Flexural modes of a clamped-free beam

The equation for the flexural dynamics of the beam is

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = 0$$

with boundary conditions $\forall t$,

$$y(0,t) = \frac{\partial}{\partial x} y(0,t) = \frac{\partial^2}{\partial x^2} y(L,t) = \frac{\partial^3}{\partial x^3} y(L,t) = 0$$

We look for special solution of the form $y(x,t) = Y(x)\cos(\omega t)$, where ω (modal pulsation) is unknown.

This leads to

$$EI\frac{d^4 Y}{dx^4} - \rho A \omega^2 \, Y = 0 \tag{1}$$

This is a linear Ordinary Differential equation, but with boundary conditions not both at $x = 0$ and at $x = 1$.

$$Y(0) = Y'(0) = 0 \qquad Y''(1) = Y'''(1) = 0$$

Solutions of equation (1) are all of the form

$$Y(x) = A\cos(kx) + B\sin(kx) + C\cosh(kx) + D\sinh(kx)$$

where

$$k = \left(\frac{\rho A \omega^2}{EI}\right)^{1/4}$$

The first boundary condition show that $C = -A$ and $D = -B$ so that we are looking for

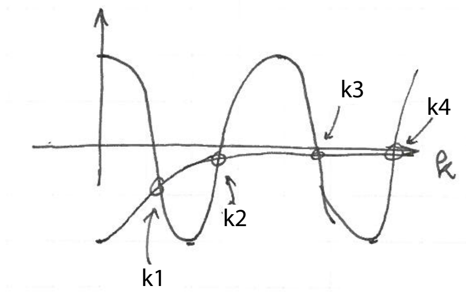
$$Y(x) = A\left[\cos kx - \sin kx\right] + C\left[\cosh kx - \sinh kx\right]$$

that satisfies the 2 remaining boundary conditions at $x = L$, which can be rewritten in the form of a hierarchy of 2 linear equations on A, B or the form of a matrix equation

$$k^2 \begin{pmatrix} -\cos kL - \cosh kL & -\sin kL - \sinh kL \\ k(\sin kL - \sinh kL) & -k(\cos kL + \cosh kL) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{2}$$

A non zero solution exists only if the determinant of this matrix is zero, which occurs only for certain values of k (and ω) : when

$$\cos kL \cosh kL = -1 \tag{3}$$



This equation, or $\cos kL = -1/\cosh kL$, has an infinite (discrete) number of solution k_n, as can be seen graphically. Each correspond to a mode. For large n, we have the approximation $k_n \sim (n - \frac{1}{2})\pi/L$.

Going back to matrix equation (2), when $k = k_n$, (and $\omega = \omega_n$) the determinant of the matrix is zero, so that the two lines of the matrix are proportional (the system is degenerate) and $A(-\cos k_n L - \cosh k_n L) + B(-\sin k_n L - \sinh k_n L) = 0$, This is the relation beteween A and B for the non-zero solutions of this system (the mode)

Finally the nth flexural mode is proportional to

$$Y_n(x) = [\cos k_n x - \cosh k_n x] - \left(\frac{\cos k_n L \cosh k_n L}{\sin k_n L \sinh k_n L}\right) [\sin k_n x - \sinh k_n x]$$

with k_n is the nth root of $\cos kL \cosh kL = -1$, and $\omega_n^2 = \frac{EI}{\rho A} k_n^4$