

# Coplanar perturbation of a crack lying on the mid-plane of a plate

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**Abstract** Several groups have studied experimentally the deformation of the front of mode I cracks propagating quasistatically along the interface between bonded plates. The theoretical interpretation of such experiments has always been based up to now on a formula of Rice (ASME J Appl Mech 52:571–579, 1985); this formula provides the first-order variation of the local mode I stress intensity factor resulting from some small, but otherwise arbitrary coplanar perturbation of the front of a semi-infinite crack in an infinite body. To be applicable to bonded plates, this formula requires that the characteristic distance of variation of this perturbation in the direction of the crack front be small compared to all other characteristic dimensions of the problem, and first of all the thickness of the plates. This condition is unfortunately frequently violated in practice. The purpose of this paper is therefore to provide a more exact formula for the variation of the local stress intensity factor, for the specific cracked geometry and boundary conditions used in experiments; this should

allow for more accurate theoretical interpretations. This is done in two steps. The first one consists in adapting Rice's (ASME J Appl Mech 52:571–579, 1985) treatment, applicable to the extreme case of plates of infinite thickness, to the other extreme one of plates of infinitesimal thickness, using the standard Love-Kirchhoff plate theory. An interesting outcome of the analysis is that the distance from the crack front to the boundary of the plate acts as a “cutoff length”, in the sense that when the distance between two points on the crack front gets larger than it, the influence of the crack advance at the first point upon the stress intensity factor at the second diminishes quickly; the plate thickness, however, plays no similar role. The second step consists in supplementing the theoretical expressions applicable to extreme values of the plate thickness with finite element computations providing results for intermediate values. These computations lead to the definition of a simple, approximate but accurate “interpolation formula” for the variation of the local stress intensity factor, applicable to plates of arbitrary thickness.

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## 1 Introduction

In the past 20 years, a number of works have been devoted to experimental studies of the deformation of

the front of plane cracks propagating quasistatically in heterogeneous materials; see [Schmittbuhl and Maloy \(1997\)](#), [Delaplace et al. \(1999\)](#), [Maloy and Schmittbuhl \(2001\)](#), [Schmittbuhl et al. \(2003\)](#), [Maloy et al. \(2006\)](#), [Dalmas et al. \(2009\)](#), [Chopin \(2010\)](#), [Santucci et al. \(2010\)](#), and the recent reviews of [Bonamy \(2009\)](#) and [Bonamy and Bouchaud \(2011\)](#). In a typical experiment, two plexiglas plates were bonded together through application of a large pressure at high temperature, an inhomogeneous distribution of toughness being generated on the interface through some suitable treatment.<sup>1</sup> The plates were then gradually separated by increasing the gap between them on one side, thus inducing propagation of a crack from this side along the interface. The transparency of the plates allowed for direct optical observation of the shape of the crack front as a function of the distance of propagation.

Theoretical interpretations of experiments of this type have been proposed by many authors; see [Schmittbuhl et al. \(1995\)](#), [Tanguy et al. \(1998\)](#), [Schmittbuhl and Vilotte \(1999\)](#), [Krishnamurthy et al. \(2000\)](#), [Roux et al. \(2003\)](#), [Schmittbuhl et al. \(2003\)](#), [Charles et al. \(2004\)](#), [Katzav and Adda-Bedia \(2006\)](#), [Bonamy et al. \(2008\)](#), [Laurson et al. \(2010\)](#), [Ponson and Bonamy \(2010\)](#) and the reviews of [Alava et al. \(2006\)](#) and [Lazarus \(2011\)](#). These interpretations were invariably based on a formula of [Rice \(1985\)](#) providing the first-order expression of the variation of the local mode I stress intensity factor resulting from some small, but otherwise arbitrary in-plane perturbation of the crack front. This formula was established for a semi-infinite crack in some infinite body. Clearly, to be applicable to a crack lying on the interface between bonded plates, it requires that the characteristic distance of variation of the perturbation of the front be small compared to all other characteristic dimensions of the problem, and first of all the thickness of the plates.

Unfortunately, this condition is often violated in actual experiments. In [Dalmas et al. \(2009\)](#)'s experiments for instance, regularly distributed tough spots of size comparable to the plate thickness were generated, resulting in notable fluctuations of the position of the crack front over this lengthscale. Even in more customary experiments which involved randomly distributed toughness inhomogeneities of much smaller size, Fourier components of the fluctuations of position of the

front of wavelength comparable to the plate thickness were present; see for instance Figure 5 of [Schmittbuhl et al.'s \(2003\)](#) paper.

Applicability of [Rice's \(1985\)](#) formula to the experiments discussed is therefore questionable. More accurate interpretations of these experiments would result from use of a formula specifically adapted to the geometric configuration employed. Establishing it seems all the more important since extensions of [Rice's \(1985\)](#) formula to different situations—a tunnel-crack loaded in mode I ([Leblond et al. 1996](#)) or in some arbitrary combination of modes ([Lazarus and Leblond 2002](#)); a system of two coplanar, parallel, identical mode I tunnel-cracks ([Pindra et al. 2010](#)); a system of two coplanar, parallel semi-infinite mode I cracks ([Legrand and Leblond 2010](#))—have revealed a notable influence of the cracked geometry considered upon the “influence function” depicting the effect of a small crack advance at some point of the front upon the variation of the stress intensity factor at some other point. The effect is particularly important upon the asymptotic behavior of this function at large distances ([Legrand and Leblond 2010](#)).

The aim of this paper is therefore to extend [Rice's \(1985\)](#) formula to coplanar perturbation of a crack lying on the interface between bonded plates, or equivalently on the mid-plane of a plate of double thickness.<sup>2</sup> This will be done in two steps. First, [Rice's \(1985\)](#) treatment, appropriate for plates of very large thickness, will be adapted to the other extreme case of plates of very small thickness. This will be done by replacing the standard 3D theory of elasticity he used by the Love-Kirchhoff plate theory, which is known to be *the* correct asymptotic theory in the limit of an infinitesimal plate thickness ([Sanchez-Hubert and Sanchez-Palencia 1992](#)). Second, the theoretical expressions applicable to extreme values of the thickness will be supplemented with finite element computations providing results for intermediate values. These results will be used to define a simple, approximate but accurate “interpolation formula” for the variation of the local stress intensity factor, applicable in all cases.

The paper is organized as follows:

- Section 2 presents the problem to be solved.

<sup>1</sup> In a variant, the interface was between a plate and a rigid substrate.

<sup>2</sup> The general consensus is that the sole role of the interface is to enforce coplanar propagation of the crack between the plates; its lower elastic stiffness is unimportant and may be ignored.

- Section 3 recalls Rice’s (1985) solution, applicable to plates of very large thickness.
- As a prerequisite for the next section, Sect. 4 expounds some generalities on cracks lying on the mid-plane of plates.
- Rice’s (1985) treatment is then adapted in Sect. 5 to the case of plates of very small thickness.
- Finally Sect. 6 presents the numerical study for plates of intermediate thickness and the interpolation formula deduced from it.

**2 Statement of the problem—notations**

Figures 1 and 2 offer different 2D views of the 3D problem considered. A plate occupies the domain  $0 \leq x < +\infty, -h \leq y \leq h, -\infty < z < +\infty$  in the 3D space. This plate contains a plane crack occupying the region  $0 \leq x \leq a(z), -\infty < z < +\infty$  in the plane  $y = 0$ . The crack front is almost straight so that the distance  $a(z)$  separating it from the  $Oz$  axis may be written in the form

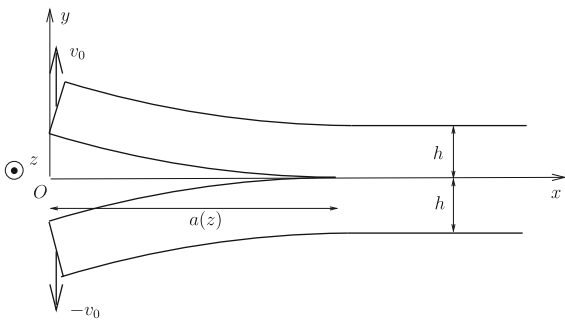
$$a(z) \equiv a + \delta a(z), \quad |\delta a(z)| \ll a. \tag{1}$$

At each point  $P$  of the crack front, we define some local axes  $Px_1, Px_2, Px_3$  with  $Px_1$  perpendicular to the front within the  $Oxz$  plane,  $Px_2$  parallel to the  $Oy$  axis and  $Px_3$  tangent to the front.

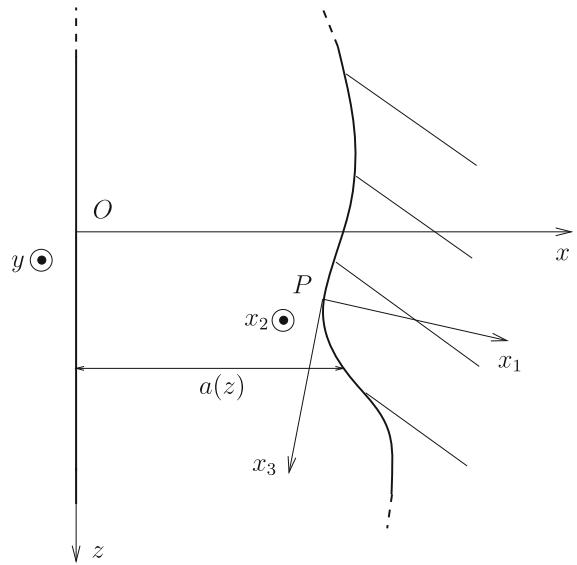
The plate is made of some isotropic elastic material with Young’s modulus  $E$  and Poisson’s ratio  $\nu$ .

The loading consists of uniform, opposite displacements  $v_0, -v_0$  in the  $y$  direction imposed on the upper and lower halves of the left boundary.

The crack is thus loaded in a state of pure mode I at all points of its front. When this front is straight, the mode I stress intensity factor is uniform along it and



**Fig. 1** The general problem—view in the  $Oxy$  plane



**Fig. 2** The general problem—view in the  $Oxz$  plane

denoted  $K$ . When it is curved, this stress intensity factor is non-uniform and denoted  $K(P)$  or  $K(z)$ , depending on the context. If the deviation from straightness is slight,  $K(z)$  may be written in the form

$$K(z) \equiv K + \delta K(z), \quad |\delta K(z)| \ll K. \tag{2}$$

The problem is to derive an expression of the local fluctuation of the stress intensity factor at the point  $z$  of the crack front,  $\delta K(z)$ , accurate to first order in the values of the perturbation of this front,  $\delta a(z')$ ,  $-\infty < z' < +\infty$ .

Use will be made of Fourier transforms of functions in the direction of the crack front. The definition adopted for the Fourier transform  $\widehat{\phi}(k)$  of an arbitrary function  $\phi(z)$  will be

$$\widehat{\phi}(k) = \int_{-\infty}^{+\infty} \phi(z)e^{ikz} dz \Leftrightarrow \phi(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{\phi}(k)e^{-ikz} dk. \tag{3}$$

**3 The Rice solution for an infinite 3D body**

Rice (1985) derived the expression of  $\delta K(z)$  for a semi-infinite crack in an infinite body. Such a geometric schematization is acceptable in the present case only if the

characteristic distance of variation  $\lambda$  of the perturbation of the crack front<sup>3</sup> satisfies the conditions

$$\lambda \ll h, \quad \lambda \ll a, \tag{4}$$

the first of which implies that the plate must be “very thick”. Provided that these conditions are met,  $\delta K(z)$  is given by the formula

$$\frac{\delta K(z)}{K} = \frac{1}{2\pi} PV \int_{-\infty}^{+\infty} \frac{\delta a(z') - \delta a(z)}{(z' - z)^2} dz' \tag{5}$$

(subject to conditions (4))

where the symbol  $PV$  denotes a Cauchy principal value.

The equivalent formula in Fourier’s space is readily shown to be

$$\frac{\widehat{\delta K}(k)}{K} = -\frac{|k|}{2} \widehat{\delta a}(k) \quad \text{(subject to conditions (4)).} \tag{6}$$

#### 4 Generalities on cracks lying on the mid-plane of very thin plates

We now consider the limit of a “very thin” plate, that is one satisfying the conditions

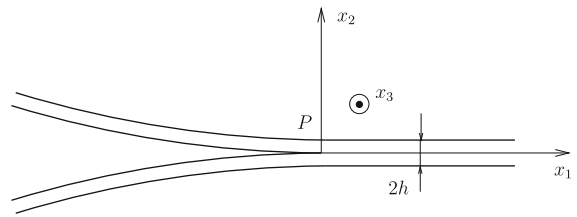
$$h \ll \lambda, \quad h \ll a. \tag{7}$$

There are two ways to consider this limit, according to whether one observes the plate as it is or adopting a magnified view of the vicinity of the crack front. Within the framework of matched asymptotic expansions, the first possibility corresponds to considering the “exterior problem”, and the second one the “interior problem”. For the sake of simplicity and clarity, we shall avoid a fully rigorous, but heavily formal study of these two problems, and be content with a more qualitative and intuitive presentation.

##### 4.1 Exterior problem

Consider the vicinity of some arbitrary point  $P$  of the crack front, with the local axes  $Px_1, Px_2, Px_3$  defined in Fig. 2. Figure 3 shows the “exterior” view of this region. In a formal asymptotic expansion of the equations of the problem, this view would correspond to using the variables  $x_1, x_2/h, x_3$ , that is, considering a dilation of coordinates in the sole direction  $x_2$ .

<sup>3</sup> That is, in the case of a sinusoidal perturbation, simply the wavelength  $2\pi/|k|$ .



**Fig. 3** The exterior problem

In the limit  $h \rightarrow 0$ , the mechanical situation is equivalent to a system of two symmetric plates occupying the region  $x_1 \leq 0$  and clamped into some rigid frame at  $x_1 = 0$ . These plates must be described using Love-Kirchhoff’s plate theory, since it is known to be the correct asymptotic one in the limit  $h \rightarrow 0$  (Sanchez-Hubert and Sanchez-Palencia (1992), p. 158 sqq.). Within this theory, the displacement  $u_2(x_1, x_3) \equiv v(x_1, x_3)$  of the upper plate in the  $x_2 \equiv y$  direction is of the following form, in the plane  $x_3 = 0$  and behind and near the point  $P$ :

$$v(x_1, 0) = \frac{1}{2} \frac{\partial^2 v}{\partial x_1^2}(0, 0) x_1^2 + O(x_1^3) \quad (x_1 \leq 0).$$

Using rather Love-Kirchhoff’s deformation tensor (the opposite of the curvature tensor of the plate)<sup>4</sup>

$$\mathbf{C} \equiv -\mathbf{grad}(\mathbf{grad} v), \tag{8}$$

this expression may be rewritten as

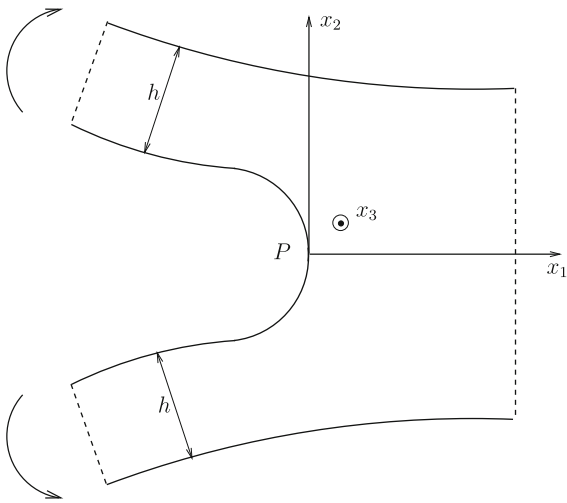
$$v(x_1, 0) = -\frac{C_{11}(P)}{2} x_1^2 + O(x_1^3) \quad (x_1 \leq 0). \tag{9}$$

##### 4.2 Interior problem

Now Fig. 4 shows the “interior” view of the vicinity of the point  $P$ . In a formal asymptotic expansion, this view would correspond to using the variables  $x_1/h, x_2/h, x_3$ , that is, considering a dilation of coordinates in the directions  $x_1$  and  $x_2$  but *not*  $x_3$ . (Dilating coordinate  $x_3$  would mean letting  $\lambda$  go to zero together with  $h$ ).

In the limit  $h \rightarrow 0$ , the 3D elasticity problem becomes one of plane strain in the plane  $Px_1x_2$ , since the characteristic distance of variation  $\lambda$  of the position of the crack front becomes infinitely larger than

<sup>4</sup> The more customary notation  $\mathbf{K}$  for Love-Kirchhoff’s deformation tensor is avoided in order to prevent any possible confusion with the stress intensity factor  $K$ .



**Fig. 4** The interior problem

the typical dimension  $h$  of the region studied. In this problem, the component  $C_{11}(P)$  of Love-Kirchhoff's deformation tensor in the exterior problem acts as a boundary condition imposed on the far left of the upper half of the geometry, which is thus subjected to some prescribed bending moment (see Fig. 4). The linearity of the problem implies that the stress intensity factor  $K(P)$  at point  $P$  must depend linearly on this boundary condition, so that there must be a constant  $\gamma$  such that

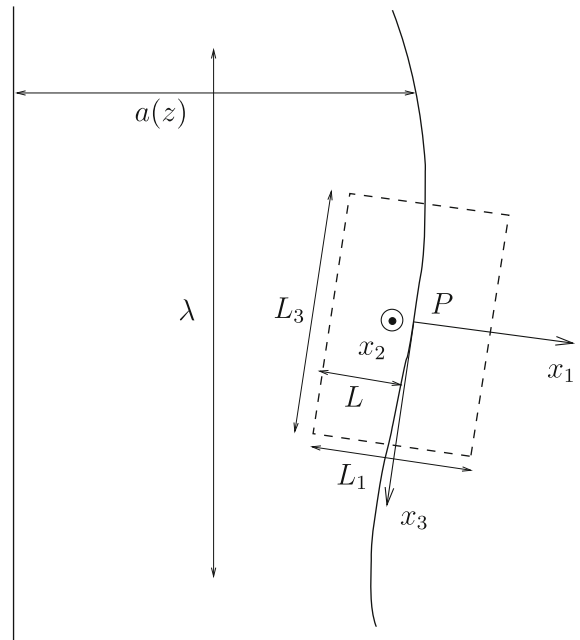
$$K(P) = -\frac{C_{11}(P)}{2\gamma}. \tag{10}$$

### 4.3 Calculation of the constant $\gamma$

Although the calculation of the constant  $\gamma$  is not strictly indispensable in the present context, we shall present it for completeness.

The standard way of evaluating  $\gamma$  would be to explicitly solve the interior problem. It so happens, however, that this is unnecessary in the present case. Indeed, as will now be seen,  $\gamma$  may be calculated analytically and explicitly by combining elementary energetic considerations and Irwin's formula connecting the local energy-release-rate  $G(P)$  to the local stress intensity factor  $K(P)$ .

Let us consider a rectangular region centered at the point  $P$  of the crack front, with sides parallel to the  $Px_1$  and  $Px_3$  axes (Fig. 5). Conditions (7) make it possible to choose, for the dimensions  $L_1$  and  $L_3$  of this rectangle along these axes, values much larger than  $h$ ,



**Fig. 5** Definition of a domain for the calculation of the local energy-release rate

while still much smaller than  $\lambda$  and  $a$ . The conditions  $L_1, L_3 \gg h$  allow for the use of Love-Kirchhoff's theory to evaluate the total energy of the region (the result will be asymptotically exact in the limit  $h \rightarrow 0$ ), while the conditions  $L_1, L_3 \ll \lambda, a$  make it possible to consider the strain and stress states as uniform within its left, deformed part.

Let  $\mathbf{M}$  denote the tensor of bending and torsion moments, related to the deformation tensor  $\mathbf{C}$  through Love-Kirchhoff's constitutive law

$$\mathbf{M} = D [(1 - \nu)\mathbf{C} + \nu(\text{tr } \mathbf{C})\mathbf{1}] \tag{11}$$

where

$$D \equiv \frac{Eh^3}{12(1 - \nu^2)} \tag{12}$$

is the bending stiffness. In the left part of the upper plate, the values of the components of the deformation tensor are  $C_{11} = C_{11}(P), C_{13} = C_{33} = 0$  (because of the clamping conditions  $v = 0, \partial v / \partial x_1 = 0$  at  $x_1 = 0$ ). It then follows from Eq. 11 that  $M_{11} = DC_{11}(P)$ , so that the value of the surface density of elastic energy of the upper plate is  $\frac{1}{2} \mathbf{M} : \mathbf{C} = \frac{1}{2} M_{11} C_{11} = \frac{D}{2} [C_{11}(P)]^2$ . Accounting for the presence of two plates, one concludes that the value of the total elastic energy of the rectangular region is

$$W = D [C_{11}(P)]^2 LL_3 \tag{13}$$

where  $L$  is the length of the crack within the rectangle, that is, the distance between its left side and the crack front (see Fig. 5).

Now the opposite of the potential energy of prescribed forces,  $\phi$ , is just the work done in the deformed configuration by the bending moments imposed on the left side of the plates. Therefore, by the work theorem,

$$\phi = 2W. \quad (14)$$

It follows from Eqs. 13 and 14 that

$$-\frac{\partial}{\partial L}(W - \phi) = \frac{\partial W}{\partial L} = D [C_{11}(P)]^2 L_3. \quad (15)$$

But the left-hand side of Eq. 15 is equal to  $\int_{-L_3/2}^{L_3/2} G(x_3) dx_3 = G(P)L_3$ , by the very definition of the local energy-release-rate. It follows that

$$G(P) = D [C_{11}(P)]^2. \quad (16)$$

Using now Irwin's formula, one gets from there the value of the local stress intensity factor:

$$K(P) = \sqrt{\frac{E}{1-\nu^2}} G(P) = -\frac{Eh^{3/2}}{2\sqrt{3}(1-\nu^2)} C_{11}(P), \quad (17)$$

where Eq. 12 has been used and account has been taken of the obvious positiveness of  $K(P)$  and negativity of  $C_{11}(P)$ . This means that  $K(P)$  and  $C_{11}(P)$  are related by a formula of type (10), with

$$\gamma \equiv \sqrt{3} \frac{1-\nu^2}{Eh^{3/2}}. \quad (18)$$

#### 4.4 Synthesis of results

Combining Eqs. 9 and 10, one sees that the asymptotic expression of the displacement of the upper plate in the  $y$  direction, behind and near the point  $P$  of the crack front, in the exterior problem, reads

$$v(x_1, 0) = \gamma K(P) x_1^2 + O(x_1^3) \quad (x_1 \leq 0) \quad (19)$$

with  $\gamma$  given by Eq. 18. This expression is the equivalent, for cracks lying on the mid-plane of very thin plates, of the standard asymptotic formula for the crack opening displacement near the crack front in classical 3D fracture mechanics.<sup>5</sup> It will play a central role in the next section.

<sup>5</sup> It may be noted incidentally that Eq. 19 is just a straightforward 3D extension of that applicable to 2D problems, which underlies the derivation of some solutions of thin elastic cracked bodies using beam theory; see Tada et al. (2000) for specific examples.

## 5 The solution for a very thin plate

We shall now adapt Rice's (1985) treatment of the in-plane perturbation of a semi-infinite crack in some infinite body to the in-plane perturbation of a crack lying on the midplane of a very thin plate (that is, satisfying conditions (7)).

In fact Rice's (1985) paper contained two independent derivations of the same final formula: one based on the theory of Bueckner's 3D weight functions, and the other on a direct calculation of the Neuber-Papkovich potentials of the 3D elasticity problem. The first approach would be inapplicable here, because of lack of development of an analog of Bueckner's theory for thin plates. On the other hand the second approach can be adapted to a thin plate, as will be seen; in fact the adaptation will even turn out to be simpler than the original, in that it will directly deal with the displacement field instead of the Neuber-Papkovich potentials. It will suffice to consider the exterior problem defined in Section 4, making use of Eq. (19); the study of the interior problem will not be necessary.

### 5.1 Original problem for an arbitrary perturbation of the crack front

The "exterior equations" of the problem defined in Sect. 2, with a perturbation of the crack front of arbitrary amplitude (that is, without hypothesis (1)), within Love-Kirchhoff's plate theory, read on the upper plate:

$$\left\{ \begin{array}{l} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) v(x, z) = 0 \\ v(0, z) = v_0 \\ \frac{\partial^2 v}{\partial x^2}(0, z) = 0 \\ v(a(z), z) = 0 \\ \frac{\partial v}{\partial n}(a(z), z) = 0 \end{array} \right\} \begin{array}{l} \text{for } 0 \leq x \leq a(z), -\infty < z < +\infty \\ \text{for } -\infty < z < +\infty \\ \text{for } -\infty < z < +\infty \\ \text{for } -\infty < z < +\infty \end{array} \quad (20)$$

where the symbol  $\partial/\partial n$  denotes the normal derivative to the front. Equation 20<sub>1</sub> is the equilibrium equation, Eq. 20<sub>2,3</sub> are the boundary conditions on the left side, and Eq. 20<sub>4,5</sub> are the clamping conditions on the crack front.

5.2 First-order problem for an infinitesimal perturbation of the crack front

Consider now a perturbation of small amplitude (obeying hypothesis (1)), and denote  $\delta v(x, z)$  the variation, under constant loading, of the normal displacement of the upper plate arising from the variation of position  $\delta a(z)$  of the crack front around the mean value  $a$ . The equilibrium equation and boundary conditions of the left side satisfied by  $\delta v(x, z)$  are readily obtained through differentiation of Eq. 20<sub>1,2,3</sub> at constant  $v_0$ :

$$\left\{ \begin{array}{l} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)^2 \delta v(x, z) = 0 \\ \delta v(0, z) = 0 \\ \frac{\partial^2 \delta v}{\partial x^2}(0, z) = 0 \end{array} \right\} \text{ for } \begin{array}{l} 0 \leq x \leq a, -\infty < z < +\infty \\ -\infty < z < +\infty. \end{array} \quad (21)$$

On the other hand the conditions verified by  $\delta v(x, z)$  on the crack front cannot be obtained from Eq. 20<sub>4,5</sub> in such a simple way. To derive them, it is convenient to temporarily write the perturbation of the front in the form

$$\delta a(z) \equiv \epsilon \phi(z)$$

where  $\epsilon$  is a small parameter and  $\phi(z)$  a given function. We then introduce the power expansion of the displacement in the original problem, up to the third order, in the plane perpendicular to the front at the point  $(a + \epsilon \phi(z), z)$ , with explicit indications of dependence of all quantities upon  $\epsilon$ :

$$\begin{aligned} v(\epsilon; x, z + \epsilon(a - x)\phi'(z)) &= \gamma K(\epsilon; z) [a + \epsilon \phi(z) - x]^2 \\ &+ \chi(\epsilon; z) [a + \epsilon \phi(z) - x]^3 \\ &+ O([a + \epsilon \phi(z) - x]^4) \quad (0 \leq x \leq a + \epsilon \phi(z)). \end{aligned} \quad (22)$$

(In the left-hand side, the correction of the  $z$  coordinate arises from the fact that the normal vector to the perturbed front is not exactly  $\mathbf{e}_x$  but  $\mathbf{e}_x - \epsilon \phi'(z)\mathbf{e}_z$ , see Fig. 2; also, in the right-hand side, the distance from the point  $(x, z + \epsilon(a - x)\phi'(z))$  to the point  $(a + \epsilon \phi(z), z)$  of the crack front is considered to be  $a + \epsilon \phi(z) - x$ , with an error of second order in  $\epsilon$ ). Differentiating this equation with respect to  $\epsilon$  at  $\epsilon = 0$ , one gets

$$\begin{aligned} \frac{\partial v}{\partial \epsilon}(0; x, z) + \frac{\partial v}{\partial z}(0; x, z)(a - x)\phi'(z) &= 2\gamma K(0; z)(a - x)\phi(z) + \gamma \frac{\partial K}{\partial \epsilon}(0; z)(a - x)^2 \\ &+ 3\chi(0; z)(a - x)^2\phi(z) + O((a - x)^3) \end{aligned}$$

or more simply, since for  $\epsilon = 0$  the problem becomes 2D in the  $Oxy$  plane and  $v, K$  and  $\chi$  independent of  $z$ :

$$\begin{aligned} \frac{\partial v}{\partial \epsilon}(0; x, z) &= 2\gamma K(0)(a - x)\phi(z) \\ &+ \gamma \frac{\partial K}{\partial \epsilon}(0; z)(a - x)^2 \\ &+ 3\chi(0)(a - x)^2\phi(z) + O((a - x)^3). \end{aligned}$$

Multiplying both sides by  $\epsilon$  and identifying  $\epsilon \partial v / \partial \epsilon$  to  $\delta v$ ,  $\epsilon \phi$  to  $\delta a$  and  $\epsilon \partial K / \partial \epsilon$  to  $\delta K$ , one gets from there

$$\delta v(x, z) = 2\gamma K(a - x)\delta a(z) + \gamma \delta K(z)(a - x)^2 + 3\chi(a - x)^2\delta a(z) + O((a - x)^3) \quad (0 \leq x \leq a) \quad (23)$$

where henceforth superfluous indications that quantities are taken at  $\epsilon = 0$  have been discarded. It follows that

$$\left\{ \begin{array}{l} \delta v(a, z) = 0 \\ \frac{\partial \delta v}{\partial x}(a, z) = -2\gamma K \delta a(z) \end{array} \right\} \text{ for } -\infty < z < +\infty. \quad (24)$$

These are the boundary conditions on the crack front looked for.

5.3 Solution of the first-order problem in Fourier's space

To solve Eqs. 21, 24, take their Fourier transform in the direction  $z$  and get

$$\left\{ \begin{array}{l} \left( \frac{\partial^2}{\partial x^2} - k^2 \right)^2 \widehat{\delta v}(x, k) = 0 \\ \widehat{\delta v}(0, k) = 0 \\ \frac{\partial^2 \widehat{\delta v}}{\partial x^2}(0, k) = 0 \end{array} \right\} \text{ for } 0 \leq x \leq a, -\infty < k < +\infty \quad (25)$$

$$\left\{ \begin{array}{l} \widehat{\delta v}(a, k) = 0 \\ \frac{\partial \widehat{\delta v}}{\partial x}(a, k) = -2\gamma K \widehat{\delta a}(k) \end{array} \right\} \text{ for } -\infty < k < +\infty.$$

The solution of Eq. 25<sub>1</sub> must be looked for in the form of an exponential  $e^{\mu x}$ . Then  $\mu$  satisfies the characteristic equation  $(\mu^2 - k^2)^2 = 0$ , which admits two double solutions,  $\mu = k$  and  $\mu = -k$ . It follows that  $\widehat{\delta v}(x, k)$  must be a linear combination of the functions  $e^{kx}, xe^{kx}, e^{-kx}, xe^{-kx}$ , or equivalently of the form

$$\widehat{\delta v}(x, k) = (Ax + B) \sinh(kx) + (Cx + D) \cosh(kx) \quad (26)$$

where  $A, B, C$  and  $D$  are constants to be determined.<sup>6</sup>

Now Eq. 25<sub>2,3</sub> immediately imply that

$$A = D = 0, \tag{27}$$

and the values of  $B$  and  $C$  then follow from Eq. 25<sub>4,5</sub>:

$$\begin{cases} B = 2\gamma K a \widehat{\delta a}(k) \frac{\cosh(ka)}{\cosh(ka) \sinh(ka) - ka} \\ C = -2\gamma K \widehat{\delta a}(k) \frac{\sinh(ka)}{\cosh(ka) \sinh(ka) - ka} \end{cases} \tag{28}$$

Then, by Eq. 26,

$$\widehat{\delta v}(x, k) = 2\gamma K \widehat{\delta a}(k) \times \frac{a \cosh(ka) \sinh(kx) - x \sinh(ka) \cosh(kx)}{\cosh(ka) \sinh(ka) - ka} \tag{29}$$

#### 5.4 Fourier transform of the variation of the stress intensity factor

The next task is to relate the Fourier transform of the variation of the stress intensity factor,  $\widehat{\delta K}(k)$ , to that of the perturbation of the front,  $\widehat{\delta a}(k)$ . To do so, differentiate Eq. 23 twice with respect to  $x$  at  $x = a$ :

$$\begin{aligned} \frac{\partial^2 \delta v}{\partial x^2}(a, z) &= 2\gamma \delta K(z) + 6\chi \delta a(z) \\ \Rightarrow \delta K(z) &= \frac{1}{2\gamma} \frac{\partial^2 \delta v}{\partial x^2}(a, z) - \frac{3\chi}{\gamma} \delta a(z), \end{aligned}$$

then take the Fourier transform of this equation:

$$\widehat{\delta K}(k) = \frac{1}{2\gamma} \frac{\partial^2 \widehat{\delta v}}{\partial x^2}(a, k) - \frac{3\chi}{\gamma} \widehat{\delta a}(k). \tag{30}$$

Now, by Eq. 29,

$$\frac{\partial^2 \widehat{\delta v}}{\partial x^2}(a, k) = -4\gamma k K \widehat{\delta a}(k) \frac{\sinh^2(ka)}{\cosh(ka) \sinh(ka) - ka} \tag{31}$$

Also, to calculate the constant  $\chi$ , note that by Eq. 22 applied to the unperturbed configuration of the crack front ( $\epsilon = 0$ ),

$$\gamma K \equiv \frac{1}{2} \frac{d^2 v}{dx^2}(a); \quad \chi \equiv -\frac{1}{6} \frac{d^3 v}{dx^3}(a) \tag{32}$$

where  $v(x)$  is the unperturbed normal displacement of the upper plate. Now it is easy to calculate this displacement from the equations of the unperturbed (2D) problem (20) with  $a(z) \equiv Cst. \equiv a$ :

$$v(x) = v_0 \left( 1 - \frac{3x}{2a} + \frac{x^3}{2a^3} \right), \tag{33}$$

<sup>6</sup> The constant  $D$  should not be confused with the bending stiffness defined by Eq. 12, which no longer plays a role.

so that

$$\gamma K = \frac{3v_0}{2a^2}; \quad \chi = -\frac{v_0}{2a^3} = -\frac{\gamma K}{3a}. \tag{34}$$

(The value of  $K$  in Eq. 34<sub>1</sub>, with  $\gamma$  given by Eq. 18, coincides with that provided by Tada et al. (2000), p. 419, for the problem in question). Combining Eqs. 30, 31 and 34, one gets

$$\begin{aligned} \frac{\widehat{\delta K}(k)}{K} &= \frac{1}{2\gamma K} \frac{\partial^2 \widehat{\delta v}}{\partial x^2}(a, k) - \frac{3\chi}{\gamma K} \widehat{\delta a}(k) \\ &= -\frac{2k \sinh^2(ka)}{\cosh(ka) \sinh(ka) - ka} \widehat{\delta a}(k) + \frac{\widehat{\delta a}(k)}{a} \\ &= \left( 1 - \frac{2ka \sinh^2(ka)}{\cosh(ka) \sinh(ka) - ka} \right) \frac{\widehat{\delta a}(k)}{a} \end{aligned}$$

or equivalently

$$\frac{\widehat{\delta K}(k)}{K} = F(ka) \frac{\widehat{\delta a}(k)}{a} \quad (\text{subject to conditions (7)}) \tag{35}$$

where  $F(p)$  is a function of the “reduced wavenumber”

$$p \equiv ka \tag{36}$$

defined by

$$F(p) \equiv \frac{2p \cosh(2p) - \sinh(2p)}{2p - \sinh(2p)}. \tag{37}$$

Several features of the function  $F(p)$  are noteworthy:

- First, expanding  $\cosh(2p)$  and  $\sinh(2p)$  in powers of  $p$ , one easily shows that

$$F(0) = -2. \tag{38}$$

- Second, the very definition (37) of  $F(p)$  implies that it is indefinitely differentiable except perhaps at the origin. But near the origin, it is obviously the ratio of two power series of  $p$ , and this ratio has a finite limit for  $p \rightarrow 0$ , as just noted. Hence it is also indefinitely differentiable at the origin. In conclusion,  $F(p)$  is indefinitely differentiable on the whole real line.

- Finally it is easy to show that

$$F(p) = -2|p| + 1 + O(p^2 e^{-2|p|}) \quad \text{for } p \rightarrow \pm \infty. \tag{39}$$

There is a connection here with the work of Vandembroucq and Roux (1997) on biharmonic functions satisfying certain conditions on the boundary of a slightly perturbed domain. Although the results of these authors, obtained by a conformal mapping technique,



do not directly apply to the present problem because of some differences in the boundary conditions, a link may be established in the limiting case where the depth  $a$  of the crack goes to infinity; the asymptotic behavior of the function  $F(p)$  described by Eq. 39 fully agrees with Vandembroucq and Roux (1997)'s findings in this case.

There is also a connection with a very recent, yet unpublished work of Ponson et al. (2011) devoted to the problem of coplanar perturbation of a crack resulting from a ‘‘peeling test’’, in which some thin flexible membrane bonded onto some rigid substrate is gradually pulled from it. This problem was more complex than that envisaged here in that it required use of the Föppl-von Karman nonlinear plate theory in order to account for the effect of the in-plane tension of the deformed membrane; on the other hand the geometry was simpler in that the crack was semi-infinite. The solutions obtained may be compared by replacing the lower deformable plate by a rigid one and letting the depth  $a$  of the crack go to infinity in the present work, and dropping nonlinearities in Ponson et al.'s (2011) work. The two solutions do coincide with these adaptations and simplifications.

### 5.5 Variation of the stress intensity factor

Although Eqs. 35, 37 for the Fourier transform  $\widehat{\delta K}(k)$  may be used directly, an explicit formula for the variation of the stress intensity factor in the physical space,  $\delta K(z)$ , rather than in Fourier's space, may also be useful.

To derive such a formula, take the inverse Fourier transform of Eq. 35:

$$\begin{aligned} \frac{\delta K(z)}{K} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widehat{\delta K}(k)}{K} e^{-ikz} dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(ka) \frac{\widehat{\delta a}(k)}{a} e^{-ikz} dk. \end{aligned}$$

Now use the Fourier decomposition of  $\widehat{\delta a}(k)$  in this expression:

$$\frac{\delta K(z)}{K} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} F(ka) \frac{\delta a(z')}{a} e^{ikz'} e^{-ikz} dz' \right) dk.$$

It would seem tempting at this stage to change the order of integration; the inverse Fourier transform of the

function  $F(p)$  would then appear as a result of the first integration on  $k$ . This is however not feasible because the function  $F(p)$ , being nonzero at infinity as is apparent in Eq. 39, does not admit a Fourier transform.

A more elaborate reasoning is thus necessary. First, rewrite the preceding expression in the form

$$\begin{aligned} \frac{\delta K(z)}{K} &= I_1 + I_2, \\ \begin{cases} I_1 \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} F(ka) \frac{\delta a(z)}{a} e^{ik(z'-z)} dz' \right) dk \\ I_2 \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} F(ka) \frac{\delta a(z') - \delta a(z)}{a} e^{ik(z'-z)} dz' \right) dk. \end{cases} \end{aligned} \tag{40}$$

The integrals  $I_1$  and  $I_2$  may now be calculated as follows:

- With regard to  $I_1$ , performing the integration on  $z'$ , one gets

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(ka) \frac{\delta a(z)}{a} 2\pi \delta(k) e^{-ikz} dk \\ &= F(0) \frac{\delta a(z)}{a} = -2 \frac{\delta a(z)}{a} \end{aligned} \tag{41}$$

where Eq. 38 has been used.

- With regard to  $I_2$ , note first that the function

$$k \mapsto -i \int_{-\infty}^{+\infty} \frac{\delta a(z') - \delta a(z)}{z' - z} e^{ik(z' - z)} dz'$$

vanishes at infinity, being a Fourier transform, and that its derivative is the function

$$k \mapsto \int_{-\infty}^{+\infty} [\delta a(z') - \delta a(z)] e^{ik(z' - z)} dz'.$$

Therefore integration by parts of the expression of  $I_2$  yields

$$\begin{aligned} I_2 &= \frac{i}{2\pi} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} F'(ka) \frac{\delta a(z') - \delta a(z)}{z' - z} e^{ik(z' - z)} dz' \right) dk \\ &= \frac{i}{2\pi} \int_{-\infty}^{+\infty} F'(ka) \left( PV \int_{-\infty}^{+\infty} \frac{\delta a(z') - \delta a(z)}{z' - z} e^{ik(z' - z)} dz' \right) dk \end{aligned}$$

where the introduction of the  $PV$  is allowed by the fact that the integral is convergent in the usual sense.

Now note that the function

$$k \mapsto -i PV \int_{-\infty}^{+\infty} \frac{\delta a(z') - \delta a(z)}{(z' - z)^2} e^{ik(z' - z)} dz'$$

again vanishes at infinity, being a Fourier transform, and that its derivative is the function

$$k \mapsto PV \int_{-\infty}^{+\infty} \frac{\delta a(z') - \delta a(z)}{z' - z} e^{ik(z'-z)} dz'.$$

Therefore a second integration by parts of the expression of  $I_2$  yields

$$I_2 = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} F''(ka) \times \left( PV \int_{-\infty}^{+\infty} \frac{\delta a(z') - \delta a(z)}{(z' - z)^2} e^{ik(z'-z)} dz' \right) adk.$$

The order of integration may now be changed since the function  $F''(p)$ , being zero at infinity as implied by Eq. 39, does admit a Fourier transform unlike  $F(p)$ . One thus gets

$$I_2 = -PV \int_{-\infty}^{+\infty} \frac{\delta a(z') - \delta a(z)}{(z' - z)^2} \times \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} F''(ka) e^{ik(z'-z)} adk \right) dz'$$

or equivalently

$$I_2 = PV \int_{-\infty}^{+\infty} f\left(\frac{z - z'}{a}\right) \frac{\delta a(z') - \delta a(z)}{(z' - z)^2} dz', \tag{42}$$

where  $f(u)$  is a function of the “reduced distance”

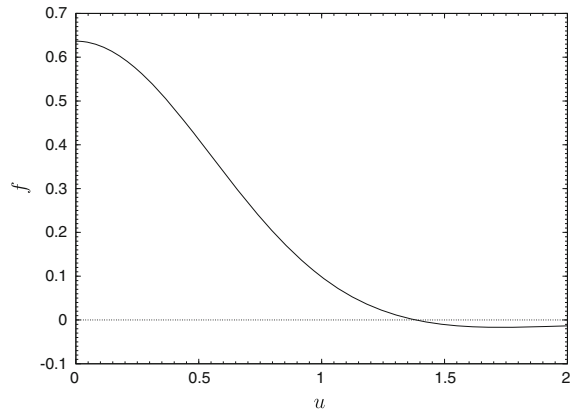
$$u \equiv \frac{z - z'}{a} \tag{43}$$

defined by<sup>7</sup>

$$f(u) \equiv -\frac{1}{2\pi} \int_{-\infty}^{+\infty} F''(p) e^{-ipu} dp = -\frac{1}{\pi} \int_0^{+\infty} F''(p) \cos(pu) dp; \tag{44}$$

note that  $f(u)$  is just in fact the opposite of the inverse Fourier transform of  $F''(p)$ .

<sup>7</sup> Use is been made here of the fact that the function  $F(p)$  being even, see Eq. 37, its second derivative is also even.



**Fig. 6** The function  $f(u)$  for a very thin plate

Combining Eqs. 40, 41 and 42, one finally gets

$$\frac{\delta K(z)}{K} = -2 \frac{\delta a(z)}{a} + PV \int_{-\infty}^{+\infty} f\left(\frac{z - z'}{a}\right) \frac{\delta a(z') - \delta a(z)}{(z' - z)^2} dz' \tag{45}$$

(subject to conditions (7))

with  $f(u)$  given by Eq. 44.

Formula (45) is analogous to that derived by Rice (1985) for perturbation of a semi-infinite crack in an infinite body, Eq. 5, except for two differences. First, the formula involves a term proportional to  $\delta a(z)$  in addition to the integral term, which represents the variation of the stress intensity factor for a uniform advance of the crack front ( $\delta a(z') \equiv \delta a(z), \forall z'$ ).<sup>8</sup> Second, the “influence function” in the integral term is more complex since it involves an extra function  $f((z - z')/a)$ .

Figure 6 provides a graphical representation of the function  $f(u)$  obtained by performing the second integral in Eq. 44 numerically, using an explicit expression of  $F''(p)$  deduced from Eq. 37. The slightly negative values of  $f(u)$  for large values of  $u$  in this figure have been checked to be significant and not arise from numerical errors. Also, one may note the following additional features of this function:

- First, by Eq. 44<sub>2</sub>,

$$f(0) = -\frac{1}{\pi} [F'(+\infty) - F'(0)] = \frac{2}{\pi} \tag{46}$$

<sup>8</sup> In fact a term of this kind does appear in Rice’s (1985) formula but must logically be discarded when this formula is applied to a cracked plate, its contribution becoming negligible then.

where use has been made of the properties  $F'(0) = 0$  (consequence of the fact that the function  $F(p)$  is even) and  $F'(+\infty) = -2$  (consequence of Eq. 39). Note that the value of  $f(0)$  is higher than its equivalent for a semi-infinite crack in an infinite body, that is, the constant prefactor of  $\frac{1}{2\pi}$  in Rice's (1985) formula (5), by a factor of 4. In other words, the influence function is 4 times larger for a thin plate than for an infinite body, when the distance between the points considered is much smaller than  $a$  (but still of course much larger than  $h$ ).

- Second, it has been noted in Sect. 5.4 that the function  $F(p)$  is indefinitely differentiable on the entire real line, and the same is of course true of its second derivative. It then follows from a classical theorem that *the opposite  $f(u)$  of the inverse Fourier transform of  $F''(p)$  vanishes at infinity more quickly than any power function of  $u$* . The behavior at large distances of the influence function  $f((z - z')/a)(z' - z)^{-2}$  in Eq. 45 is thus very different from that of the simple influence function  $(z' - z)^{-2}$  in Eq. 5. Clearly, in the cracked geometry envisaged here, the distance  $a$  from the left boundary to the mean position of the crack front acts as a "cutoff length" above which the influence function quickly vanishes.

## 6 Numerical study for a plate of arbitrary thickness

### 6.1 Generalities

We shall now envisage the general case of a plate of arbitrary thickness. Therefore, neither conditions (4) nor conditions (7) will be imposed. We shall however introduce the hypothesis that

$$h \ll a. \tag{47}$$

Indeed this condition is always fulfilled in practice, and considerable simplifications result from there. Note immediately for instance that with such a hypothesis,  $\lambda \ll h$  suffices to warrant conditions (4) and therefore applicability of Rice's (1985) formulae (5) and (6); whereas  $\lambda \gg h$  suffices to warrant conditions (7) and therefore applicability of formulae (35) and (45).

We shall look for some "interpolation formula", in Fourier's space, between the expression (6), appropriate

for  $\lambda \ll h$ , and the expression (35), appropriate for  $\lambda \gg h$ , in the form

$$\frac{\delta \widehat{K}(k)}{K} = \left[ - (1 - \theta(kh)) \frac{|k|}{2} + \theta(kh) \frac{F(ka)}{a} \right] \widehat{\delta a}(k), \tag{48}$$

where  $\theta(q)$  denotes some even function of the "reduced wavenumber"

$$q \equiv kh \tag{49}$$

to be determined, obeying the properties

$$\theta(0) = 1; \quad \theta(\pm\infty) = 0. \tag{50}$$

Formula (48) does possess the desired features: if  $\lambda \ll h$ ,  $|kh| = 2\pi h/\lambda \gg 1$  so that  $\theta(kh) \simeq 0$  and the expression of  $\delta \widehat{K}(k)/K$  reduces to (6); and if  $\lambda \gg h$ ,  $|kh| = 2\pi h/\lambda \ll 1$ ,  $\theta(kh) \simeq 1$  and it reduces to (35).

Formula (48) may seem to be criticizable on the grounds that since the problem involves three characteristic lengths,  $a$ ,  $h$  and  $\lambda \equiv 2\pi/|k|$ , and thus two dimensionless geometric parameters, for instance  $kh$  and  $ka$ , the function  $\theta$  should depend on both of these parameters instead of just one,  $kh$ . However the dependence of  $\theta$  upon  $ka$  may safely be disregarded: indeed

- if  $\lambda$  is much smaller than  $a$ , then the latter length is much larger than the other two by hypothesis (47); it is thus safe to consider it as infinite and replace  $\theta(kh, ka)$  by  $\theta(kh, \pm\infty)$ ;
- if  $\lambda$  is comparable to  $a$  or larger, then it is necessarily much larger than  $h$  by hypothesis (47); then  $\theta(kh, ka) \simeq 1$  by Eq. (50)<sub>1</sub>, whatever the value of  $ka$ .

Hence with hypothesis (47), one may safely consider the interpolation function  $\theta$  as a function of the sole variable  $kh$ , as assumed in Eq. 48.

### 6.2 Principle of numerical calculations

The function  $\theta(q)$  will be determined by performing finite element elastic computations of cracked plates of finite dimensions. The depth  $a$  of the crack and the length of the unbroken ligament ahead of the crack front will be chosen much larger than the thickness  $h$  of the plate. Two configurations of the front will be considered, one straight and one with slight sinusoidal in-plane undulations of wavelength  $\lambda$ . Because of the

periodicity of the problem in the  $z$  direction, only one slice of material lying between the planes  $z = 0$  and  $z = \lambda$  will be meshed, periodic boundary conditions being imposed on these planes.

The oscillations of the local stress intensity factor along the crack front,  $\delta K(z)$ , will be determined by comparing the values of this factor in the original and perturbed configurations of the front. One test of the relevance of the first-order perturbation approach will be that these oscillations will have to be sinusoidal like those of the local crack advance,  $\delta a(z)$ ; this test will serve to fix a maximum admissible value of the amplitude of variation of  $\delta a(z)$ . For acceptable values of this amplitude, the value of the ratio  $\widehat{\delta K}(k)/\widehat{\delta a}(k)$  ( $k = \pm 2\pi/\lambda$ ) will be deduced from that of the ratio of the amplitudes of variation of  $\delta K(z)$  and  $\delta a(z)$ .

Various values of  $\lambda$  will be considered, all of them comparable to  $h$  and therefore much smaller than  $a$ . Under such conditions  $|ka| \gg 1$  so that  $F(ka)/a \sim -2|k|$  by Eq. 39. Hence Eq. 48 reduces to

$$\frac{\widehat{\delta K}(k)}{K} = -\frac{1}{2} [1 + 3\theta(kh)] |k| \widehat{\delta a}(k) \quad (\text{subject to the condition } |ka| \gg 1), \quad (51)$$

which shows that the function  $\theta(kh)$  is related to the quantity

$$X \equiv -\frac{\widehat{\delta K}(k)/K}{|k| \widehat{\delta a}(k)} \quad (52)$$

through the equivalent relations

$$X = \frac{1}{2} [1 + 3\theta(kh)] \Leftrightarrow \theta(kh) = \frac{2X - 1}{3}. \quad (53)$$

Equations 52 and 53 will serve to determine numerical values of the quantity  $X$  and the function  $\theta(kh)$  from those of the ratio  $\widehat{\delta K}(k)/\widehat{\delta a}(k)$ . Note in particular that by Eqs. 50 and 53,  $X$  must take the values 2 and  $\frac{1}{2}$  in the limits  $|kh| \ll 1$  ( $\lambda \gg h$ ) and  $|kh| \gg 1$  ( $\lambda \ll h$ ) respectively.

### 6.3 Presentation of numerical calculations

All computations are performed with the finite element code CAST3M developed by the French *Commissariat à l'Énergie Atomique* (CEA).

Two types of computations are performed with different boundary conditions on the upper and lower halves of the left boundary of the cracked plate: opposite vertical displacements versus opposite bending

moments. Boundary conditions of the first type precisely correspond to those employed in actual experiments. Boundary conditions of the second type are representative of those prevailing on the left side of a rectangular subregion of the plate of dimensions much smaller than  $a$ , though much larger than  $h$ .<sup>9</sup> (That dimensions satisfying these two conditions may be found is a consequence of hypothesis (47)). Hence the two types of conditions are expected to yield identical results for sufficiently large plates; this will serve as a test that the dimensions considered are sufficient.

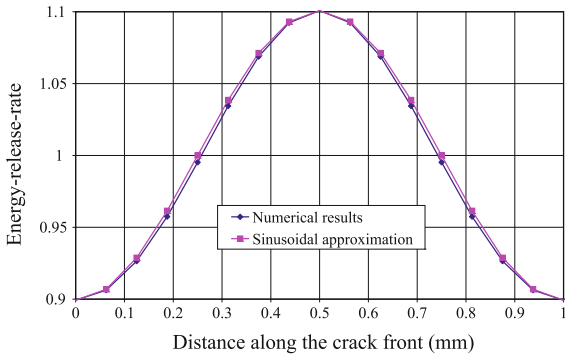
The two types of computations also use different dimensions, meshes and types of elements, although the plate thickness is 1 mm in all cases. In these involving prescribed displacements, the distance  $a$  between the left side and the unperturbed crack front is 100 mm or 200 mm, depending on whether  $\lambda$  is smaller or larger than 10 mm; the mesh consists of 449,760 (for  $\lambda < 10$  mm) or 646,440 (for  $\lambda > 10$  mm) bilinear 8-node parallelepipedic elements and 471,713 or 688,568 nodes. In those involving prescribed moments, which require a smaller mesh since they focus on a subregion of the plate, the distance  $a$  between the left side and the unperturbed front is 50 mm, and the mesh consists of 182,400 bilinear 6-node prismatic elements and 106,896 nodes. In both cases the elements are strictly parallelepipedic or prismatic only for the unperturbed straight configuration of the front; for the perturbed configuration the same mesh is used with a slight distortion of elements.

The value of Poisson's ratio is the same in all cases, namely  $\nu = 0.3$ .

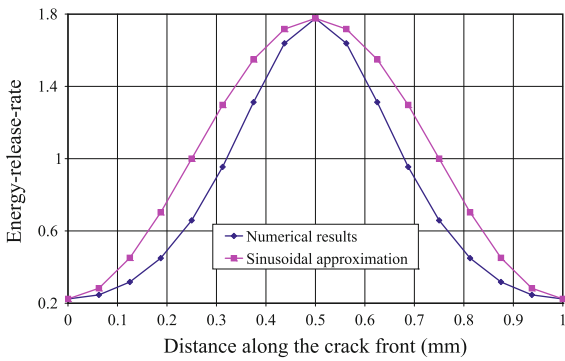
Local values of the stress intensity factor along the crack front are evaluated using Destuynder et al. (1983)'s "G -  $\theta$  method", the accuracy of which is well established.

In a first step, a straight front is considered. The unperturbed stress intensity factors calculated for the two types of boundary conditions are compared to the exact analytic values provided by Tada et al. (2000), pp. 417 and 419, for the problems in question. The relative error found decreases from about 10% for  $a/h = 10$  to less than 0.1% for the largest values of this ratio. (The explanation of the relatively large difference for the smallest value of  $a/h$  does not lie in numerical

<sup>9</sup> It is implicitly assumed here that the crack advance does not induce any variation of the bending moments exerted on the two halves of the left side; this is true in all calculations since  $\lambda \ll a$ .



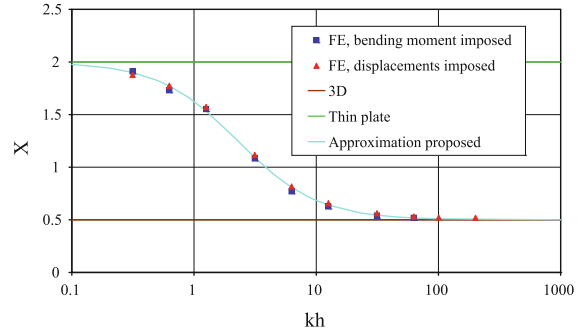
**Fig. 7** Distribution of the normalized energy-release-rate along the crack front for  $\delta a_{\max}/\lambda = 0.01$



**Fig. 8** Distribution of the normalized energy-release-rate along the crack front for  $\delta a_{\max}/\lambda = 0.1$

errors but in the fact that Tada et al. (2000)’s solutions are asymptotic ones in the limit  $a/h \rightarrow +\infty$ ).

In a second step, wavy fronts of sinusoidal shape with different amplitudes  $\delta a_{\max}$  are considered. Figures 7 and 8 show, in the typical case where  $\lambda = h$ , the distribution of the local energy-release-rate, normalized by half the sum of its maximum and minimum values, along the crack front, for values of  $\delta a_{\max}$  equal to 1 and 10% of  $\lambda$ . In each case the blue curve represents numerical results and the red one the sinusoidal approximation of these results having the same minima and maxima. The oscillation of the energy-release-rate can be observed to be sinusoidal for the smaller amplitude, but not so for the larger one. This shows that the linear perturbation approach is applicable only to the smaller one, which is therefore adopted in subsequent calculations.



**Fig. 9** The quantity  $X$  for a plate of arbitrary thickness

### 6.4 Results

Figure 9 shows numerical values of the quantity  $X$  defined by Eq. 52, as a function of  $kh$ . (Only positive values of  $kh$  are considered here since the function  $X(kh)$  is obviously even). The two types of boundary conditions can be observed to yield identical results, which shows that the dimensions of the plates considered are large enough. Also, the quantity  $X$  goes to the values 2 (thin plate solution) and  $\frac{1}{2}$  (3D solution) in the limits  $kh \rightarrow 0$  and  $kh \rightarrow +\infty$  respectively, as expected.

Figure 9 shows that a decrease of  $kh$  from values much larger than unity to values much smaller than it, or equivalently an increase of  $\lambda$  from values much smaller than  $h$  to values much larger than it, induces an increase of  $X$ . This clearly shows that in contrast to the distance between the crack front and the boundary of the plate, the plate thickness does not play the role of a “cut-off length” above which the ratio  $\widehat{\delta K}(k)/\widehat{\delta a}(k)$  should quickly decrease. (There is in fact nothing surprising in this conclusion: in Love-Kirchhoff’s thin plate theory, the deformation of a plate is never confined by the thickness but by the boundary conditions).

It is also shown in Fig. 9 that the numerical results may be accurately represented by the following very simple approximate formula for the function  $\theta(q)$ , which is therefore adopted hereafter:

$$\theta(q) \simeq \frac{1}{1 + \frac{1}{3}|q|^{4/3}}. \tag{54}$$

### 6.5 Final formulae in Fourier’s space and the physical space

We shall now provide final formulae for the variation of the stress intensity factor, distinguishing between the

cases  $a < +\infty$  and  $a = +\infty$  (which often represents an acceptable approximation in actual experiments). The formulae will be parametrized by the dimensionless quantity

$$\xi \equiv \frac{a}{h} \quad (\gg 1 \text{ by condition (47)}). \tag{55}$$

In Fourier’s space, formula (48) for  $\widehat{\delta K}(k)$  may be written in the form

$$\frac{\widehat{\delta K}(k)}{K} = G\left(\frac{a}{h}; kh\right) \frac{\widehat{\delta a}(k)}{h} \tag{56}$$

(subject to condition (47))

where

$$G(\xi; q) \equiv \begin{cases} -[1 - \theta(q)] \frac{|q|}{2} + \theta(q) \frac{F(\xi q)}{\xi} & \text{if } \xi < +\infty \\ -\frac{1}{2} [1 + 3\theta(q)] |q| & \text{if } \xi = +\infty. \end{cases} \tag{57}$$

In Eq. 57, the expression of the function  $\theta(q)$  is given by Eq. 54, and that of the function  $F(p)$  by Eq. 37 (or 39) when  $\xi$  is infinite).

The following special features of the function  $G(\xi; q)$  are of interest:

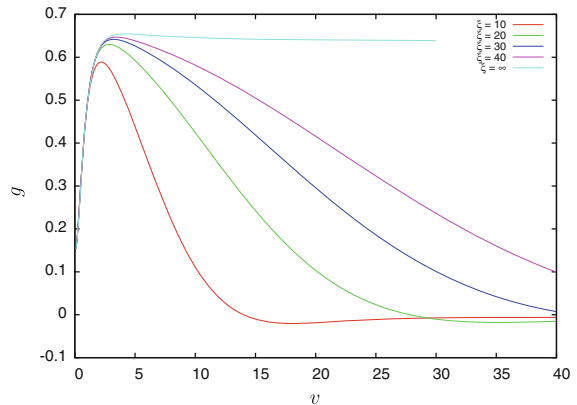
$$G(\xi; 0) = -\frac{2}{\xi}; \quad G(\xi; q) \sim -\frac{|q|}{2} \quad \text{for } q \rightarrow \pm\infty. \tag{58}$$

Of course, the values of  $G(\xi; \cdot)$  at the origin and near infinity are connected to those of  $\widehat{\delta K}(k)$  for crack front perturbations of very large wavelength in a thin plate (Eqs. 35 and 38), and of very small wavelength in an infinite body (Eq. 6), respectively.

In the physical space, the formula for  $\delta K(z)$  may be obtained through Fourier inversion of that for  $\widehat{\delta K}(k)$ . The reasoning is completely analogous to that presented in Sect. 5.5 for the case of a very thin plate (the only difference being that the reference length used here is  $h$  instead of  $a$ ), and therefore will not be repeated. The resulting formula reads

$$\frac{\delta K(z)}{K} = -2 \frac{\delta a(z)}{a} + PV \int_{-\infty}^{+\infty} g\left(\frac{a}{h}; \frac{z - z'}{h}\right) \frac{\delta a(z') - \delta a(z)}{(z' - z)^2} dz' \tag{59}$$

where  $g(\xi; v)$  is a function of the dimensionless parameter  $\xi$  and the “reduced distance”



**Fig. 10** The function  $g(\xi; v)$  for a plate of arbitrary thickness

$$v \equiv \frac{z - z'}{h} \tag{60}$$

defined by

$$g(\xi; v) \equiv -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\partial^2 G}{\partial q^2}(\xi; q) e^{-iqv} dq = \begin{cases} -\frac{1}{\pi} \int_0^{+\infty} \frac{\partial^2 G}{\partial q^2}(\xi; q) \cos(qv) dq & \text{if } \xi < +\infty \\ \frac{2}{\pi} - \frac{1}{\pi} \int_0^{+\infty} \frac{\partial^2 G}{\partial q^2}(+\infty; q) \cos(qv) dq & \text{if } \xi = +\infty. \end{cases} \tag{61}$$

The extra term  $\frac{2}{\pi}$  in the final expression of  $g(+\infty; v)$  arises from the fact that the function  $\partial G/\partial q(+\infty; q)$  being discontinuous at the point  $q = 0$ , with  $\partial G/\partial q(+\infty; 0^\pm) = \mp 2$  (see Eqs. 50<sub>1</sub> and 57<sub>2</sub>), its derivative  $\partial^2 G/\partial q^2(+\infty; q)$  in the integral from  $-\infty$  to  $+\infty$  must be understood in the sense of generalized functions, with an additional term  $-4\delta(q)$ .

Figure 10 shows the function  $g(\xi; v)$  obtained numerically from Eq. 61<sub>2</sub> for various values of  $\xi$ , including infinity.<sup>10</sup> Again, like in Figure 6, the slightly negative values of  $g(\xi; v)$  for moderate  $\xi$  and large  $v$  have been checked to be significant. The numerical results may also be supplemented by analytical ones in some special cases. First, at the origin, by Eq. 61,

$$g(\xi; 0) = \begin{cases} -\frac{1}{\pi} \left[ \frac{\partial G}{\partial q}(\xi; +\infty) - \frac{\partial G}{\partial q}(\xi; 0^+) \right] & \text{if } \xi < +\infty \\ \frac{2}{\pi} - \frac{1}{\pi} \left[ \frac{\partial G}{\partial q}(+\infty; +\infty) - \frac{\partial G}{\partial q}(+\infty; 0^+) \right] & \text{if } \xi = +\infty \end{cases} = \frac{1}{2\pi} \text{ in all cases} \tag{62}$$

<sup>10</sup> The curve corresponding to  $\xi = 10$  is shown for illustrative purposes but involves some error since the necessary hypothesis (47) is barely satisfied for such a value.

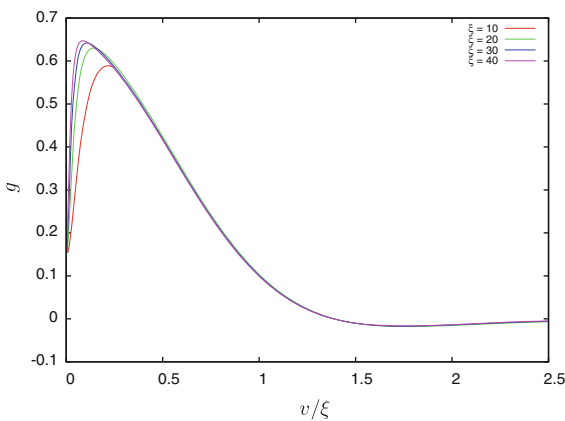
where use has been made of the properties  $\partial G/\partial q(\xi; +\infty) = -\frac{1}{2}$  in all cases, whereas  $\partial G/\partial q(\xi; 0^+) = 0$  or  $-2$  depending on whether  $\xi < +\infty$  or  $\xi = +\infty$ . Equation 62 means that the influence function  $g(\frac{a}{h}; \frac{z-z'}{h})(z'-z)^{-2}$  in Eq. 59 behaves like  $\frac{1}{2\pi(z'-z)^2}$  at short distances, like in an infinite body, of course (see Eq. 5). Also,

$$\lim_{v \rightarrow \pm\infty} g(\xi; v) = \begin{cases} 0 & \text{if } \xi < +\infty \\ \frac{2}{\pi} & \text{if } \xi = +\infty \end{cases} \quad (63)$$

since the integrals in Eq. 61<sub>2</sub> go to zero when  $v$  goes to  $\pm\infty$ . All of these features agree with what can be observed in Fig. 10.

It is easy to understand the behavior of the function  $g(\xi; v)$ . When the distance  $|z - z'|$  increases from values much smaller than  $h$  to values much larger than it, but still much smaller than  $a$  (that is, when the normalized distance  $|v|$  increases from values much smaller than unity to values much larger than it, but still much smaller than  $\xi$ ), this function increases from the value for an infinite body,  $\frac{1}{2\pi}$ , to that for a thin plate,  $\frac{2}{\pi}$ . However, since the distance  $a$  has been observed to play the role of a cutoff length, when  $|z - z'|$  becomes comparable to or much larger than it (that is, when  $|v|$  becomes comparable to or much larger than  $\xi$ ), the function decreases back to zero. Of course, the larger the value of  $a$  or  $\xi$ , the slower the decrease; for  $a = +\infty$  or  $\xi = +\infty$ , there is no decrease at all.

The role of cutoff length played by the distance  $a$  is probably best illustrated by Fig. 11, which shows  $g(\xi; v)$  as a function of the variable  $v/\xi \equiv (z - z')/a$ . (The use of this new variable obviously forbids to consider the case  $\xi = +\infty \Leftrightarrow a = +\infty$ ). This figure



**Fig. 11** Plot of  $g(\xi; v)$  versus  $v/\xi \equiv (z - z')/a$

clearly shows that the decrease of  $g(\xi; v)$  occurs above some critical distance comparable to  $a$ .

### 7 Conclusion

The aim of this paper was to provide a first-order formula for the local variation of the mode I stress intensity factor resulting from some small but otherwise arbitrary coplanar perturbation of the front of a planar crack, in the specific case where this crack results from gradual separation of two bonded plates from one side. This configuration is frequently used in experiments, and it was our hope to provide an alternative to Rice’s (1985) formula for the theoretical interpretation of such experiments. Indeed, although this formula has systematically been used for this purpose up to now, it was derived for a semi-infinite crack in some infinite body, and thus applies to the configuration considered only when the characteristic distance of variation of the perturbation of the front is much smaller than the plate thickness.

In a first step, Rice (1985)’s treatment, appropriate for infinitely thick plates, was adapted to the other extreme case of plates of infinitesimal thickness. This was done by replacing the standard 3D theory of elasticity he used by Love-Kirchhoff’s theory of plates. The validity of this procedure is ensured by the fact that the latter theory is known to be *the* correct asymptotic one for very thin plates. An interesting outcome of the analysis is that the distance between the crack front and the boundary of the plate plays the role of a “cutoff length” for elastic interactions along the crack front; that is, the influence of the crack advance at some point of the front upon the stress intensity factor at some other point quickly vanishes when the separation between these points becomes larger than this distance. The plate thickness, on the other hand, plays no similar role.

In a second step, the analytical formulae applicable to extreme, infinite or infinitesimal, values of the plate thickness, were completed by finite element computations for a number of intermediate values. In addition to confirming the validity of the analytical formulae for the two limit cases, these computations allowed us to define an approximate but accurate formula for the local variation of the stress intensity factor applicable to plates of arbitrary thickness.

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