a simple and unified framework for the description of a wide class of material responses which are traditionally treated as unrelated. Regarding each of the two issues, local minimization and convexity properties, taken separately, our paper is far from being exhaustive. For instance, in the context of *global minimization*, a variety of other forms for the cohesive energy have been considered in [3,6,7,16]. The local minimization in fracture problems with *concave* cohesive energies have been addressed in [8,13,14,18–21,23,24,35,36,38,45,46,54,55].

The paper is organized as follows. In Sect. 2 we introduce the basic assumptions concerning the structure of the bulk and cohesive terms in the energy functional, and formulate our parametric loading constraint. In Sect. 3 we interpret the cohesive term as conservative and formulate the elastic model for the bar. The examples of response patterns generated in the elastic framework are discussed in Sect. 4. The inelastic model is formulated and studied in Sect. 5. In Sect. 6 we give examples of unloading paths generated by the inelastic model and compare the two models. The Appendix contains some auxiliary existence and nonexistence results in the context of global minimization which are useful to understand some of our examples.

2 Basic assumptions

Consider a bar with constant cross section, subject to axial deformation. Assume that every axial displacement u is continuous and differentiable, except at a finite or countable set of jump points x at which the right limit $u(x+)$ and the left limit $u(x-)$ exist but do not coincide. The difference $(u(x+)-u(x-))$ is the *jump amplitude* at x. It will be denoted by $[u](x)$, while $S(u)$ will denote the set of all jump points of u.

Due to the presence of the jumps, there are two possible ways for stretching the bar. One is to increase the bulk deformation u', and one is to increase the jump amplitudes. With the bulk deformation we associate the *bulk energy density* w, which is a function of $u'(x)$ at $x \notin S(u)$, and with the jumps we associate the *cohesive energy* θ, which is a function of $[u](x)$ at $x \in S(u)$. We take w strictly convex, with

$$w(0) = 0, \quad w'(0) = 0, \tag{2.1}$$

and we assume the growth conditions

$$\lim_{e \to -1} w(e) = +\infty, \quad \lim_{e \to +\infty} w(e) = +\infty. \tag{2.2}$$

Notice that the assumptions (2.1) together with the convexity of w imply $w(e) \geq 0$ for all e and $w(e) = +\infty$ for all $e \leq -1$. For the function θ we assume that

$$\theta(0) = 0, \quad \theta(q) = +\infty \quad \text{for all } q < 0. \tag{2.3}$$

We require that θ is lower semicontinuous and monotonic non-decreasing in $[0, +\infty)$, so that $\theta(q) \geq 0$ for all $q > 0$, and that the limits

$$\theta(0+) = \lim_{q \to 0+} \theta(q), \quad \theta'(0+) = \lim_{q \to 0+} \frac{\theta(q)}{q}, \tag{2.4}$$

exist. Observe that both limits are non-negative, and that $\theta'(0+) = +\infty$ when $\theta(0+) > 0$. The strain energy of a bar of length l subject to a displacement u is

$$E(u) = \int_0^l w(u'(x))dx + \sum_{S(u)} \theta([u](x)). \tag{2.5}$$

As the domain of definition of E we take the set of all real-valued functions u from $(0, l)$ for which there is a finite or countable partition of $(0, l)$ into pairwise disjoint intervals, such that u has a C^1 extension to the closure of each interval. This set will be denoted by $V(0, l)$.

We assume that there are no applied loads, and that the endpoints of the bar are subject to the displacements

$$u(0) = 0, \quad u(l) = \beta l, \tag{2.6}$$

with $\beta \geq 0$. For simplicity, but without loss in generality, we exclude the possibility that a jump may occur exactly at one of the endpoints. Then, within a rigid translation, the assumed boundary conditions are equivalent to

$$\beta l = \int_0^l u'(x)dx + \sum_{S(u)} [u](x). \tag{2.7}$$

We denote by $V_\beta(0, l)$ the set of all $u \in V(0, l)$ which satisfy the boundary condition in the form (2.7) for the given β. We point out that, since there are no applied loads, $E(u)$ is the total energy of the bar in its configuration u.

3 The elastic model

3.1 Equilibrium configurations

For a given $\beta \geq 0$, an *equilibrium configuration* for β is a function $u \in V_\beta(0, l)$ with $E(u) < +\infty$, such that

$$\lim_{\lambda \to 0+} \frac{1}{\lambda}(E(u + \lambda\eta) - E(u)) \geq 0 \tag{3.1}$$

for all $\eta \in V_0(0, l)$, that is, for all perturbations η which leave unchanged the total length of the bar

$$\int_0^l \eta'(x)dx + \sum_{S(\eta)} [\eta](x) = 0. \tag{3.2}$$

Inequality (3.1) generalizes to non-smooth functionals the condition of the vanishing of the first variation. Here the non-smoothness of E is due to the assumptions (2.2), and (2.3) made on w and θ.

The following local characterization of equilibrium is the counterpart of Euler's equation for a smooth functional. In it, we assume for simplicity that w is smooth in $(-1, +\infty)$ and that θ is smooth in $(0, +\infty)$, keeping the possibility of a discontinuity of θ at the origin.

Proposition 3.1 *Let $w \in C^1(-1, +\infty)$ and $\theta \in C^1(0, +\infty)$ satisfy the assumptions made in Sect. 2. Then a displacement field $u \in V(0, l)$ is an equilibrium configuration if and only if there is a non-negative constant u' such that*

$$0 \leq w'(u') \leq \theta'(0+), \tag{3.3}$$

$$u'(x) = u' \quad \textit{for all } x \in (0, l) \backslash S(u), \tag{3.4}$$

$$\theta'([u](x)) = w'(u') \quad \textit{for all } x \in S(u). \tag{3.5}$$

Proof Let u be an equilibrium configuration. For a smooth perturbation η the set $S(\eta)$ is empty, so that $S(u+\eta) = S(u)$ and $[u+\eta](x) = [u](x)$ at all $x \in S(u)$. Therefore,

$$E(u+\lambda\eta) - E(u) = \int_0^l (w(u'(x) + \lambda\eta'(x)) - w(u'(x))\, dx = \lambda \int_0^l w'(u'(x))\eta'(x)\, dx + o(\lambda). \quad (3.6)$$

The integral of η' being zero by (3.2), a standard argument in the Calculus of Variations leads to the conclusion that (3.1) holds only if $w'(u'(x))$ is constant. But $w'(u'(x))$ constant implies $u'(x)$ constant, because w strictly convex implies w' strictly increasing. Then (3.4) holds, and the expression (2.5) of the energy and the boundary condition (2.7) reduce to

$$E(u) = l w(u') + \sum_{S(u)} \theta([u](x)), \quad \beta = u' + l^{-1} \sum_{S(u)} [u](x), \quad (3.7)$$

respectively. Now take a perturbation η with constant derivative η' and with a single jump $[\eta]$, so that $l\eta' + [\eta] = 0$ by (3.2). If $[\eta]$ is located at a point x_o of $S(u)$, one has

$$\begin{aligned} E(u+\lambda\eta) - E(u) &= l w(u' + \lambda\eta') + \theta([u](x_o) + \lambda[\eta]) - l w(u') - \theta([u](x_o)) \\ &= \lambda(w'(u') l\eta' + \theta'([u](x_o))[\eta]) + o(\lambda) = \lambda(\theta'([u](x_o)) - w'(u'))[\eta] + o(\lambda). \end{aligned} \quad (3.8)$$

Then the equilibrium condition (3.1) implies

$$(\theta'([u](x_o)) - w'(u'))[\eta] \geq 0, \quad (3.9)$$

and condition (3.5) follows from the arbitrariness of $[\eta]$. If x_o is not in $S(u)$ one has $[u](x_o) = 0$, and the derivative of θ at 0 does not exist. However, from the first equality in (3.8) it follows that]qed

$$E(u+\lambda\eta) - E(u) = \theta(\lambda[\eta]) + \lambda l w'(u')\eta' + o(\lambda) = \theta(\lambda[\eta]) - \lambda w'(u')[\eta] + o(\lambda). \quad (3.10)$$

If $\theta(0+) > 0$, the limit in (3.1) is equal to $+\infty$, and inequality (3.3) is trivially satisfied. If $\theta(0+) = 0$, the same limit is equal to $+\infty$ if $[\eta] < 0$, and to

$$(\theta'(0+) - w'(u'))[\eta] \quad (3.11)$$

if $[\eta] > 0$. Then condition (3.3) follows.

It remains to prove that u' and $w'(u')$ are non-negative. If u has no jumps, then $u' = \beta \geq 0$ by $(3.7)_2$. If u has a jump $[u]$, this jump must be positive, because otherwise $E(u) = +\infty$. Then from the equilibrium condition (3.5) one has $w'(u') = \theta'([u]) \geq 0$ because θ is non-decreasing. In turn, $w'(u') \geq 0$ implies $u' \geq 0$ because $w(0) = 0$ and w' is strictly increasing.

Now, let us prove that any u which satisfies conditions (3.3)–(3.5) is an equilibrium configuration. For any perturbation $\eta \in V_0(0, l)$, we have

$$\begin{aligned} E(u+\lambda\eta) - E(u) &= \int_0^l (w(u' + \lambda\eta'(x)) - w(u'))dx + \sum_{S(\eta)\cap S(u)} (\theta([u+\lambda\eta](x)) - \theta([u](x))) + \sum_{S(\eta)\setminus S(u)} \theta(\lambda[\eta](x)) \\ &= \lambda(w'(u') \int_0^l \eta'(x)\, dx + \sum_{S(\eta)\cap S(u)} (\theta'([u](x))\, [\eta](x)) + \sum_{S(\eta)\setminus S(u)} \theta(\lambda[\eta](x)) + o(\lambda) \\ &= \lambda w'(u') \left(\int_0^l \eta'(x)\, dx + \sum_{S(\eta)\cap S(u)} [\eta](x) \right) + \sum_{S(\eta)\setminus S(u)} \theta(\lambda[\eta](x)) + o(\lambda) \\ &= \sum_{S(\eta)\setminus S(u)} \big((\theta(\lambda[\eta](x)) - \lambda w'(u')[\eta](x)\big) + o(\lambda), \end{aligned} \quad (3.12)$$

with the last three steps due to (3.2), (3.4) and (3.5), respectively. If one of the jumps $[\eta](x)$ in $S(\eta)\backslash S(u)$ is negative, the corresponding energy θ is equal to $+\infty$ and condition (3.1) is trivially satisfied. So let us consider the case of all $[\eta](x)$ in $S(\eta)\backslash S(u)$ positive. If $\theta(0+) = 0$, from (3.3) one has

$$E(u + \lambda\eta) - E(u) = \lambda(\theta'(0+) - w'(u')) \sum_{S(\eta)\backslash S(u)} [\eta](x)) + o(\lambda) \geq o(\lambda), \tag{3.13}$$

and condition (3.1) is satisfied. If $\theta(0+) > 0$, one has $\theta(\lambda[\eta](x)) = \theta(0+) + O(\lambda)$, the limit in (3.1) becomes equal to $+\infty$, and (3.1) is trivially satisfied. □

We recall that $w'(u')$ is the axial force at all x not in $S(u)$, and $\theta'([u](x))$ is the axial force at all x in $S(u)$. Equation (3.4) tells us that in an equilibrium configuration the axial force is the same at all points not in the jump set, while (3.5) tells us that the same axial force acts at all jump points. Thus, at all points of the bar the axial force takes the same value, which we denote by σ. Let us set

$$\sigma_c = \theta'(0+), \quad \beta_c = (w')^{-1}(\theta'(0+)). \tag{3.14}$$

Then, by inequalities (3.3), the axial force has the bounds

$$0 \leq \sigma \leq \sigma_c. \tag{3.15}$$

Moreover, because w convex implies $(w')^{-1}$ monotonic, inequality (3.3) is preserved under the transformation $(w')^{-1}$. Therefore, the bulk deformation u' is bounded by

$$0 \leq \; u' \leq \beta_c. \tag{3.16}$$

We say that σ_c and β_c are the *ultimate force* and the *ultimate bulk deformation*, respectively. They are determined by $\theta'(0+)$, that is, by the behavior of θ in a right neighborhood of the origin; in particular, for $\sigma_c = 0$ one has a no-tension material, and for $\sigma_c = +\infty$ one has a material with no ultimate force.

3.2 Local minimizers

The issue whether an equilibrium configuration is a local minimizer, depends on the norm chosen for the space $V(0,l)$. In the presence of jumps, an appropriate choice is the norm of the *total variation*

$$||u|| := \int_0^l |u'(x)| \, dx + \sum_{S(u)} |[u](x)|. \tag{3.17}$$

This norm attributes a finite distance to configurations with the same jumps but different jump sets. Therefore, the convergence of a sequence in the space $V(0,l)$ equipped with this norm is possible only if either all jump amplitudes converge to zero, or if in all elements of the sequence, except at most a finite number, the jumps are located at the same points. We emphasize that the choice of this norm reflects a specific assumption on the nature of the continuum. For instance, it penalizes translational modes for the structural defects represented here by displacement discontinuities.

We say that an equilibrium configuration u is a *local minimizer* for E if there is a $\delta > 0$ such that

$$E(u + \eta) \geq E(u) \tag{3.18}$$

for all perturbations $\eta \in V_0(0,l)$ with $||\eta|| < \delta$. We call *metastable* the configurations which are local minimizers, and *unstable* those which are not. The following Proposition shows that the stability test (3.18) can be restricted to perturbations which have a constant derivative η'.

Proposition 3.2 *Let η be a perturbation in $V_0(0,l)$, and let $\bar{\eta}$ be the perturbation with the same jump points and the same jump amplitudes as η, and with the constant derivative*

$$\bar{\eta}' = \frac{1}{l} \int_0^l \eta'(x) dx. \tag{3.19}$$

Then $\bar{\eta}$ belongs to $V_0(0,l)$, $||\bar{\eta}||$ is not greater than $||\eta||$, and $E(u + \eta) \geq E(u + \bar{\eta})$ for all equilibrium configurations u.

Proof That $\bar{\eta}$ satisfies condition (3.2) follows trivially from its definition. From the same definition one has

$$||\bar{\eta}|| - ||\eta|| = \left| \int_0^l \eta'(x)\, dx \right| - \int_0^l |\eta'(x)|\, dx \leq 0. \tag{3.20}$$

Moreover, for all u with constant derivative u',

$$E(u+\eta) - E(u+\bar{\eta}) = \int_0^l w(u' + \eta'(x))\, dx - l w\left(u' + \frac{1}{l}\int_0^l \eta'(x) dx\right), \tag{3.21}$$

and since w is convex the right-hand side is non-negative by Jensen's inequality. □

A further possibility of restricting the class of perturbations comes in the two following cases:

(i) $\theta(0+) > 0$,
(ii) $\theta(0+) = 0$ and $w'(u') < \theta'(0+)$.

In both cases, the stability test (3.18) can be restricted to perturbations with constant derivative, whose jump set is included in the jump set of u.

Proposition 3.3 *Let u be an equilibrium configuration, and let one of the two above conditions (i) or (ii) be satisfied. Let η be a perturbation in $V_0(0,l)$, and let $\tilde{\eta}$ be the perturbation in which all jumps in $S(\eta) \cap S(u)$ are left unchanged, and all those in $S(\eta)\backslash S(u)$ are eliminated. Moreover, let $\tilde{\eta}$ have the constant derivative*

$$\tilde{\eta}' = \frac{1}{l}\left(\int_0^l \eta'(x)dx + \sum_{S(\eta)\backslash S(u)} [\eta]_i\right). \tag{3.22}$$

Then $\tilde{\eta}$ belongs to $V_0(0,l)$, $||\tilde{\eta}||$ is not greater than $||\eta||$, and $E(u+\eta) \geq E(u+\tilde{\eta})$ for all equilibrium configurations u.

Proof That $\tilde{\eta}$ satisfies condition (3.2) again follows from its definition. From the same definition and from (3.22) one has

$$||\tilde{\eta}|| - ||\eta|| = \left|\int_0^l \eta'(x)dx + \sum_{S(\eta)\backslash S(u)} [\eta]_i dx\right| - \int_0^l |\eta'(x)|dx - \sum_{S(\eta)\backslash S(u)} |[\eta]_i| \leq 0. \tag{3.23}$$

Moreover, for all u with *a* constant derivative u', again by the convexity of w and by (3.22),

$$\begin{aligned} E(u+\eta) - E(u+\tilde{\eta}) &= \int_0^l w(u' + \eta'(x))dx - l w(u' + \tilde{\eta}') + \sum_{S(\eta)\backslash S(u)} \theta([\eta]_i) \\ &\geq l w'(u' + \tilde{\eta}') \int_0^l (\eta'(x) - \tilde{\eta}')dx + \sum_{S(\eta)\backslash S(u)} \theta([\eta]_i) \\ &= \sum_{S(\eta)\backslash S(u)} (\theta([\eta]_i) - w'(u' + \tilde{\eta}')[\eta]_i). \end{aligned} \tag{3.24}$$

If $\theta(0+) > 0$, the right-hand side is positive for sufficiently small $||\eta||$. If $\theta(0+) = 0$ and $w'(u') < \theta'(0+)$, by the continuity of w' there is a $\delta > 0$ such that $w'(u' + \tilde{\eta}') < \theta'(0+)$ for all $\tilde{\eta}' < \delta$. But if $w'(u' + \tilde{\eta}') < \theta'(0+)$ there is a $\delta_1 > 0$ such that

$$\theta([\eta]) > w'(u' + \tilde{\eta}')[\eta] \tag{3.25}$$

for all $[\eta] < \delta_1$. Then we may conclude that $E(u+\tilde{\eta}) - E(u+\eta) \geq 0$ for all η with $||\eta|| < \min\{\delta, \delta_1\}$. □

When $S(u)$ is the empty set, we have the following

Corollary 3.4 *All equilibrium configurations u without jumps and with* $w'(u') < \theta'(0+)$ *are metastable.*

Indeed, if $S(u)$ is empty then $\tilde{\eta}'$ reduces to the $\bar{\eta}$ defined in Proposition 3.2.

We now give an example showing that when $\theta(0+) = 0$ and $w'(u') = \theta'(0+)$ the stability test cannot be restricted to the perturbations with constant η' and with $S(\eta) \subset S(u)$. We recall that for a strictly convex function $\theta : (0, +\infty) \to (0, +\infty)$ with $\theta(0+) = 0$ one has

$$\theta(a + b) > \theta(a) + \theta(b) \tag{3.26}$$

for all $a, b > 0$. In what follows, we say that θ is *strictly convex in* $[0, c]$ if the above inequality holds for all $a, b > 0$ with $< a + b < c$.

Proposition 3.5 *Let* θ *be strictly convex in* $[0, c]$ *and with* $\theta(0+) = 0$*, let* β_c *be as in* (3.14)*, and let u be an equilibrium configuration for* β *in* $(\beta_c, \beta_c + l^{-1}c)$*, with* $w'(u') = \theta'(0+)$*. Then u is unstable.*

Proof Let x_o be a jump point for u. If $w'(u') = \theta'(0+)$, then $u' = (w')^{-1}(\theta'(0+)) = \beta_c$ and

$$\beta = u' + \frac{1}{l} \sum_{S(u)} [u](x) \geq \beta_c + \frac{1}{l}[u](x_o), \tag{3.27}$$

because all jump amplitudes are positive. Then $\beta \geq \beta_c$, and $[u](x_o) < c$ if $\beta < \beta_c + l^{-1}c$. Take ε in (0,c) and a perturbation η with $\eta' = 0$, with a jump of amplitude $-\varepsilon/2$ at x_o, and with a jump of amplitude $\varepsilon/2$ at a point not in the jump set of u. Then η belongs to $V_0(0, l)$ and $||\eta|| = \varepsilon < c$. Moreover,

$$E(u + \eta) - E(u) = \theta([u](x_o) - \varepsilon/2) + \theta(\varepsilon/2) - \theta([u](x_o)). \tag{3.28}$$

For $\beta < \beta_c + l^{-1}c$, from (3.27) one has $[u](x_o) < c$, and $E(u + \eta) < E(u)$ follows from (3.26). Then for every ε in $(0, c)$ there is an η in $V_0(0, l)$ with $||\eta|| = \varepsilon$ and with energy less than $E(u)$. Therefore, u is unstable. □

When $\theta(0+) = 0$ and $w'(u') = \theta'(0+)$, metastability depends on the *initial subadditivity* of θ. Recalling that a function $\theta : [0, +\infty) \to [0, +\infty)$ is *subadditive* if

$$\theta(a + b) \leq \theta(a) + \theta(b) \tag{3.29}$$

for all a, b, we say that θ is *initially subadditive* if there is a $c > 0$ such that the above inequality holds for all a, b with $a + b < c$, and that it is *initially strictly subadditive* if there is a $c > 0$ such that the above inequality is strict for all positive a, b with $a + b < c$.

Proposition 3.6 *Let* $\theta(0+) = 0$*, and let u be an equilibrium configuration. If* θ *is initially subadditive, then for each* $\eta \in V_0(0, l)$ *there is an* $\hat{\eta} \in V_0(0, l)$ *with* $\hat{\eta} = \eta'$*, with* $S(\hat{\eta}) = S(u) \cup \{x_o\}$*, and with* x_o *not in* $S(u)$*, such that* $||\hat{\eta}|| \leq ||\eta||$ *and* $E(u + \eta) \geq E(u + \hat{\eta})$*.*

Proof Let $\hat{\eta}$ be a perturbation with $\hat{\eta}' = \eta'$, with $[\hat{\eta}]_i = [\eta]_i$ at all jump points in $S(u)$, and with

$$[\hat{\eta}](x_o) = \sum_{S(\eta)\setminus S(u)} [\eta]_i. \tag{3.30}$$

Then,

$$||\hat{\eta}|| - ||\eta|| = \left| \sum_{S(\eta)\setminus S(u)} [\eta]_i \right| - \sum_{S(\eta)\setminus S(u)} |[\eta]_i| \leq 0. \tag{3.31}$$

Moreover,

$$E(u + \eta) - E(u + \hat{\eta}) = \sum_{S(\eta)\setminus S(u)} \theta([\eta]_i) - \theta\left(\sum_{S(\eta)\setminus S(u)} [\eta]_i \right). \tag{3.32}$$

For sufficiently small $||\eta||$, the right-hand side is non-negative by the initial subadditivity of θ.

3.3 Finite dimensional minimization

In the following Proposition we prove that all local energy minimizers which satisfy one of the following conditions

(i) $\theta(0+) > 0$,
(ii) $\theta(0+) = 0$ and $w'(u') < \theta'(0+)$,
(iii) $\theta(0+) = 0$ and θ is initially strictly subadditive,

have a finite number of jumps. The singular case of $\theta(0+) = 0$, $w'(u') = \theta'(0+)$, and θ not initially strictly subadditive will be considered later in Sect. 4.4.

Proposition 3.7 *Let u be a local energy minimizer. If one of the three above conditions is satisfied, then u has a finite number of jumps.*

Proof Let u be an energy minimizer with an infinite number of jumps. Because the energy of u is finite, all jump amplitudes $[u]_i$ are positive. Then from (3.17) follows the inequality

$$||u|| \geq \sum_{S(u)} [u]_i, \tag{3.33}$$

which tells us that the sum of the jump amplitudes is finite. Then for every $\varepsilon > 0$ there is a finite subset S_ε of $S(u)$ such that

$$\sum_{S_\varepsilon} [u]_i > \sum_{S(u)} [u]_i - \varepsilon, \tag{3.34}$$

and, consequently, the sum of the remaining jumps is less than ε. Take η with $\eta' = 0$, $[\eta]_i = 0$ at S_ε, $[\eta]_i = -[u]_i$ at $S(u)\backslash S_\varepsilon$, and $[\eta] = \sum_{S_\varepsilon} [u]_i$ at some $x_o \notin S(u)$. Then $\eta \in V_0(0, l)$ and

$$||\eta|| = 2 \sum_{S(u)\backslash S_\varepsilon} [u]_i < 2\varepsilon. \tag{3.35}$$

Moreover,

$$E(u+\eta) - E(u) = \theta\left(\sum_{S(u)\backslash S_\varepsilon} [u]_i\right) - \sum_{S(u)\backslash S_\varepsilon} \theta([u]_i). \tag{3.36}$$

If u is a local minimizer, the left-hand side is non-negative for sufficiently small ε. But if one of the above conditions (i)–(iii) holds, the right-hand side is negative. Indeed, if $\theta(0+) > 0$, from the monotonicity of θ it follows that $\theta([u]_i) > \theta(0+)$, and therefore the right-hand side is equal to $-\infty$ if the number of jumps is infinite. If $\theta(0+) = 0$ and $w'(u') < \theta'(0+)$, define

$$a := \min\{[u] > 0 \mid \theta'([u]) = w'(u')\}.$$

By the equilibrium condition (3.5), the amplitudes of all jumps of u are greater or equal than a. Then the right-hand side is again equal to $-\infty$ if the number of jumps is infinite. Finally, if θ is strictly subadditive in $[0,c]$, the right-hand side is negative for sufficiently small ε. In all cases, the contradiction inherent in inequality (3.36) proves that u has a finite number of jumps. □

When one of conditions (i)–(iii) holds, the search for a local minimum reduces to finite-dimensional minimization. Indeed, by Propositions 3.5 and 3.6, the stability analysis can be restricted to perturbations with $S(\eta)$ included in $S(u)$ plus at most one point not in $S(u)$. The minimum problem for the bar then becomes identical with that of a chain of $N+2$ atoms connected by $N+1$ springs, whose elongations are lu' and the N jump amplitudes $[u]_i$, and whose energies are $lw(u')$ and $\theta([u]_i)$, respectively.

Thanks to this analogy, necessary or sufficient conditions for a local minimum can be deduced from those obtained for finite chains of springs, see, e.g. [51]. Such conditions involve the Hessian matrix of the second partial derivatives of the energy

$$[H_{ij}(u)] = \begin{bmatrix} Q_0 + Q_1 & Q_0 & Q_0 & . & Q_0 \\ Q_0 & Q_0 + Q_2 & Q_0 & . & Q_0 \\ Q_0 & Q_0 & Q_0 + Q_3 & . & Q_0 \\ . & . & . & . & . \\ Q_0 & Q_0 & Q_0 & . & Q_0 + Q_N \end{bmatrix}, \tag{3.37}$$

where $Q_0 = l^{-1} w''(u')$ and $Q_h = \theta''([u]_h)$. Standard stability results are that u is metastable if the Hessian matrix is positive definite, and that if u is metastable then the Hessian matrix is positive semidefinite. Here we give a characterization of the positive semidefiniteness and positive definiteness of the matrix (3.37) under the assumption, due to the strict convexity of w, that Q_0 is positive.

Proposition 3.8 *If $Q_0 > 0$, the matrix $[H_{ij}(u)]$ is positive semidefinite if and only if*:

- *there is at most one negative Q_i,*
- *if there is a negative Q_i then all remaining Q_i are positive and such that*

$$\frac{1}{Q_0} + \sum_{i=1}^{N} \frac{1}{Q_i} \leq 0. \tag{3.38}$$

Moreover, the same matrix is positive definite if and only if:

- *there is at most one non-positive Q_i,*
- *if there is a negative Q_i then inequality (3.38) is satisfied strictly.*

Proof We recall that a matrix is positive semidefinite if and only if its principal minors are non negative. For the matrix (3.37), this condition is expressed by the inequalities

$$\begin{gathered} Q_0 + Q_h \geq 0, \\ Q_0 Q_h + Q_0 Q_k + Q_h Q_k \geq 0, \\ Q_0 Q_h Q_k + Q_0 Q_h Q_l + Q_0 Q_k Q_l + Q_h Q_k Q_l \geq 0, \\ \dots \end{gathered} \tag{3.39}$$

to be satisfied by all $h, k, l, \dots$ in $\{1, 2, \dots, N\}$. They are trivially satisfied if all Q_i are non-negative. On the contrary, if both Q_h and Q_k are negative then inequality $(3.39)_2$ is violated, because $Q_0 Q_h < 0$ and $Q_k(Q_0 + Q_h) \leq 0$ by inequality $(3.39)_1$. The same conclusion holds if $Q_h < 0$ and $Q_k = 0$. This proves that if there is a negative Q_i then all remaining Q_i must be positive.

Let Q_h be negative and all remaining Q_i positive. Divide the first inequality in (3.39) by $Q_0 Q_h$, the second by $Q_0 Q_h Q_k$, the third by $Q_0 Q_h Q_k Q_l$, and so on. Because all divisors are negative, we get

$$\frac{1}{Q_0} + \sum_{i=1}^{M} \frac{1}{Q_i} \leq 0 \tag{3.40}$$

for all $1 \leq M \leq N$ and for all sets of M components Q_k which include the negative Q_h. In particular, inequality (3.38) is obtained for $M = N$. It implies all inequalities (3.40).

For positive definiteness, a necessary and sufficient condition is that all inequalities (3.39) be strict. This is trivially verified for $Q_h \geq 0$ and all remaining Q_k positive. On the other hand, this is never verified if two of the Q_i, say Q_h and Q_k, are non-positive. Indeed, this contradicts the strict form $Q_0 Q_h + (Q_0 + Q_h) Q_k > 0$ of inequality $(3.39)_2$. Finally, consider the case of $Q_h < 0$ and all remaining Q_k positive. If the strict inequality (3.38) holds, the strict inequalities (3.40) follow. Each of them, multiplied by the product of the corresponding Q_i, provides one of the strict inequalities (3.39). Then the Hessian matrix is positive definite. Conversely, if the strict inequality (3.38) is violated, the last of the strict inequalities (3.39) is also violated, and therefore the matrix is not positive definite. □

It is interesting to observe that

$$\det[H_{ij}(u)] = \prod_{i=1}^{N} Q_i \left(1 + \sum_{i=1}^{N} \frac{Q_0}{Q_i}\right), \tag{3.41}$$

see [51, Section 4]. Consequently, Proposition 3.8 can be re-stated as follows:

Proposition 3.9 *If $Q_0 > 0$, the matrix $[H_{ij}(u)]$ is positive semidefinite if and only if there is at most one negative Q_i and det $[H_{ij}(u)] \geq 0$. Moreover, $[H_{ij}(u)]$ is positive definite if and only if there is at most one non-positive Q_i and det $[H_{ij}(u)] > 0$.*

3.4 Equilibrium curves and response curves

By Proposition 3.1, necessary conditions for equilibrium are

- the bulk deformation u' is constant and non-negative,
- all jump amplitudes $[u]_i$ are non-negative.

The set of all functions u in $V(0, l)$ which satisfy these two conditions will be called the *configuration space* and will be denoted by X. Each element of this space is a vector with $N + 1$ non-negative components, where N is the number of the jumps of u. The set X is in general infinite dimensional, since the number of the jumps may be infinite.

Not all elements of X are equilibrium configurations. Indeed, an equilibrium configuration must also satisfy the relation (3.5) between jump amplitudes and bulk deformation and the bounds (3.3) on u'. The value of β for which u is an equilibrium configuration is specified by the boundary condition $(3.7)_2$.

We prove below that the equilibrium configurations form smooth curves in X, which we call *equilibrium curves,* parametrized by a parameter t so that we can write $u = u(t)$ and $\beta = \beta(t)$. Let u be an equilibrium configuration for β, so that

$$w'(u') = \theta'([u]_i), \quad u' + l^{-1} \sum_i [u]_i = \beta. \tag{3.42}$$

Since u' and $[u]_i$ are now functions of t, denoting by a superimposed dot the differentiation with respect to t we have

$$w''(u')\,\dot{u}' = \theta''([u]_i)[\dot{u}]_i, \quad \dot{u}' + l^{-1} \sum_i [\dot{u}]_i = \dot{\beta}. \tag{3.43}$$

With Q_0 and Q_i as in (3.37), the first equation yields

$$l\dot{u}' \frac{Q_0}{Q_i} = [\dot{u}]_i, \tag{3.44}$$

and summing with respect to i and using the second equation in (3.43) one gets

$$l\dot{u}' \sum_i \frac{Q_0}{Q_i} = \sum_i [\dot{u}]_i = l(\dot{\beta} - \dot{u}'). \tag{3.45}$$

This implies

$$\dot{u}' = \left(1 + \sum_i \frac{Q_0}{Q_i}\right)^{-1} \dot{\beta}, \tag{3.46}$$

and, by (3.44),

$$[\dot{u}]_i = \frac{Q_0}{Q_i} \left(1 + \sum_i \frac{Q_0}{Q_i}\right)^{-1} l\dot{\beta}, \tag{3.47}$$

The fact that $\dot{u}'$ and $[\dot{u}]_i$ are uniquely determined proves that there is a unique direction of evolution for u as a function of β. Therefore, there is a unique equilibrium curve through the given u. This curve can be determined by integration of (3.46), and (3.47) with respect to β.

The following Proposition shows that, at least in the case of $\theta(0+) = 0$, each equilibrium curve consists of segments characterized by a constant number N of jump points, and that a transition from one segment to another may only occur at points at which $w'(u') = \theta'(0+)$.

Proposition 3.10 *Let $\theta(0+) = 0$. On a given equilibrium curve, let u, u_ε be equilibrium configurations for β, β_ε. Then a new jump $[u_\varepsilon]_o$ can be created in the transition between β and β_ε only if $w'(u') = \theta'(0+)$.*

Proof By the continuity of the equilibrium curve, the distance

$$||u - u_\varepsilon|| = l|u' - u'_\varepsilon| + \sum_{S(u)} |[u] - [u_\varepsilon]| + |[u_\varepsilon]_o|. \tag{3.48}$$

tends to zero when $\varepsilon \to 0$. Then $\theta([u_\varepsilon]_o) \to \theta(0+) = 0$, and $\theta'([u_\varepsilon]_o) \to \theta'(0+)$. But $\theta'([u_\varepsilon]_o)$ is equal to $w'(u'_\varepsilon)$ by equilibrium, and $w'(u'_\varepsilon) \to w'(u')$ because $u'_\varepsilon \to u'$ and w is smooth. Therefore, $w'(u') = \theta'(0+)$. □

For the axial force σ, by differentiation of the relation $\sigma = w'(u')$ we obtain

$$\dot{\sigma} = w''(u')\dot{u}' = Q_0\, l\, \dot{u}' \tag{3.49}$$

and therefore, from (3.46),

$$\dot{\sigma} = Q_0 l \left(1 + \sum_i \frac{Q_0}{Q_i}\right)^{-1} \dot{\beta} \tag{3.50}$$

The force-elongation response curve $\sigma = \tilde{\sigma}(\beta)$ is then determined by integration. The slope of this curve is $\dot{\sigma}/\dot{\beta} = d\tilde{\sigma}(\beta)/d\beta$. If we exclude the singular case $\det[H_{ij}(u)] = 0$, from (3.41) we have

$$\frac{d\tilde{\sigma}(\beta)}{d\beta} = (\det[H_{ij}(u)])^{-1} \prod_{i=1}^{N} Q_i \tag{3.51}$$

This expression establishes a relationship between the slope of the response curve and metastability. Indeed, Proposition 3.9 tells us that $\det[H_{ij}(u)] \geq 0$ for all metastable configurations. For $\det[H_{ij}(u)] > 0$, the slope of a metastable branch of the response curve is positive at the equilibrium configurations with all Q_h positive, and negative at those with one negative Q_h.

Similarly, by differentiation of $(3.7)_1$ we get

$$\dot{E}(u) = l w'(u')\dot{u}' + \sum_{S(u_0+)} \theta'([u]_i)[\dot{u}]_i, \tag{3.52}$$

and using the equalities $\sigma = w'(u') = \theta'([u]_i)$ and the incremental boundary condition $(3.43)_2$ we conclude that $\dot{E}(u) = l\sigma\dot{\beta}$, that is,

$$\frac{d\tilde{E}(\beta)}{d\beta} = l\tilde{\sigma}(\beta). \tag{3.53}$$

We see that, to within the factor l, the response curve $\sigma = \tilde{\sigma}(\beta)$ is the slope of the *energy-elongation response curve* $E = \tilde{E}(\beta)$. In particular, the relation

$$\frac{d^2\tilde{E}(\beta)}{d\beta^2} = l\frac{d\tilde{\sigma}(\beta)}{d\beta} = l(\det[H_{ij}(u)])^{-1} \prod_{i=1}^{N} Q_i \tag{3.54}$$

shows that a metastable branch of the curve $E = \tilde{E}(\beta)$ is convex at the equilibrium configurations with all Q_h positive, and concave at those with one negative Q_h.

3.5 Non-equilibrium transitions

It is possible that a metastable branch ends at some value of β. In this case the continuity of the minimizers with respect to the loading parameter is lost, and a dynamic transition necessarily occurs. In general, such transitions involve either the formation or the disappearance of jump points, accompanied by a transformation of bulk energy into surface energy or vice versa. To be consistent with the rate-independent character of our model, we assume that each transition is instantaneous at the time scale of the loading, and therefore takes place at a fixed value β_o of the loading parameter β. We leave open the question of which configuration, among all possible metastable configurations at β_o, is indeed reached as the result of the transition. We only assume that the transition has a dissipative character. Since the power supplied from the exterior is zero, this implies that the difference between the energies before and after the transition is irreversibly lost during the transition. Non-equilibrium transitions of this type have been considered before (e.g. [15,27,46]). Some authors (e.g. [35,43,45,52,54]) proposed the *vanishing viscosity* approach as the selection principle for the final state. Within this approach, the non-equilibrium transitions can be interpreted in the framework of overdamped viscoelasticity under the assumption that the viscosity coefficient tends to zero.

4 The elastic model: examples

To illustrate the variety of the material responses described by the elastic model, we consider several explicit expressions of the cohesive energy θ. For the bulk energy density w we do not assume any specific form. In the figures we refer to an energy which near the origin has the quadratic expression $w(u') = \frac{1}{2}ku'^2$, with k a positive material constant (Young's modulus).

4.1 Griffith's energy

Consider the function represented in Fig. 1a,

$$\theta([u]) = \begin{cases} \gamma & \text{if} \quad [u] > 0, \\ 0 & \text{if} \quad [u] = 0, \\ +\infty & \text{if} \quad [u] < 0, \end{cases} \tag{4.1}$$

where γ is a positive material constant (fracture toughness). An energy of this form is called Griffith's energy. For reference see, e.g., [4].

For Griffith's energy, $\theta'([u])$ is zero for all $[u] > 0$. Then by the equilibrium condition (3.5) for all equilibrium configurations with jumps one has

$$w'(u') = \theta'([u]_h) = 0, \tag{4.2}$$

and, therefore, $u' = 0$ and $\sigma = 0$. The total energy is then $E(u) = \gamma N$, where N is the number of the jumps of u. Because $\theta(0+) = \gamma > 0$, from Sect. 3.3 it follows that N is finite. The metastability of u does not follow from the analysis of the Hessian matrix (3.37). Indeed, since $Q_h = \theta''([u_h]) = 0$ for all positive $[u]_h$, the matrix is positive semidefinite, but not positive definite. However, the metastability of u can be proved directly with the following argument.

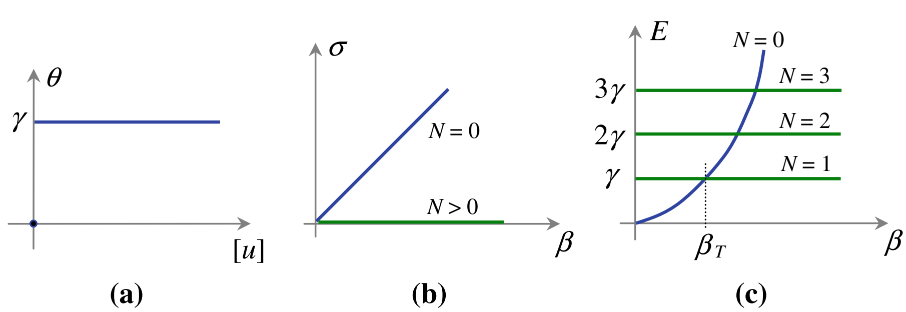

Fig. 1 Griffith's energy **(a)**, and overall response curves **(b)**, **(c)**

Because $\theta(0+) > 0$, by Proposition 3.3 it is possible to consider only perturbations η with constant derivative η' and with jumps $[\eta]_h$ located on the jump set of u. For such perturbations we have

$$E(u+\eta) - E(u) = l(w(u'+\eta') - w(u')) + \sum_{S(u)} (\theta([u]_h + [\eta]_h) - \theta([u]_h)), \tag{4.3}$$

and because $u' = 0$ by (4.2) and $\theta([u]_h + [\eta]_h) = \theta([u]_h) = \gamma$ by (4.1), we conclude that $E(u+\eta) - E(u) = lw(\eta') \geq 0$ by the positiveness of w.

For configurations without jumps one has $u' = \beta$ and $E(u) = lw(\beta)$. The condition $\theta(0+) > 0$ implies $\theta'(0+) = +\infty$, so that, by (3.15), and (3.16), there is no ultimate stress and no ultimate bulk deformation. Moreover, by Corollary 3.4, all equilibrium configurations without jumps are metastable.

The preceding analysis tells us that there are two types of equilibrium configurations, those with $N = 0$ and those with $N > 0$, and that both are metastable. They form equilibrium paths in the configuration space. The corresponding response curves are

$$\tilde{\sigma}(\beta) = w'(\beta), \quad \tilde{E}(\beta) = lw(\beta), \tag{4.4}$$

for $N = 0$, and

$$\tilde{\sigma}(\beta) = 0, \quad \tilde{E}(\beta) = \gamma N, \tag{4.5}$$

for $N > 0$. They are shown in Fig. 1b,c.

By Proposition A.2 in the Appendix, if $\theta(0+) > 0$ there exist global minimizers for the energy for all $\beta > 0$. The global minimum is the smallest of the local minima, that is, the smallest between $lw(\beta)$ and γN. According to (4.5) and (4.6), the global minimum is attained at configurations with $N = 0$ for small β and at configurations with $N = 1$ for large β. The transition occurs at the elongation $\beta_T = w^{-1}(\gamma l^{-1})$, determined by the intersection of the curves $N = 0$ and $N = 1$ in Fig. 1c. For $\beta = \beta_T$, the two equilibrium configurations with $N = 0$ and $N = 1$ have the same energy but are at a finite distance in the configuration space X. Any path joining the two configurations necessarily involves the opening of a jump. The nature of the corresponding energy barrier was discussed, e.g., in [39].

Therefore, a bar subject to a growing elongation β from the natural configuration remains indefinitely on the curve $N = 0$. This conclusion reflects the well-known inadequacy of Griffith's model in predicting the onset of fracture, see, e.g., [12–14,27].

4.2 Barenblatt's energy

In this model we consider a function θ whose restriction to $[0,+\infty)$ is C^2 and strictly concave, that is, $\theta''([u]) < 0$ for all $[u] > 0$. We also assume that $\theta(0+) = 0$ and $0 < \theta'(0+) < +\infty$. A function of this type is shown in Fig. 2a. Its derivative θ' is strictly decreasing, as shown in Fig. 2b. Fracture models based on Barenblatt's energy are numerous, see [2,8,11,13,14,19,20,23,24,31,33,40,55] to mention just a few.

As a consequence of Proposition 3.8, when $\theta'' < 0$ a metastable configuration can have at most one jump point. Therefore, it is described by a pair $(u', [u])$ of non-negative numbers. Consequently, the configuration

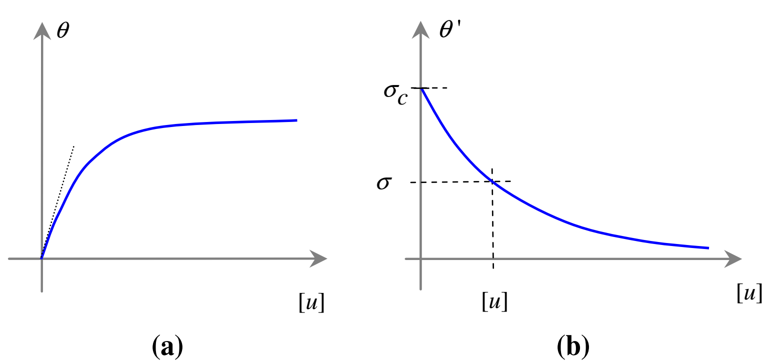

Fig. 2 Barenblatt's energy (**a**), and its derivative (**b**)

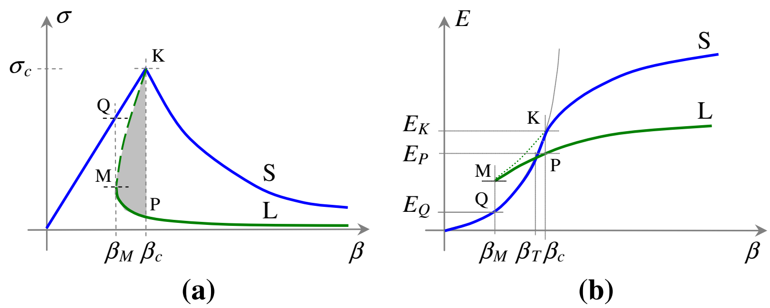

Fig. 3 Barenblatt's energy. Force-elongation **(a)** and energy-elongation **(b)** response curves for the elastic model, for bars with small (S) and large (L) length. Thick lines denote metastable branches, and thin lines denote unstable branches. The gray area in Fig. 3a represents the energy dissipated in the non-equilibrium transition from K to P

space X is two-dimensional. Because there is a one-to-one correspondence between σ and u', the configuration space can be identified with the first quadrant of the $(\sigma, [u])$ plane in Fig. 2b.

For Barenblatt's energy there can be only two types of metastable branches, with $N = 0$ and with $N = 1$. The corresponding response curves $\sigma = \tilde{\sigma}(\beta)$ and $E = \tilde{E}(\beta)$ are shown in Fig. 3. They reveal two fundamental advantages with respect to Griffith's model. The first is that now there is a prediction for the fracture onset. Indeed, the ultimate bulk deformation β_c being finite because $\theta'(0+)$ is finite, the bar must be fractured when $\beta > \beta_c$. The second advantage is that the model predicts two qualitatively different responses depending on the length l of the bar. In this way, it captures the *size effect* which is a typical feature of fracture.

To see this, let us look at the response curve $\sigma = \tilde{\sigma}(\beta)$. It consists of two branches, with $N = 0$ and $N = 1$, respectively. Since both w' and θ' are strictly monotonic, the branch $N = 1$ is given by

$$\beta = u' + l^{-1}[u] = (w')^{-1}(\sigma) + l^{-1}(\theta')^{-1}(\sigma). \tag{4.6}$$

Of the two terms on the right-hand side, the first determines the branch $N = 0$. The two possible shapes of the branch $N = 1$, one for *small* (curve S) and one for *large* (curve L) values of l, are shown in Fig. 3. They are determined by the sign of the slope of the response curve at its endpoint K, which is negative for S and positive for L. From Eq. (3.49) it follows that

$$\frac{d}{d\beta}\tilde{\sigma}(\beta_c) = l\frac{Q_0 Q_1}{Q_0 + Q_1} = l\frac{w''(\beta_c)\theta''(0+)}{w''(\beta_c) + l\theta''(0+)}, \tag{4.7}$$

where $\theta''(0+)$ is the limit of $\theta''([u])$ when $[u] \to 0+$. Then l is *small* if $d\tilde{\sigma}(\beta_c)/d\beta < 0$, that is, if

$$l < -\frac{w''(\beta_c)}{\theta''(0+)}, \tag{4.8}$$

and *large* otherwise. From the figure we see that the curve S has a negative slope for all β, while for the curve L the slope starts positive at K and then turns to negative at M. According to the conclusions in Sect. 3.4 the whole curve S is metastable, while for the curve L only the part with a negative slope is metastable.

We observe that for small l there is a single σ associated with each β. One may distinguish an elastic regime $\beta < \beta_c$, in which there are no jumps and the force grows up to the ultimate value σ_c, and a fractured regime $\beta > \beta_c$, in which there is one jump and the force decreases smoothly. On the contrary, for large l there is an interval (β_M, β_c) at which both regimes are possible. The two regimes are far from each other in the configuration space, and are separated by an energy barrier.

Under growing β the bar is forced to follow the branch $N = 0$ up to the endpoint K at which $\beta = \beta_c$. At this point, a non-equilibrium transition of the type described in Sect. 3.5 occurs, along the non-equilibrium path represented by the vertical segment (KP) in the figure. The dissipation involved in this transition is equal to the difference of the energies E_K and E_P in Fig. 3b, and is represented by the gray area in Fig. 3a. In the alternative behaviors exhibited by the two curves S and L one may recognize the qualitative features of *ductile fracture* and of *brittle fracture*, respectively.

By Proposition A.2 in the Appendix, the existence of global minimizers is guaranteed when θ is strictly concave, since strict concavity implies strict subadditivity. Then for small l the global minimizer coincides with the unique local minimizer, while for large l it coincides with the local minimizer of smallest energy.

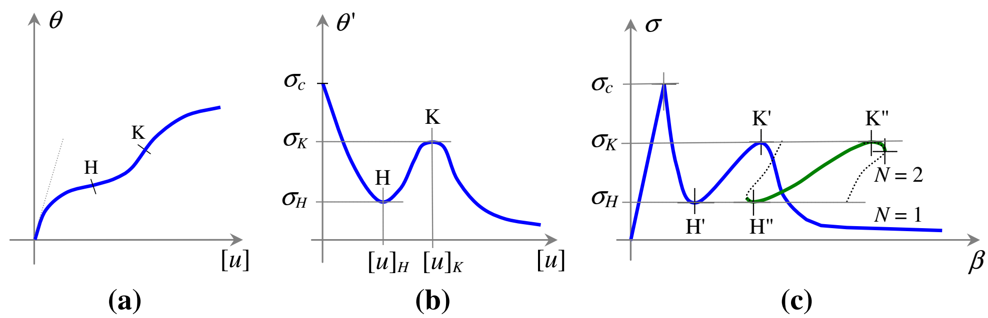

Fig. 4 Bi-modal energy **(a)**, its derivative **(b)**, and construction of the response curves $\sigma = \tilde{\sigma}(\beta)$ for $N = 1$ and $N = 2$ **(c)**. In **(c)**, the *solid lines* denote metastable branches and the *dashed lines* denote unstable branches

From the response curve in Fig. 3b we see that there is a value β_T of β at which the two local minimizers have the same energy. Just as explained in Griffith's case, this is not the value of β at which the transition from $N = 0$ to $N = 1$ occurs, because there are energy barriers opposing this transition.

The present example reveals some unrealistic features of the response of the model at unloading. At small l the response is totally reversible, in the sense that the curves $N = 0$ and $N = 1$ can be traversed back and forth. This implies fracture healing, a phenomenon rarely observed at the macroscopic level, even though it may be relevant for micro-fabricated adhesives or biological systems. For large l, when unloading from $\beta > \beta_c$ the system follows backwards the curve $N = 1$ up to its endpoint M. At this point a new non-equilibrium process occurs, which carries the system back to the curve $N = 0$. The energy $E_M - E_Q$ dissipated in this transition is shown in Fig. 3b. The concomitant upward jump in stress shown in Fig. 3a is not usually observed in macroscopic experiments.

4.3 Bi-modal energies

A *bi-modal* energy, introduced by Del Piero and Truskinovsky [23,24], is a function θ with two inflection points H and K separating a central convex branch from two lateral concave branches, as shown in Fig. 4a. From Fig. 4b we see that the derivative θ' has a central ascending branch and two lateral descending branches, with a local minimum σ_H at $[u]_H$ and a local maximum σ_K at $[u]_K$. We assume that

$$0 < \sigma_H < \sigma_K < \sigma_c < +\infty. \tag{4.9}$$

Due to the presence of the first concave branch, θ is initially strictly subadditive. Then by Proposition A.2 in the Appendix, global minimizers do exist for all $\beta > 0$. According to the stability analysis made in Sect. 3.3, a metastable equilibrium configuration may have an arbitrary number of jumps located on the ascending branch (HK) of the curve θ', but at most one jump located outside.

The response curves can be constructed in the same way as for Barenblatt's model. The construction of the branches $N = 1$ and $N = 2$ of the response curve $\sigma = \tilde{\sigma}(\beta)$ is shown in Fig. 4c. In the branch $N = 1$ one can distinguish three parts, separated by the points H′, K′, and corresponding to the location of the jump in the three branches of the curve θ'. In the curve for $N = 2$ there are also three parts, separated by the points H″, K″, since one of the jumps may be located on any of the three branches of the curve θ', while the other one can only be located on the ascending branch. According to the stability conditions given in Proposition 3.10, the central branches (H′K′) and (H″K″) are metastable, while the side branches are metastable only in their parts with a negative slope.

Just as for Barenblatt's energy, the shape of the response curves depends on the length l of the bar. In the example in Fig. 4, l has been chosen small enough to ensure the metastability of the whole branch $N = 1$, while the branch $N = 2$ has unstable parts. Under increasing elongation the bar follows the branch $N = 1$, because the transition to larger N is forbidden by energy barriers, just as in the case of Barenblatt's energy. In the situation shown in the figure the bar exhibits a *ductile* behavior, characterized by a smooth transition between the unfractured phase $N = 0$ and the fractured phase $N = 1$.

For large l we have the response curves shown in Fig. 5. Under increasing β, after attaining a peak at σ_c the axial force oscillates between the values σ_H and σ_K, with a downward jump at the creation of each new jump point. The decrease in the slope of the curves with increasing N is a sign of the weakening of the bar.

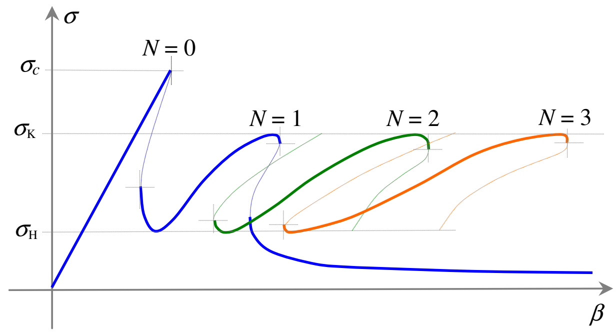

Fig. 5 Bi-modal cohesive energy. Response curves $\sigma = \tilde{\sigma}(\beta)$ for $N = 1, 2, 3$ for *large l*. Thick lines denote metastable branches, and thin lines denote unstable branches

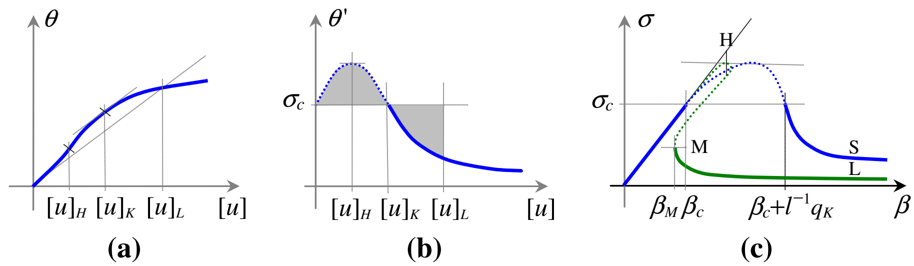

Fig. 6 Convex-concave energy **(a)**, its derivative **(b)**, and metastable equilibrium paths for small (S) and for large (L) values of l **(c)**. Thick lines denote metastable branches, and thin lines denote either non-equilibrium or unstable branches

Thus, a bi-modal cohesive energy reproduces the qualitative features of a process of *damage*, with an initial elastic phase followed by a progressive weakening process.

4.4 Convex-concave energies

As a final example we consider a cohesive energy strictly convex in a right neighborhood $(0,[u]_H)$ of the origin, and concave outside (e.g. [6,7,19,20,24]). An energy of this type is shown in Fig. 6a, where $[u]_K$ and $[u]_L$ are the values of $[u]$ at which $\theta'([u]_K) = \sigma_c$ and $\theta([u]_L) = \sigma_c[u]_L$, respectively. From the graph of θ' in Fig. 6b we see that the equilibrium condition $\theta'([u]) \leq \sigma_c$ is violated by all $[u]$ in $(0, [u]_K)$. Therefore, in an equilibrium configuration with jumps all jumps are larger than $[u]_K$. Moreover, because θ'' is negative for all $[u] > [u]_K$, from the stability analysis in Sect. 3.2 it follows that a metastable configuration may have at most one jump. Therefore, there may be only two types of metastable branches, one with $N = 0$ and one with $N = 1$.

The response curves $\sigma = \tilde{\sigma}(\beta)$ can be constructed in the same way as in the preceding examples. To show their dependence upon the size of the bar, two curves for $N = 1$, one for large and one for small l, are drawn in Fig. 6c. We see that none of them is accessible from the branch $N = 0$. Indeed, for large l, just as in the case of Barenblatt's energy, the branch $N = 0$ and the metastable part of the branch $N = 1$ are separated by an energy barrier. But here, unlike in Barenblatt's case, the barrier does not disappear at the critical point H $= (\beta_c, \sigma_c)$.

For small l, the two branches are separated by an interval $(\beta_c, \beta_c + l^{-1}[u]_K)$ in which there are no metastable equilibrium configurations. This is precisely the situation of non-existence of local minimizers considered in Proposition 3.8. We also recall that, by Proposition A.4 in the Appendix, there are no global minimizers when $\beta_c l < [u]_H$ and $\beta \in (\beta_c, l^{-1}[u]_H)$.

In non-existence cases like this one, it seems reasonable to look whether the energy is bounded from below and, in the affirmative, whether the infimum can be attained at some *generalized solution*, obtained by suitably extending the domain $V(0, l)$ of the functional. For this purpose, let us evaluate the infimum of E when $l < [u]_L/\beta_c$ and $\beta \in (\beta_c, l^{-1}[u]_L)$. By the boundary condition $(3.7)_2$, for $\beta < l^{-1}[u]_L$ the jump amplitudes

$[u]_h$ cannot be larger than $[u]_L$. Then $\theta([u]_h) > \theta'(0+)[u]_h$, and from the convexity of w it follows that

$$\begin{aligned} E(u) &= lw\left(\beta - l^{-1}\sum_{S(u)}[u]_h\right) + \sum_{S(u)}\theta([u]_h) \\ &\geq lw(\beta_c) + lw'(\beta_c)\left(\beta - l^{-1}\sum_{S(u)}[u]_h - \beta_c\right) + \theta'(0+)\sum_{S(u)}[u]_h. \end{aligned} \tag{4.10}$$

Because $w'(\beta_c) = \theta'(0+) = \sigma_c$, we get the lower bound

$$E(u) \geq lw(\beta_c) + l\sigma_c(\beta - \beta_c) =: \varphi_\beta. \tag{4.11}$$

Take the sequence $N \mapsto u^N$ in which $(u^N)' = \beta_c$ for all N, and each u^N has N jumps of equal amplitude

$$[u^N]_h = \frac{l}{N}(\beta - \beta_c). \tag{4.12}$$

The corresponding energies

$$E(u^N) = lw(\beta_c) + N\theta\left(\frac{l}{N}(\beta - \beta_c)\right) \tag{4.13}$$

converge to φ_β when $N \to \infty$. Therefore, φ_β is the infimum for E, and $N \mapsto u^N$ is a minimizing sequence.

The nature of the generalized solution to be associated with φ_β depends on the locations of the jump points in the approximating sequence. These locations, however, are not determined by the model due to the neglect of the interaction between the cracks. Without going deeper into the omitted physics of the problem we can still consider two interesting possibilities.

One for instance may place the jump points at equal distances

$$x_h^N = \left(h - \frac{1}{2}\right)\frac{l}{N}, \quad h \in \{1, 2, \ldots, N\}, \tag{4.14}$$

so that

$$u^N(x) = \beta_c x + (\beta - \beta_c)\frac{l}{N}\sum_{h=1}^{N} H(x - x_h^N), \tag{4.15}$$

where H is the step function with values $H(\xi) = 0$ if $\xi < 0$ and $H(\xi) = 1$ if $\xi > 0$. With this choice, the sequence $N \mapsto u^N$ converges uniformly to the function $u(x) = \beta x$. This is the macroscopic deformation to be associated with the generalized solution. In it, the jumps are not macroscopically visible, but their presence is revealed by the difference between the macroscopic elongation βl and the elongation $\beta_c l$ due to the bulk part of the deformation. Therefore, the generalized solution can be regarded as a *deformation with microstructure*. Its description requires two functions, the macroscopic deformation u and the bulk deformation g which, in the present example, is equal to β_c. In [22], the pair (u, g) was called a *structured deformation*, and the inequality $u' \neq g$ was regarded as the sign of the presence of microstructure.

Another possible location for the jump points is obtained by setting $N' = 2^N$ and taking the jump sets

$$\begin{aligned} &\{l\} \text{ for } N' = 1, \quad \left\{\frac{1}{3}l, l\right\} \text{ for } N' = 2, \quad \left\{\frac{1}{9}l, \frac{1}{3}l, \frac{7}{9}l, l\right\} \text{ for } N' = 3, \\ &\left\{\frac{1}{27}l, \frac{1}{9}l, \frac{7}{27}l, \frac{1}{3}l, \frac{19}{27}l, \frac{7}{9}l, \frac{25}{27}l, l\right\} \text{ for } N' = 4, \ldots \end{aligned} \tag{4.16}$$

In this way one gets a sequence $N' \mapsto u^{N'}$ converging uniformly to

$$u_C(x) := \beta_c x + (\beta - \beta_c)lC(x), \tag{4.17}$$

where C is the Cantor function.

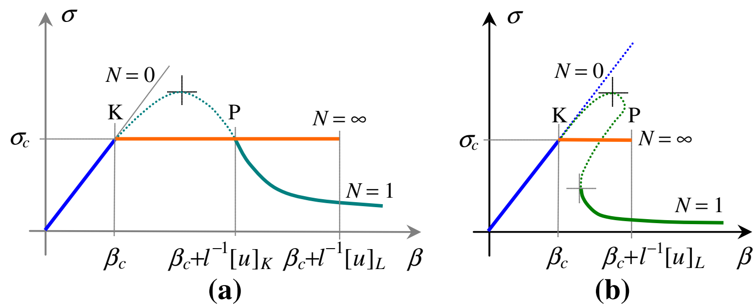

Fig. 7 Convex-concave energy. Response curves $\sigma = \tilde{\sigma}(\beta)$ for small **(a)** and for large **(b)** values of l

A difference between the choices (4.14) and (4.16) for the jump sets is that, while the first determines a generalized solution described by a pair (u, g) of functions, the generalized solution determined by the second is fully described by the function u_C alone. This can be explained using the language of measure theory, see, e.g., [1]. Let us consider the elements u^N of the approximating sequence as elements of $\mathrm{L}^1(0, l)$ whose derivatives are measures. Then $(u^N)'$ is the absolutely continuous part of the distributional derivative Du^N, and the jumps are the singular part, which is concentrated over a finite set. With the choice (4.16), when $N' \to \infty$ the absolutely continuous parts converge to zero, and the singular parts converge to a singular measure which acts on a set, the Cantor set, which is neither finite nor countable. Therefore, in the limit element the singular part of the measure transforms from a *jump part* to a *Cantor part*. With the choice (4.14), the absolutely continuous parts of the derivatives Du^N still converge to zero, and the jumps disappear in the limit. As $N' \to \infty$, the jump parts are again replaced by a singular measure, whose description requires now the introduction of a second function g.

If we extend the domain of the functional E to include functions whose derivatives have a Cantor part, we have to consider segments of the response curves made of generalized configurations. In our example, generalized configurations occur for β in $(\beta_c, \beta_c + l^{-1}[u]_L)$; their bulk deformation is equal to β_c for all β, and the axial force is equal to σ_c. The corresponding equilibrium curve is the horizontal line $N = \infty$ in Fig. 7a,b, for the two cases of small and large l, respectively.

It seems quite natural to interpret the generalized configurations in terms of plastic response, with β_c and $\beta - \beta_c$ playing the role of the elastic and plastic parts of the total deformation β. A comparison of Fig. 7a, b also provides an illustration of how the *ductility* of the response, that is, the range of the elongations covered by the generalized solution, can be influenced by the size of the bar. However, the elastic model does not provide a complete description of plastic behavior, since it does not include a correct description of the response at unloading.

5 The inelastic model

The elastic model described so far predicts a non-dissipative evolution along metastable equilibrium branches, together with dissipative non-equilibrium transitions between metastable branches. The examples given in the preceding sections show that this model describes quite satisfactorily a process of loading, in which the prescribed elongation βl grows monotonically. Less satisfactory is the response at unloading. Indeed, the elastic model predicts that some segments of the loading path be traversed backwards at unloading, while the experiments generally show completely different responses at loading and at unloading. The inadequacy of the elastic model to describing the material response at unloading is the main motivation for introducing the *inelastic model* which we discuss below.

5.1 The basic assumption

The basic assumption in the inelastic model is that the cohesive energy required to produce a jump opening $[u]$, instead of being recoverable as in the elastic model, is totally dissipated. In the case of a bar under tension, this corresponds to replacing the cohesive energy $\theta([u])$ by a rate-independent *dissipation potential* $D([u], [\dot{u}])$, where the dot denotes time derivative. To maintain a close link between our elastic and inelastic models, we assume that

$$D([u], [\dot{u}]) = \theta'([u])[\dot{u}]. \tag{5.1}$$

The linear dependence on $[\dot{u}]$ characterizes the material as *rate independent*. Then $\theta'([u])[\dot{u}]$ is also the expression of the *dissipation rate*, and $\theta'([u])$ is the *generalized force* associated with the jump $[u]$, see e.g. [30].

From time integration of (5.1) it follows that, to within an additive constant,

$$\int_0^t \theta'([u](x,s)[\dot{u}](x,s)ds = \theta([u](x,t)), \tag{5.2}$$

so that $\theta([u](x))$ is the total *dissipation* experienced at the jump point x by the time t in the past history of $[u](x)$, starting from $[u](x) = 0$. We see that, for this particular model, the total dissipation at x depends on history in a very special way, since it is a function of the current value $[u](x,t)$ of the jump opening.

With this assumption, the jump openings $[u]$ acquire the status of *state variables*. They are subject to the incremental restriction

$$\theta'([u])[\dot{u}] \geq 0, \tag{5.3}$$

which expresses the non-negativeness of the dissipation rate. Their evolution in time is governed by evolution equations of the incremental type. The solution of these equations can be obtained through incremental minimization of the *total energy*, which we define as the sum of the stored elastic strain energy plus the dissipated energy, see e.g., [26,41,42].

In our model, the total energy has the same expression as the elastic strain energy in the elastic model

$$E(u) = \int_0^l w(u'(x))dx + \sum_{S(u)} \theta([u](x)), \tag{5.4}$$

and is defined over the same domain $V(0,l)$. However, the second term on the right-hand side must now obey the dissipation inequality (5.3). The optimization procedure is described in Sect. 5.4. It requires a preliminary discussion of the changes brought by the introduction of the dissipation inequality in the definitions of the equilibrium configurations and of metastability.

5.2 Equilibrium configurations

In the inelastic model, an equilibrium configuration it is still defined by the inequality

$$\lim_{\lambda \to 0+} \frac{1}{\lambda}(E(u+\lambda\eta) - E(u)) \geq 0 \tag{5.5}$$

as in the elastic model. But now this inequality does not hold for all perturbations η which leave unchanged the total length of the bar, but only for those which are compatible with the dissipation inequality (5.3):

$$\theta'([u])[\eta] \geq 0. \tag{5.6}$$

In what follows we assume that θ is strictly increasing, so that the preceding inequality reduces to

$$[\eta] \geq 0. \tag{5.7}$$

Since the increments $[\dot{u}]$ are particular perturbations, the reduced inequality implies that the jump amplitudes can never decrease. Though this does not always agree with experimental observations, we first analyze this case for reasons of simplicity. Only later, in Sect. 5.7, we shall remove this restriction by introducing the case of complete fracture. Other dissipation mechanisms, with different assumptions on the evolution of the state variables, have been considered in the literature, see [10,17,33,47].

The main effect of inequality (5.7) is that of drastically increasing the number of the equilibrium configurations.

Proposition 5.1 *Let w and θ be as in Proposition 3.1. Then a displacement field $u \in V(0,l)$ is an equilibrium configuration for the inelastic model if and only if there is a non-negative constant u' which satisfies the equilibrium conditions (3.3), (3.4) plus the inequality*

$$\theta'([u](x)) \geq w'(u') \quad \text{for all} \quad x \in S(u). \tag{5.8}$$

Proof The proof of conditions (3.3) and (3.4) is the same as in Proposition 3.1. Where the proof differs is in the consideration of a perturbation η with constant derivative η' and with a single jump $[\eta]$ located at a point x_o of $S(u)$. Indeed, due to inequality (5.7), condition (3.9) now yields inequality (5.8) instead of Eq. (3.5). □

Recalling the definition of the configuration space X given in Sect. 3.4, we see that the equilibrium configurations for the inelastic model are the elements u of X which satisfy the inequalities $(3.3)_2$ and (5.8). For convenience, we collect them in the single inequality

$$w'(u') \leq \min\{\theta'(0+), \theta'([u](x)) | x \in S(u)\}. \tag{5.9}$$

The set of all functions in X which satisfy this inequality will be denoted by Ξ and will be called the *equilibrium set* for the inelastic model. It is clear that all u which are equilibrium configurations for the elastic model are equilibrium configurations for the inelastic model. Moreover, they are *boundary points* for Ξ, since for them inequality (5.9) is satisfied as an equality. Therefore, the set Ξ has internal points, and the equilibrium set of the elastic model is included in the boundary of Ξ.

5.3 Metastable configurations

In the inelastic model we keep the definition of a metastable configuration as a local minimizer for E with respect to the norm (3.17). The fact that, due to the dissipation inequality, the class of perturbations is smaller for the new model, has the consequence that all metastable configurations for the elastic model are metastable for the inelastic model. For the new equilibrium configurations not on the boundary of Ξ, there is a quite general sufficient condition for metastability.

Proposition 5.2 *Let u be an equilibrium configuration for which the number of the jump amplitudes is finite and inequality (5.9) is strict. Then u is metastable.*

Proof By the convexity of w and by the boundary condition (3.2),

$$\begin{aligned} E(u+\eta) - E(u) &= \int_0^l (w(u' + \eta'(x) - w(u'))dx + \sum_{S(\eta)} (\theta([u]_i + [\eta]_i) - \theta([u]_i)) \\ &\geq \sum_{S(\eta)} (\theta([u]_i + [\eta]_i) - \theta([u]_i) - w'(u')[\eta]_i), \end{aligned} \tag{5.10}$$

with $[u]_i > 0$ in $S(\eta) \cap S(u)$ and $[u]_i = 0$ in $S(\eta) \backslash S(u)$. If inequality (5.9) is strict, for each such $[u]_i$ there is a $\delta_i > 0$ such that

$$\theta([u]_i + [\eta]_i) - \theta([u]_i > w'(u')[\eta]_i \quad \forall \eta_i \in (0, \delta_i). \tag{5.11}$$

By inequality (5.7), all $[\eta]_i$ in $S(\eta)$ are positive. Moreover, the number of the δ_i is finite, because the number of the $[u]_i$ is finite by assumption. Then the right-hand side of (5.10) is positive for all η with $||\eta||$ smaller than the smallest of the δ_i. Therefore, u is a local minimizer for E. □

For configurations on the boundary of Ξ, for which (5.9) holds as an equality, the metastability of a configuration is again decided by the Hessian matrix (3.37). But, due to inequality (5.7), the matrix now operates on jump vectors $[\eta]_i$ with non-negative components. Consequently, the conditions for metastability are weaker than those given in Sect. 3.3. More precisely, a necessary condition for metastability is that the Hessian matrix be *copositive*, that is,

$$\sum_{i,j=1}^{N} H_{ij}(u)[\eta]_i[\eta]_j \geq 0 \quad \forall \, [\eta]_i \geq 0, \tag{5.12}$$

and a sufficient condition is that the same matrix be *copositive plus*, that is, that the above inequality holds as an equality only if all $[\eta]_i$ are zero.

For the matrix (3.37), the following characterizations of copositiveness hold.

Proposition 5.3 *If $Q_0 > 0$, the matrix $H_{ij}(u)$ is copositive if and only if*

$$Q_0 + Q_i \geq 0 \quad \forall i \in \{1, \ldots, N\}, \tag{5.13}$$

and is copositive plus if and only if the above inequality is strict for all i.

Proof Necessity follows from inequality (5.12) written with all $[\eta]_i = 0$ except one. Sufficiency follows from the inequality

$$\sum_{i,j=1}^{N} H_{ij}(u)[\eta]_i[\eta]_j = \sum_{i=1}^{N}(Q_0 + Q_i)[\eta]_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} Q_0[\eta]_i[\eta]_j \geq \sum_{i=1}^{N}(Q_0 + Q_i)[\eta]_i^2, \tag{5.14}$$

which holds for all $[\eta]_i \geq 0$.

Notice that, according to the above conditions, a metastable configuration for the inelastic model may contain any number of jumps with negative Q_i.

5.4 The evolution equation for the jump openings

In Sect. 3.4 it has been proved that the equilibrium configurations for the elastic model form curves in the configuration space X, and that, with the exception of some singular points, only one of such curves passes through a given equilibrium configuration. The situation is completely different in the inelastic model, in which the equilibrium set Ξ has interior points. It is then necessary to introduce a selection principle to determine the trajectory in the configuration space corresponding to a given evolution of the loading parameter.

This selection procedure is based on the minimization of the increment of the total energy (5.4). Given a loading program $t \mapsto \beta_t$, in which the loading parameter β is a given continuous function of time, and given the initial deformation u_0 at $t = 0$, the corresponding deformation process $t \mapsto u_t$ has the property that, for each t, the deformation u_t is a local minimizer of the total energy among configurations with total elongation $l\beta_t$. The minimization procedure is subject to the following conditions:

(i) the condition of finite energy $[u_t]_i > 0$ for all $x_i \in S(u_t)$,
(ii) the dissipation inequality $[u_\tau]_i \geq [u_t]_i$ for all $x_i \in S(u_t)$ and for all $\tau > t$,
(iii) the incremental boundary condition $l\dot{u}_t' + \sum_{S(u_t)}[\dot{u}_t]_i = l\dot{\beta}_t$ for all $t \in [0, \tau]$,
(iv) the initial conditions $u_t = u_0$, $\dot{u}(t) = \dot{u}(0)$ for $t = 0$,
(v) the metastability condition $E(u_t) \leq E(u_t + \eta_t)$ for all perturbations η_t with $||\eta_t|| < \delta$ for some $\delta > 0$, which leave unchanged the length of the bar

$$\int_0^l \eta_t'(x)dx + \sum_{S(\eta_t)}[\eta_t]_i = 0, \tag{5.15}$$

and which satisfy the inequality

$$[\eta_t]_i > 0 \quad \forall x_i \in S(\eta_t). \tag{5.16}$$

We remark that the condition (v) of local minimum makes the difference with the models based on global minima, as in [27]. Local minimization was considered in a different setting by [17,25,36,37,46,54].

To determine the process $t \mapsto u_t$ we use an incremental technique. Starting from the initial configuration u_0, we first determine the configuration u_τ for a conveniently small τ. Then we repeat the procedure for any $t > 0$, to determine $u_{t+\tau}$ starting from u_t. For the first step, we introduce the expansion

$$u_t(x) = u_0(x) + t\dot{u}_0(x) + o(t), \quad 0 < t < \tau. \tag{5.17}$$

The corresponding expression for the energy is

$$E(u_t) = E(u_0) + t\dot{E}(u_0) + o(t), \tag{5.18}$$

with

$$\dot{E}(u_0) = l w'(u_0')\dot{u}_0' + \sum_{S(u_0+)} \theta'([u_0]_i)\,[\dot{u}_0]_i. \tag{5.19}$$

Note that the jump set $S(u_0+)$ not only includes the pre-existing jump points at the instant $t = 0$, but also those which are created immediately after. For the newly created jumps, $\theta'([u_0]_i)$ is equal to $\theta'(0+)$.

A necessary condition for a local minimum at the instants following $t = 0$ is that $\dot{u}_0$ be a minimizer for $\dot{E}(u_0)$. Using the incremental boundary condition (iii), we can rewrite (5.19) as

$$\dot{E}(u_0) = l w'(u_0')\dot{\beta}_0 + \sum_{S(u_0+)} (\theta'([u_0]_i) - w'(u_0'))[\dot{u}_0]_i. \tag{5.20}$$

where $\dot{\beta}_0 = d\beta(0)/dt$. Assume first that u_0 is an interior point of Ξ. Then inequality (5.9) is strict, and therefore all coefficients $(\theta'([u_0]_i) - w'(u_0'))$ are positive. Then, because all $[\dot{u}_0]_i$ are non-negative by the dissipation inequality, the minimum is attained at $[\dot{u}_0]_i = 0$. This means that the evolution is purely elastic, that is, without dissipation. Thus, we have proved that

Proposition 5.4 *If u_0 is an equilibrium configuration for which inequality (5.9) is strict, then in any process from u_0 the evolution is purely elastic for sufficiently small τ.*

If $\theta'([u_0]_i) = w'(u_0')$ for some i, from the minimization of (5.20) it follows that only for such i a positive $[\dot{u}_0]_i$ is possible. In particular, if we apply this conclusion to points at which $[u_0]_i = 0$, we have that

Proposition 5.5 *If u_0 is an equilibrium configuration for which inequality (5.9) holds as an equality, then a new jump can be created in a process from u_0 only if $w'(u_0') = \theta'(0+)$.*

The same conclusion was obtained in Proposition 3.10 for the equilibrium curves of the elastic model. Now we see that, whether inequality (5.9) is strict or not all terms in the functional (5.20) involving the $[\dot{u}_0]_i$ vanish, because either $[\dot{u}_0]_i$ or its coefficient is zero. Consequently, the $[\dot{u}_0]_i$ cannot be determined by minimizing $\dot{E}(u_0)$. It is then necessary to look at a second order necessary condition for a minimum, based on the second term in the expansion of $E(u_t)$. Notice the difference with the elastic case, in which the incremental extension of the equilibrium branch was unique, and could be obtained by differentiating the equilibrium equations.

5.5 Second-order minimization

Let us add a second-order term to the expansion (5.16) of u,

$$u_t(x) = u_0(x) + t\dot{u}_0(x) + \frac{1}{2}t^2\ddot{u}_0(x) + o(t^2), \quad 0 < t < \tau, \tag{5.21}$$

and, consequently, to the expansion (5.18) of the energy,

$$E(u_t) = E(u_0) + t\dot{E}(u_0) + \frac{1}{2}t^2\ddot{E}(u_0) + o(t^2). \tag{5.22}$$

Here $\dot{E}(u_0)$ is as in (5.20), and

$$\ddot{E}(u_0) = l w'(u_0')\ddot{u}_0' + l w''(u_0')\dot{u}_0'^2 + \sum_{S(u_0+)} \theta'([u_0]_i)\,[\ddot{u}_0]_i + \sum_{S(u_0+)} \theta''([u_0]_i)\,[\dot{u}_0]_i^2. \tag{5.23}$$

Due to the boundary condition (iii), the last functional can be rewritten in the form

$$\ddot{E}(u_0) = l w'(u_0')\ddot{\beta}_0 + w''(u_0')\left(l\dot{\beta}_0^2 - 2\dot{\beta}_0 \sum_{S(u_0+)} [\dot{u}_0]_i + \frac{1}{l}\left(\sum_{S(u_0+)} [\dot{u}_0]_i\right)^2\right)$$
$$+ \sum_{S(u_0+)} (\theta'([u_0]_i) - w'(u_0'))[\ddot{u}_0]_i + \sum_{S(u_0+)} \theta''([u_0]_i)[\dot{u}_0]_i^2. \tag{5.24}$$

We recall that, for each i, either $\theta'([u_0]_i)-w'(u_0')$ or $[\dot{u}_0]_i$ is zero. But when $[\dot{u}_0]_i = 0$ the dissipation inequality requires that $[\ddot{u}_0]_i \geq 0$. Then $[\ddot{u}_0]_i = 0$ is the value which minimizes the term $(\theta'([u_0]_i)-w'(u_0'))[\ddot{u}_0]_i$. Therefore, if we neglect the first two terms on the right, which are given, the problem reduces to the minimization of the following quadratic functional

$$I([\dot{u}_0]_i) := w''(u_0')\left(-2\dot{\beta}_0 \sum_{S(u_{0//})} [\dot{u}_0]_i + \frac{1}{l}\left(\sum_{S(u_{0//})} [\dot{u}_0]_i\right)^2\right) + \sum_{S(u_{0//})} \theta''([u_0]_i)[\dot{u}_0]_i^2, \tag{5.25}$$

where $S(u_{0//})$ denotes the set of all jump points of u_0 at which $\theta'([u_0]_i) = w'(u_0')$. In the quadratic part of this functional we recognize the Hessian matrix $H_{ij}(u_0)$. Writing again Q_0 and Q_i instead of $l^{-1}w''(u_0')$ and $\theta''([u_0]_i)$, and imposing the dissipation condition $[\dot{u}_0]_i \geq 0$ we get the minimum problem for the functional

$$I([\dot{u}_0]_i) = \begin{cases} \sum_{i,j=1}^{N} H_{ij}(u_0)[\dot{u}_0]_i[\dot{u}_0]_j - 2l\dot{\beta}_0\, Q_0 \sum_{i=1}^{N} [\dot{u}_0]_i & \text{if } [\dot{u}_0]_i \geq 0 \text{ for all } i, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.26}$$

The Kuhn–Tucker conditions for this constrained problem form the following linear complementarity problem: find two N-vectors $[\dot{u}_0]_i$, z_i such that

$$\sum_{j=1}^{N} H_{ij}(u_0)[\dot{u}_0]_j - l\dot{\beta}_0 Q_0 = z_i, \quad [\dot{u}_0]_i \geq 0, \quad z_i \geq 0, \quad [\dot{u}_0]_i z_i = 0. \tag{5.27}$$

Due to the special form (3.37) of the matrix $H_{ij}(u_0)$, the problem can be solved explicitly in the case of $Q_0 > 0$ and $H_{ij}(u_0)$ copositive plus. Consider first the case of unloading, $\dot{\beta}_0 \leq 0$. Then $I([\dot{u}_0]_i) \geq 0$ for all vectors $[\dot{u}_0]_i$ and the minimum $I([\dot{u}_0]_i) = 0$ is attained at $[\dot{u}_0]_i = 0$. Thus, at unloading there is no jump opening, and therefore no dissipation. So, we have obtained the law of *elastic unloading*:

Proposition 5.6 *Let u_0 be an equilibrium configuration for which inequality (5.9) holds as an equality. Then in any process of unloading from u_0 the evolution is purely elastic.*

In the case of loading, $\dot{\beta}_0 > 0$, we have the following result.

Proposition 5.7 *Let u_0 be an equilibrium configuration for which inequality (5.9) holds as an equality, and for which the Hessian matrix is copositive plus. If $\dot{\beta}_0 > 0$, then*

(i) *If the Q_i are either all positive or all negative, then*

$$[\dot{u}_0]_i = \frac{\frac{1}{Q_i}}{\frac{1}{Q_0} + \sum_{j=1}^{N} \frac{1}{Q_j}}\; l\dot{\beta}_0. \tag{5.28}$$

(ii) *If M of the Q_i are negative and $N-M$ are positive, then $[\dot{u}_0]_i = 0$ for all i for which $Q_i > 0$. For all i for which $Q_i < 0$, $[\dot{u}_0]_i$ is given again by (5.28), but with the sum restricted to the negative Q_j.*

Proof If $H_{ij}(u_0)$ is copositive plus, the functional $I([\dot{u}_0]_i)$ is strictly convex, and therefore it has a unique minimizer. Then if one finds a $[\dot{u}_0]_i$ which satisfies all conditions of problem (5.27), this is the solution. Rewrite the problem in the form

$$Q_i[\dot{u}_0]_i + Q_0 \sum_{h=1}^{N} [\dot{u}_0]_h - l\dot{\beta}_0 Q_0 = z_i, \quad [\dot{u}_0]_i \geq 0, \quad z_i \geq 0, \quad [\dot{u}_0]_i z_i = 0. \tag{5.29}$$

If all Q_i are positive, take all $[\dot{u}_0]_i$ positive and, consequently, all z_i equal to zero. Then rewriting the above equation for $j \neq i$ and subtracting, we find $Q_j[\dot{u}_0]_j = Q_i[\dot{u}_0]_i$ and, therefore,

$$[\dot{u}_0]_j = \frac{1}{Q_j} Q_i[\dot{u}_0]_i. \tag{5.30}$$

Substituting into (5.29) we find the $[\dot{u}_0]_i$ as in (5.28). Because all such $[\dot{u}_0]_i$ are positive and all z_i are zero, all conditions in (5.29) are satisfied and this is the solution of the problem. The same solution is obtained if all

Q_i are negative. Indeed, using the stability inequality (5.13) it is easy to see that if $H_{ij}(u_0)$ is copositive plus then the denominator in (5.28) is negative. Therefore, all $[\dot{u}_0]_i$ are positive.

Now consider the case in which M of the Q_i are negative and the remaining $N - M$ are positive. In this case the $[\dot{u}_0]_i$ cannot be all positive. Indeed, this would contradict inequality (5.30) when Q_i and Q_j have opposite signs. Then there are i, j, such that $[\dot{u}_0]_i > 0$ and $[\dot{u}_0]_j = 0$. Then $z_i = 0$ and

$$Q_i[\dot{u}_0]_i = -z_j, \tag{5.31}$$

as it follows from writing (5.29) for i and j and subtracting. Because both $[\dot{u}_0]_i$ and Q_i are non-zero, it follows that $z_j > 0$ and $Q_i < 0$. This proves that $[\dot{u}_0]_i > 0$ implies $Q_i < 0$ and, conversely, that $[\dot{u}_0]_i = 0$ for all i for which $Q_i > 0$.

As a consequence, the sum in (5.29) extends only to the h for which $Q_h < 0$. Then if we apply the first part of this Proposition to the subsystem of the M equations with negative Q_i we find the $[\dot{u}_0]_i$ as in (5.28), with the sum restricted to the negative Q_h. □

Remark 5.8 In summary, the minimization procedure yields the following results:

(i) all $[\dot{u}_0]_i$ are zero if u_0 is an interior point of Ξ and if u_0 is a boundary point and $\dot{\beta} < 0$,
(ii) If u_0 is a boundary point of Ξ, if $\dot{\beta} > 0$, and if all Q_i have the same sign, then all $[\dot{u}_0]_i$ are given by (5.28),
(iii) If u_0 is a boundary point of Ξ, if $\dot{\beta} > 0$, and if the Q_i have different signs, then $[\dot{u}_0]_i$ is zero for all i for which $Q_i > 0$, and is given by (5.28), with the sum restricted to the negative Q_i, for all i for which $Q_i < 0$.

From the incremental boundary condition we have

$$\dot{u}_0' = \dot{\beta}_0 - l^{-1} \sum_i [\dot{u}_0]_i, \tag{5.32}$$

and from the constitutive relation $\sigma = w'(u')$ we have

$$\dot{\sigma} = w''(u')\dot{u}' = Q_0 l \dot{u}'. \tag{5.33}$$

Then, from (5.28),

$$\dot{\sigma} = Q_0 l \left(1 + \sum_i \frac{Q_0}{Q_i}\right)^{-1} \dot{\beta}, \tag{5.34}$$

and, therefore,

$$\frac{d\sigma}{d\beta} = Q_0 l \left(1 + \sum_i \frac{Q_0}{Q_i}\right)^{-1}. \tag{5.35}$$

Equation (5.34) is formally identical to Eq. (3.50) of the elastic case. The difference is that this equation now applies as it is, that is, with the sum over all jump points, only to the second of the three cases listed above. In case (iii) the sum is restricted to those i for which $Q_i < 0$, and in case (i) it must be eliminated. The same holds for Eq. (5.35), which defines the response curve $\sigma = \sigma(\beta)$ for a prescribed elongation path $t \mapsto \beta(t)$.

Note that, while in the elastic model the response curve was the only possible equilibrium curve, here it is the curve selected, among an infinity of equilibrium curves, by the incremental minimization of the total energy. The coincidence of Eqs. (5.34) and (3.50) in case (ii) means that the minimizing curve in a process of loading starting from a boundary point of Ξ coincides with the unique equilibrium curve in the elastic model.

5.6 Non-equilibrium transitions

The non-equilibrium transitions in the inelastic model have the same nature as in the elastic model. We can again assume that the dynamic extension of the model is provided by conventional viscoelasticity with vanishing viscosity parameter. Here, however, due to the incremental minimization, the occurring of such transitions is associated with the loss of existence of solutions to the incremental problem (5.27). By the stability condition (5.13), this occurs when one of the sums $Q_0 + Q_i$ becomes equal to zero.

5.7 Complete fracture

A further step in the direction of a more realistic model is obtained by relaxing the assumption that the function θ is strictly increasing. Specifically, we assume that there is a critical value $q_c \geq 0$ of the jump opening, such that for all $[u] > q_c$ the function θ takes a constant value γ:

$$\theta([u]) = \gamma \quad \forall [u] > q_c. \tag{5.36}$$

Clearly this implies $\theta'([u]) = 0$ for all $[u] > q_c$. As a consequence, for $[u] > q_c$ the dissipation inequality (5.3) is satisfied without any restriction on the sign of $[\dot{u}]$. When, during a process of loading, one of the jump amplitudes $[u]_h$ reaches the critical value q_c, we say that the bar undergoes *complete fracture*. This occurs when $w'(u') = \theta'(q_c) = 0$ and, consequently, $u' = 0$ and $\sigma = 0$. That is, complete fracture occurs at zero bulk deformation and at zero axial force.

Experimental observations tell us that complete fracture determines a permanent loss of cohesion. This fact can be described by modifying the evolution law for u, by assuming that u', and, therefore, σ, remains equal to zero in any continuation of the process, both at loading and at unloading. The boundary condition (2.6) then gives

$$\sum_{S(u)} [\dot{u}](x) = \dot{\beta} l. \tag{5.37}$$

Except for this restriction, the evolution of the jump amplitudes in the regime of complete fracture is arbitrary.

6 The inelastic model: examples

For the cohesive energies considered in Sect. 4, we compare the responses of the inelastic model with those given by the elastic model.

6.1 Griffith's energy

For Griffith's energy (4.1), the assumption (5.36) of complete fracture is satisfied with $q_c = 0$. Accordingly, complete fracture is reached as soon as the first jump is created. For a completely fractured configuration one has both $u' = 0$ and $\theta'([u]_h) = 0$. Then the equilibrium condition (5.8) for the inelastic model is always satisfied as an equality just as in the elastic model. Therefore, in the special case of Griffith's energy the equilibrium configurations in the two models are the same. Moreover, because in the elastic model all equilibrium configurations are metastable, this remains true for the inelastic model.

There are again two types of response, one for $N = 0$ and one for $N > 0$, now corresponding to the unfractured regime and to the completely fractured regime, respectively. For $N = 0$ one has again the response curves (4.4), and for complete fracture one has again the response curves (4.5). Both can be traversed back and forth, so that there is no elastic unloading. Moreover, the curves $N > 0$ are inaccessible from the origin. Thus, in spite of the two different physical interpretations given to the function θ, for Griffith's energy there is no difference in the response predicted by the elastic and by the inelastic model.

In particular we note that, because $\theta'([u]_h) = 0$, inequality (5.6) does not imply (5.7). Then the jump amplitudes can increase and decrease without any constraint, as it is common experience to occur in a completely fractured body. We also note that in the special case of Griffith's energy the role of state variables is not played by the jump amplitudes, but more simply, by their number N. The corresponding evolution law is $\dot{N} \geq 0$. It expresses the impossibility for a fracture to heal.

6.2 Barenblatt's energy

For the inelastic model, the equilibrium configurations are the elements of the equilibrium set Ξ defined in Sect. 5.2. According to Proposition 5.2, if the number of the jump amplitudes $[u]_h$ is finite, all equilibrium configurations for which inequality (5.9) is strict are metastable. Moreover, by Proposition 5.3, if (5.9) is satisfied as an equality, then the inequality

$$l^{-1} w''(u') + \theta''([u]_h) \geq 0 \quad \forall h \in \{1, 2, \ldots, N\} \tag{6.1}$$

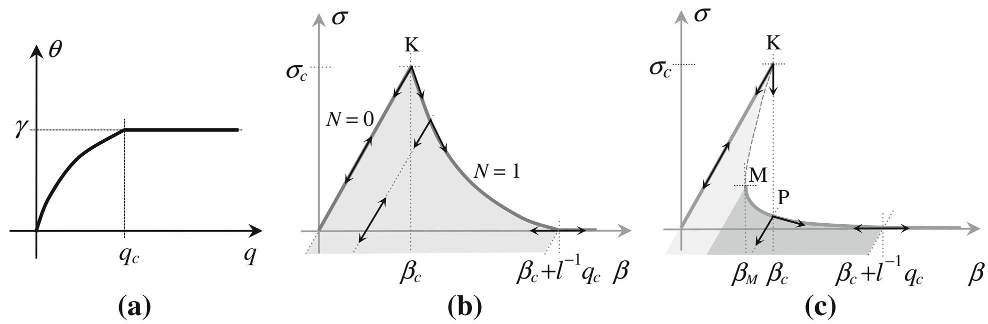

Fig. 8 Barenblatt's energy with complete fracture **(a)**. Force-elongation response curves for the dissipative model, for bars with *small* **(b)** and *large* **(c)** length. The gray areas represent the equilibrium configurations with a single jump. The light-gray area in **(c)** represents equilibrium configurations not accessible from the origin

is necessary, and the strict inequality is sufficient, for metastability. Inequality (6.1) replaces the stability condition (3.38) of the elastic model.

Let us study the evolution of u along a metastable path, under a loading program $t \mapsto \beta_t$. We slightly modify the graph of θ in Fig. 2a by assuming that there is a critical jump amplitude q_c for which the condition (5.36) of complete fracture is satisfied. The modified graph is shown in Fig. 8a. In Fig. 8b,c, the gray areas represent the equilibrium configurations with no more than one jump point, $N \leq 1$. The equilibrium configurations for the elastic model with $N \leq 1$ are represented by the boundaries of these areas, which we still call the curves $N = 0$ and $N = 1$. Equilibrium configurations with $N > 1$ are also possible, but we do not consider them here.

By Proposition 5.4, at points of the curve $N = 0$ the system evolves as in the elastic model, while at the interior points of the area it evolves along curves parallel to the curve $N = 0$. By Proposition 5.6, at points on the line $N = 1$ unloading occurs along curves parallel to the curve $N = 0$, while the evolution path at loading is determined by the incremental minimization considered in Sect. 5.3. A problem left open there was related to the creation of new jumps. Indeed, in Proposition 5.5 we proved that new jumps can be created only if $w'(u') = \theta'(0+)$, but we postponed the problem of whether, and how many, new jumps are indeed created in this particular situation. We pointed out that the answer depends on the specific form of the function θ. We now consider the case of Barenblatt's energy.

For a function θ of the Barenblatt type, the situation $w'(u') = \theta'(0+)$ may occur only at unfractured configurations. Indeed, since the derivative θ' is decreasing and the jump opening is increasing, the dissipation inequality requires that $\theta'([u]) = \theta'(0+)$, and therefore $w'(u') < \theta'(0+)$, in all configurations with jumps. The problem of how many jumps are created is answered by the solution of the minimum problem for the functional I in (5.25). For an unfractured configuration, this functional takes the form

$$I([\dot{u}]_i) = w''(u') \left(-2\dot{\beta} \sum_{i=1}^{N} [\dot{u}]_i + \frac{1}{l} \left(\sum_{i=1}^{N} [\dot{u}]_i \right)^2 \right) + \theta''(0+) \sum_{i=1}^{N} [\dot{u}]_i^2, \tag{6.2}$$

where N is the unknown number of the newly created jumps. Let us initially consider the sum of the $[\dot{u}]_i$ as fixed. Then, because $\theta''(0+)$ is negative for Barenblatt's energy, one has to minimize the sum of the squares of the $[\dot{u}]_i$; clearly, the minimum is attained for $N = 1$. Therefore, the energy minimizer has at most one jump. As a consequence, the equilibrium curves with $N > 1$ are not accessible from the origin. We then restrict our analysis to the case $N = 1$. In it, the evolution law (5.28) gives the jump increment

$$[\dot{u}] = \frac{w''(u')}{\theta''([u]) + l^{-1} w''(u')} \dot{\beta}. \tag{6.3}$$

Using the boundary condition

$$\dot{u}' + l^{-1}[\dot{u}] = \dot{\beta}, \tag{6.4}$$

it is easy to see that this value of $[\dot{u}]$ is the same obtained by differentiation of the equation $w'(u') = \theta'([u])$ of the curve $N = 1$. Therefore, Eq. (6.3) describes an evolution along that curve.

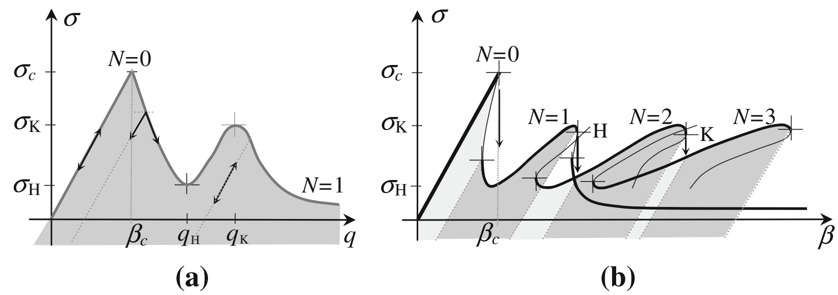

Fig. 9 Bi-modal cohesive energy. Force-elongation response curves for the dissipative model, for bars with *small* **(a)** and *large* **(b)** length. Thick lines and dark-gray areas represent metastable equilibrium configurations with a single jump. In **(b)**, the vertical arrows denote non-equilibrium transitions, and the light-gray area represents configurations not accessible from the origin

At points on the curve $N = 1$ the directions of the evolution are those shown in Fig. 8b,c: the tangent to the curve $N = 0$ at unloading, and the tangent to the curve $N = 1$ at loading. This difference in the response between loading and unloading eliminates the main defect of the elastic model. In particular, for large l the upward jump of the stress at point M predicted by the elastic model is now replaced by unloading parallel to the curve $N = 0$. In Fig. 8c, the light-gray area represents equilibrium configurations with $N = 1$ not accessible from the origin, that is, from the unfractured configuration with $u' = 0$.

We finally observe that complete fracture occurs when β reaches the value $\beta_c + l^{-1}q_c$, at which the curve $N = 1$ ends on the axis $\sigma = 0$. After this we apply the evolution law assumed in Sect. 5.5, according to which the system's response follows the horizontal line $\sigma = 0$ back and forth.

6.3 Bi-modal energies

For a bi-modal energy, the elastic model allows for metastable configurations with an arbitrary number of jumps. The same is true for the inelastic model. Here, instead of looking at all possible metastable configurations for the inelastic model, we focus on those which are accessible from the origin.

Once again we have an elastic evolution, without jump openings, until β reaches the critical value β_c. At this point, we have to consider the minimum problem for the functional (6.2) at loading, $\dot{\beta} > 0$. Because $\theta''(0+)$ is negative for a bi-modal energy, we have the same result as in Barenblatt's case, that is, that a single jump is created at $\beta = \beta_c$.

Figure 9 shows the two cases of *small* and *large l*. In the first case, Fig. 9a, there are no non-equilibrium transitions, and the evolution continues along the curve $N = 1$ at loading and parallel to the curve $N = 0$ at unloading. The gray area represents the equilibrium configurations which are accessible from the origin.

For large l, Fig. 9b, we face the problem, raised in Sect. 5.4, of which configuration is reached at the end of a non-equilibrium transition. Indeed, when the first metastable branch of the curve $N = 1$ ends at point H in Fig. 9b, there are two configurations, one with $N = 1$ and one with $N = 2$, which can be reached from that point at constant β, and our model does not say which one will be actually reached. The figure shows the second possibility: at H the system jumps to the curve $N = 2$, and follows this curve up to point K. At this point, a second jump leads to the curve $N = 3$, and so on. Once again, the configurations in the dark-gray area are accessible from the origin with this type of evolution, whereas those in the light-gray area are not.

6.4 Convex-concave energies

In the case of a convex-concave energy, we again follow the evolution of a process starting from the origin. As in the two preceding examples, during a process of loading the system follows the curve $N = 0$ up to the critical value β_c of β, and for further loading the evolution is governed by the minimum problem for the functional (6.2). For the convex-concave case $\theta''(0+)$ is positive, and therefore the result of the minimization is $N = +\infty$ instead of $N = 1$ as in the previous examples. Because for $\beta = \beta_c$ one has $Q_0 = l^{-1}w''(\beta_c)$ and

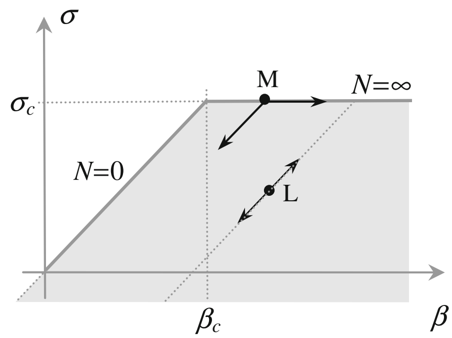

Fig. 10 Directions of incremental response for a convex-concave function θ in the inelastic model. Points located in the grey area represent equilibrium configurations

$Q_i = \theta''(0+)$ for all i, the jump amplitudes (5.28) are

$$[\dot{u}]_i = \lim_{N\to\infty} \frac{\frac{1}{\theta''(0+)}}{\frac{l}{w''(\beta_c)} + \frac{N}{\theta''(0+)}} l\dot{\beta} = 0. \tag{6.5}$$

Their sum is

$$\sum_{S(u+)} [\dot{u}]_i = \lim_{N\to\infty} \frac{\frac{N}{\theta''(0+)}}{\frac{l}{w''(\beta_c)} + \frac{N}{\theta''(0+)}} l\dot{\beta} = l\dot{\beta}, \tag{6.6}$$

and from the boundary condition (6.4) it follows that $\dot{u}' = 0$. Therefore, the evolution occurs with constant $u' = \beta_c$, and with an infinite number of jumps of equal amplitude.

The directions of evolution are shown in Fig. 10. At the point M on the line $N = \infty$, the evolution follows the horizontal path if $\dot{\beta} > 0$, and is parallel to the curve $N = 0$ if $\dot{\beta} < 0$. The presence of elastic unloading renders this curve identical to the response curve of one-dimensional perfect plasticity.

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Appendix A: Existence of global minimizers

In this Appendix we study the existence of global minimizers for the functional E in the sets $V_\beta(0, l)$. This is a special case of minimization in BV spaces, see [1] or [6].

It is known that a sufficient condition for existence is the lower semicontinuity of the functional plus the compactness of the domain of definition. In Sect. 3.4, the configuration space X has been defined as the set of all $u \in V(0, l)$ with non-negative constant derivative and with non-negative jump amplitudes. Now, we restrict our analysis to the subset X_β of X made of the functions u which satisfy the boundary condition $(3.7)_2$ for the given β. For them, from the definition (3.17) of the norm $||\cdot||$ it follows that

$$||u|| = \beta l. \tag{A.1}$$

Then X_β is bounded. Its dimension depends on the number of the jumps, and may be infinite. We prove below that if X_β is finite then it is closed, and therefore compact, and the functional E is lower semicontinuous.

Proposition A.1 *Let X_β be finite dimensional, and let $n \mapsto u_n$ be a sequence in X_β converging to u in the norm $||\cdot||$. Then u belongs to X_β and*

$$\liminf_{n\to\infty} E(u_n) \geq E(u). \tag{A.2}$$

Proof If $||u_n - u|| \to 0$, then

$$l|u_n' - u'| + \sum_{S(u)\setminus S(u_n)} [u]_i + \sum_{S(u)\cap S(u_n)} |[u_n]_i - [u]_i| + \sum_{S(u_n)\setminus S(u)} [u_n]_i \to 0. \tag{A.3}$$

All terms in the sum being non-negative, each of them must converge to zero. Then $u_n' \to u'$, and because all u_n' are non-negative, so is u'. Moreover, because the number of the jumps of each u_n is not larger than the dimension of X_β, from the convergence to zero of the three sums in (A.3) it follows that for each jump point x in $S(u)$ there is a sequence $n \mapsto [u_n]_i$ of jumps located at the same point x_i such that $[u_n]_i \to [u]_i$. Therefore, the number of the jump points in $S(u)$ is less or equal than the number of the jump points in the $S(u_n)$. Then u belongs to X and has the same dimension as X_β. Moreover, the convergence of u_n' to u' and of each of the $[u_n]_i$ to the corresponding $[u]_i$ ensures that u satisfies the boundary condition $(3.7)_2$. Then u belongs to X_β. Now consider the difference

$$E(u_n)-E(u)=l(w(u_n') - w(u'))+\sum_{S(u)\setminus S(u_n)} \theta([u]_i)+\sum_{S(u)\cap S(u_n)} (\theta([u_n]_i) - \theta([u]_i))+\sum_{S(u_n)\setminus S(u)} \theta([u_n]_i). \tag{A.4}$$

In it, all terms converge to non-negative numbers by the lower semicontinuity of w and θ. Then (A.2) holds, and therefore E is lower semicontinuous. □

It has been proved in Sect. 3.3 that the number of jumps of an energy minimizer is finite if $\theta(0+) > 0$, or if $\theta(0+) = 0$ and θ is initially strictly subadditive. In both cases the set X_β can be reduced to finite-dimensional by Proposition 3.7. Then by the preceding Proposition X_β is bounded and closed, and therefore compact, and E is lower semicontinuous. We have thus proved the following existence result.

Proposition A.2 *Let either $\theta(0+) > 0$, or $\theta(0+) = 0$ and θ initially strictly subadditive. Then E has global minimizers in $V_\beta(0, l)$ for all $\beta \geq 0$.*

We now prove an upper bound for the number of the jumps.

Proposition A.3 *Let $\theta(0+) = 0$, and let θ be strictly subadditive in [0,c] for some $c > 0$. Then a global minimizer for E has a number of jumps not greater than*

$$N_c = 1 + 2\frac{\beta l}{c}. \tag{A.5}$$

Proof Let $u \in X_\beta$ be a global minimizer for E. If u has two jumps with amplitudes $[u]_h$ and $[u]_k$ less or equal than $c/2$, then $[u]_h + [u]_k \leq c$. Then $\theta([u]_h + [u]_k) < \theta([u]_h) + \theta([u]_k)$ because θ is strictly subadditive in [0,c]. That is, the energy decreases if the two jumps are replaced by a single jump of amplitude $[u]_h + [u]_k$. Then u cannot be a global minimizer. Therefore, u cannot have more than one jump of amplitude less or equal than $c/2$. If N is the number of the jumps of u, it follows that

$$\sum_{i=1}^{N} [u]_i \geq (N-1)\frac{c}{2}, \tag{A.6}$$

But by the boundary condition $(3.7)_2$ the sum of the $[u]_i$ is equal to $l\,(\beta - u')$, with u' non-negative. Therefore,

$$\beta l \geq \sum_{i=1}^{N} [u]_i \geq (N-1)\frac{c}{2}, \tag{A.7}$$

and the desired bound (A.7) follows.

If θ is not initially subadditive, there may be no minimizers. Of interest is the following nonexistence result.

Proposition A.4 *Let θ be strictly convex in [0, c], and let l be less than $c\beta_c^{-1}$, with β_c given by* $(3.14)_2$. *Then there are no global minimizers for E if $\beta \in (\beta_c, l^{-1}c)$.*

Proof Let u be a global minimizer for E, and let N be the number of the jumps of u. If $N = 0$, then $u' = \beta$ and $\beta \leq \beta_c$ by (3.16). If $N > 0$, assume that one of the jump amplitudes $[u]_h$ is less than c. Take v in X_β, with $v' = u'$, with $[u]_h$ replaced by two jumps of amplitude $[u]_h/2$, and with all remaining jumps as in u. Then,

$$E(v) - E(u) = 2\theta([u]_h/2) - \theta([u]_h). \tag{A.8}$$

The right-hand side is negative by the strict inequality (3.26). Therefore, u is not a global minimizer for E if $[u]_h \leq c$. Then all jump amplitudes of u must be larger than c. But from the boundary condition $(3.7)_2$ with $u' \geq 0$ it follows that

$$\beta l \geq \sum_{S(u)} [u]_h > c. \tag{A.9}$$

Therefore, there are no minimizers for $\beta_c < \beta < l^{-1}c$.

In Sect. 4.4, the infimum of E has been determined and a minimizing sequence has been constructed in the case of θ initially strictly convex. The limit element of this sequence, which does not belong to X, is the generalized solution of the global minimum problem.

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