Soft Solids – Lecture 3

3 ECTS - 35h
Wednesday : 2 PM – 5:30 PM
ENS – Room L363

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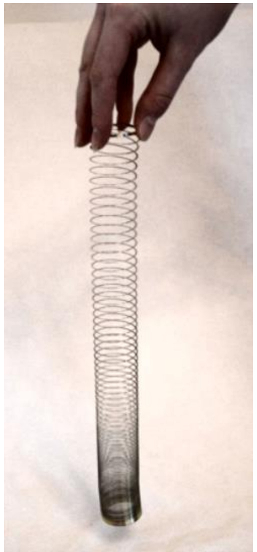
WE HAVE SEEN… 1D STRETCHABLE SOLIDS

- **Strain & stretch** $x' = \mathcal{F}(x)$ $\varepsilon(x) = \dfrac{d\ell' - d\ell}{d\ell} = \dfrac{du}{dx}(x) = \mathcal{F}'(x) - 1$

$$\lambda(x) \equiv \frac{d\ell'}{d\ell}(x) = \mathcal{F}'(x) = 1 + \varepsilon(x)$$



- **Stress**

$$\sigma(x) = \frac{F(x)}{S}$$

- **The FPD**

$$\frac{d\sigma}{dx}(x) + f(x) = \rho a(x)$$

- **Hooke's elastic law**

$$\sigma = \frac{d\mathcal{U}_{el}}{d\varepsilon}(\varepsilon) = E\varepsilon$$

- **The elastic energy**

$$\mathcal{U}_{el} = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\frac{\sigma^2}{E}$$

- **Elasto-dynamic equation:**

$$E\frac{d^2u}{dx^2}(x) + f(x) = \rho\frac{d^2u}{dx^2}$$

- **Waves and oscillations**

$$c = \sqrt{\frac{E}{\rho}} \qquad t^* \sim \frac{L}{c}$$

TODAY'S MENU: 3D STRETCHABLE SOLIDS

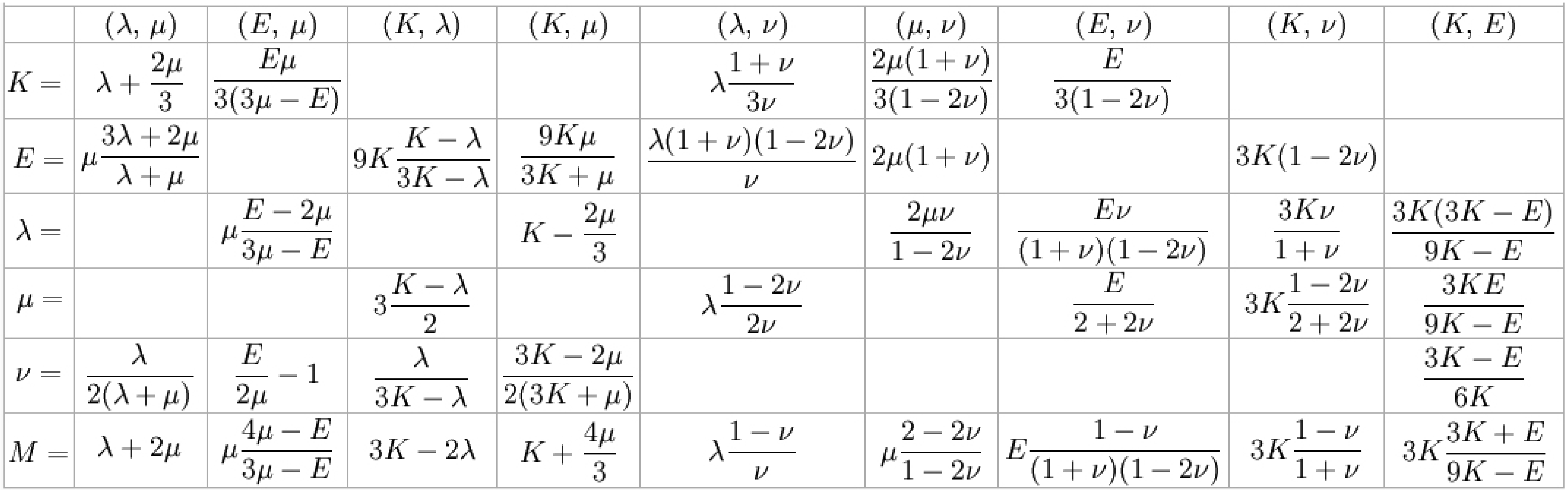
- **Strain & stress** $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\vec{T}(\vec{x}) = \bar{\bar{\sigma}}(\vec{x})\,\vec{n}$
- **The FPD** $\overrightarrow{\mathrm{div}}\,\bar{\bar{\sigma}} + \vec{f} = \rho\,\vec{a}$
- **Hooke's elastic law** $\bar{\bar{\sigma}} = \lambda(\mathrm{Tr}\,\bar{\bar{\varepsilon}})\bar{\bar{I}} + 2\mu\bar{\bar{\varepsilon}}$
- **The elastic energy** $\mathcal{U}_{el}(\bar{\bar{\varepsilon}}) = \frac{\lambda}{2}(\mathrm{Tr}\,\bar{\bar{\varepsilon}})^2 + \mu\mathrm{Tr}\,(\bar{\bar{\varepsilon}}^2)$
- **Variational principle** $dU_p(\vec{u}) = dU_{el}(\vec{u}) - dW(\vec{u}) = 0$
- **Lamé-Navier equation:** $\rho\frac{\partial^2\vec{u}}{\partial t^2} = \vec{f} + \mu\,\Delta\vec{u} + \left(K + \frac{\mu}{3}\right)\overrightarrow{\mathrm{grad}}(\mathrm{div}\,\vec{u})$
- **Analogy with Navier Stokes equation:**

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho\,\vec{v}\cdot\bar{\bar{\mathrm{grad}}}\,\vec{v} = \rho\,\vec{g} - \overrightarrow{\mathrm{grad}}\,p + \eta\,\Delta\vec{v} + \left(\kappa + \frac{\eta}{3}\right)\overrightarrow{\mathrm{grad}}\,(\mathrm{div}\,\vec{v})$$

MODULES ELASTIQUES - MATERIAU ISOTROPE

Deux modules au choix entre les suivants:

K	Module de compressibilité
E	Module d'Young – Traction uniaxiale
ν	Rapport de Poisson – Contraction latérale en traction uniaxiale
λ	Premier coefficient de Lamé
μ	Second coefficient de Lamé – Module de cisaillement
M	Module oedométrique (Compression confinée, onde compression)



	$(\lambda,\,\mu)$	$(E,\,\mu)$	$(K,\,\lambda)$	$(K,\,\mu)$	$(\lambda,\,\nu)$	$(\mu,\,\nu)$	$(E,\,\nu)$	$(K,\,\nu)$	$(K,\,E)$
$K=$	$\lambda+\frac{2\mu}{3}$	$\frac{E\mu}{3(3\mu-E)}$			$\lambda\frac{1+\nu}{3\nu}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{E}{3(1-2\nu)}$		
$E=$	$\mu\frac{3\lambda+2\mu}{\lambda+\mu}$		$9K\frac{K-\lambda}{3K-\lambda}$	$\frac{9K\mu}{3K+\mu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$2\mu(1+\nu)$		$3K(1-2\nu)$	
$\lambda=$		$\mu\frac{E-2\mu}{3\mu-E}$		$K-\frac{2\mu}{3}$		$\frac{2\mu\nu}{1-2\nu}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{3K\nu}{1+\nu}$	$\frac{3K(3K-E)}{9K-E}$
$\mu=$			$3\frac{K-\lambda}{2}$		$\lambda\frac{1-2\nu}{2\nu}$		$\frac{E}{2+2\nu}$	$3K\frac{1-2\nu}{2+2\nu}$	$\frac{3KE}{9K-E}$
$\nu=$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{E}{2\mu}-1$	$\frac{\lambda}{3K-\lambda}$	$\frac{3K-2\mu}{2(3K+\mu)}$					$\frac{3K-E}{6K}$
$M=$	$\lambda+2\mu$	$\mu\frac{4\mu-E}{3\mu-E}$	$3K-2\lambda$	$K+\frac{4\mu}{3}$	$\lambda\frac{1-\nu}{\nu}$	$\mu\frac{2-2\nu}{1-2\nu}$	$E\frac{1-\nu}{(1+\nu)(1-2\nu)}$	$3K\frac{1-\nu}{1+\nu}$	$3K\frac{3K+E}{9K-E}$

MODULES D’YOUNG ET RAPPORT DE POISSON

$$E = \frac{\sigma}{\varepsilon_{\parallel}}$$

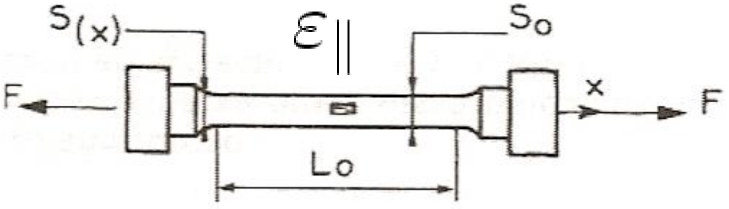
Module d’Young

$$\nu = -\frac{\varepsilon_{\perp}}{\varepsilon_{\parallel}}$$

Rapport de Poisson

$$-1 < \nu < 1/2$$

Traction uniaxiale



$$\left\{\begin{aligned} \varepsilon_{11} &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] \\ \varepsilon_{22} &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] \\ \varepsilon_{33} &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] \\ \varepsilon_{12} &= \frac{\sigma_{12}}{2\mu} \qquad \varepsilon_{13} = \frac{\sigma_{13}}{2\mu} \qquad \varepsilon_{23} = \frac{\sigma_{23}}{2\mu} \end{aligned}\right.$$

$$\overline{\overline{\varepsilon}} = \frac{1+\nu}{E}\overline{\overline{\sigma}} - \frac{\nu}{E}\mathrm{Tr}(\overline{\overline{\sigma}})\ \overline{\overline{I}}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

MODULES DE COMPRESSIBILITE ET CISAILLEMENT

Décomposition en sphérique et déviateur

$$\varepsilon_{ij} = \frac{\varepsilon_{kk}}{3}\delta_{ij} + \left(\varepsilon_{ij} - \frac{\varepsilon_{kk}}{3}\delta_{ij}\right)$$

$$\overline{\overline{\varepsilon}} = \overline{\overline{S}}_{\varepsilon} - \overline{\overline{D}}_{\varepsilon} = \varepsilon_m \overline{\overline{\delta}} + \varepsilon_d \overline{\overline{\pi}}_{\varepsilon}$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} = K\varepsilon_{kk}\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{\varepsilon_{kk}}{3}\delta_{ij}\right)$$

$$K = \lambda + \frac{2}{3}\mu$$

Module de compressibilité

$$\overline{\overline{\sigma}} = \overline{\overline{S}}_{\sigma} + \overline{\overline{D}}_{\sigma} = \sigma_m \overline{\overline{\delta}} + \sigma_d \overline{\overline{\pi}}_{\sigma}$$

$$\boxed{\overline{\overline{S}}_{\sigma} = 3K\overline{\overline{S}}_{\varepsilon} \qquad \overline{\overline{D}}_{\sigma} = 2\mu\overline{\overline{D}}_{\varepsilon} \qquad \overline{\overline{\pi}}_{\sigma} = \overline{\overline{\pi}}_{\varepsilon}}$$

Découplage

$$\sigma_m = 3K\varepsilon_m = K(dV/V)$$

Module de compressibilité

$$K = \frac{-dP}{\frac{dV}{V}} = -V\frac{dP}{dV}$$

$$\sigma_d = 2\mu\varepsilon_d$$

Module de cisaillement

ELASTICITE' 2D (SYMETRIE EN x_3)

1) Déformation plane (objets très épaisses en x_3)

$u_3 = 0 \qquad u_1, u_2$ indep. de $x_3 \qquad \varepsilon_{ij} = 1/2(\partial u_i/x_j + \partial u_j/x_i)$

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ indépendant de x_3

$$\varepsilon_{33} = 1/E(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})) = 0$$

Hooke

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

$$\overline{\overline{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \overline{\overline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \nu(\sigma_{11} + \sigma_{22}) \end{pmatrix}$$

$$\begin{cases} \varepsilon_{11} = \frac{1}{E'}(\sigma_{11} - \nu'\sigma_{22}) \\ \varepsilon_{22} = \frac{1}{E'}(\sigma_{22} - \nu'\sigma_{11}) \\ \varepsilon_{12} = \frac{\sigma_{12}}{2\mu} \end{cases} \qquad E' = \frac{E}{1-\nu^2} \qquad \nu' = \frac{\nu}{1-\nu}$$

Modules effectifs

ELASTICITE' 2D (SYMETRIE EN x3)

2) Contrainte plane (objets très fins en x3)

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

$$\sigma_{11}, \sigma_{22}, \sigma_{12} \quad \text{indépendant de } x_3$$

$$\begin{cases} \varepsilon_{11} = \dfrac{1}{E}(\sigma_{11} - \nu\sigma_{22}) \\ \varepsilon_{22} = \dfrac{1}{E}(\sigma_{22} - \nu\sigma_{11}) \\ \varepsilon_{12} = \dfrac{\sigma_{12}}{2\mu} \qquad \mu = \dfrac{E}{2(1+\nu)} \end{cases}$$

Hooke

$$\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) = -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22})$$

$$\varepsilon_{23} = 0 \qquad \varepsilon_{13} = 0$$

$$\overline{\overline{\sigma}} = \begin{pmatrix} \boxed{\begin{matrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 0 \end{pmatrix} \qquad \overline{\overline{\varepsilon}} = \begin{pmatrix} \boxed{\begin{matrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22}) \end{pmatrix}$$

$$E' = E \qquad \nu' = \nu$$

Modules effectifs

3.5.1 L'EQUATION DE LAMÉ-NAVIER

Equation élasto-dynamique pour un milieu élastique linéaire isotrope:

$$\begin{cases} \overrightarrow{\mathrm{div}}\ \overline{\overline{\sigma}} + \overrightarrow{f} - \rho\overrightarrow{\gamma} = 0 & \text{Loi fondamentale de la dynamique} \\ \overline{\overline{\sigma}} = \lambda(\mathrm{Tr}\ \overline{\overline{\varepsilon}})\overline{\overline{I}} + 2\mu\overline{\overline{\varepsilon}} & \text{Loi de Hooke} \\ \overline{\overline{\varepsilon}} = \dfrac{\overrightarrow{\overrightarrow{\mathrm{grad}}}\overrightarrow{u} + {}^{t}\,\overrightarrow{\overrightarrow{\mathrm{grad}}}\overrightarrow{u}}{2} & \text{Déformation en termes de déplacement} \end{cases}$$

$$(\lambda+\mu)\ \overrightarrow{\mathrm{grad}}(\mathrm{div}\overrightarrow{u}) + \mu\ \Delta\overrightarrow{u} + \overrightarrow{f} = \rho\frac{\partial^2\overrightarrow{u}}{\partial t^2}$$

Une expression alternative de la même équation:

$$(\lambda+\mu)\ \overrightarrow{\mathrm{rot}}(\overrightarrow{\mathrm{rot}}\,\overrightarrow{u}) + (\lambda+2\mu)\Delta\overrightarrow{u} + \overrightarrow{f} = \rho\frac{\partial^2\overrightarrow{u}}{\partial t^2}$$

3.5.2 ONDES DE VOLUME (λ << L*)

En l'absence de forces de volume on peut propager deux types d'ondes:

$$(\lambda+\mu)\ \overrightarrow{\mathrm{rot}}(\overrightarrow{\mathrm{rot}}\,\vec{u}) + (\lambda+2\mu)\Delta\vec{u} + \cancel{\vec{f}} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \qquad \vec{f} = \vec{0}$$

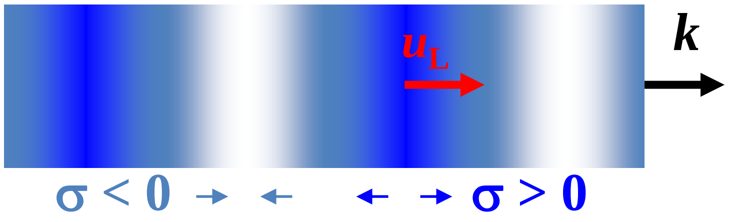
$$(\lambda+\mu)\ \overrightarrow{\mathrm{grad}}(\mathrm{div}\,\vec{u}) + \mu\ \Delta\vec{u} + \cancel{\vec{f}} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \qquad \vec{u} = \vec{u}_L + \vec{u}_T$$

Ondes **L**ongitudinales (compression)

$$\mathrm{div}\,\vec{u}_L \neq 0 \qquad \overrightarrow{\mathrm{rot}}\,\vec{u}_L = 0$$

$$(\lambda+2\mu)\Delta\vec{u_L} = \rho\frac{\partial^2\vec{u_L}}{\partial t^2}$$

$$V_L = \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{\frac{E'}{\rho}} \qquad E' = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = M$$

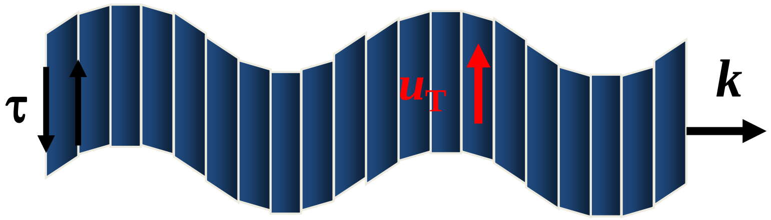


Ondes **T**ransversales (cisaillement)

$$\mathrm{div}\,\vec{u}_T = 0 \qquad \overrightarrow{\mathrm{rot}}\,\vec{u}_T \neq 0$$

$$\mu\Delta\vec{u_T} = \rho\frac{\partial^2\vec{u_T}}{\partial t^2}$$

$$V_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E'}{\rho}} \qquad E' = \frac{E}{2(1+\nu)}$$



u Polarisation ***k*** Vecteur d'onde ***V*** Célérité de l'onde

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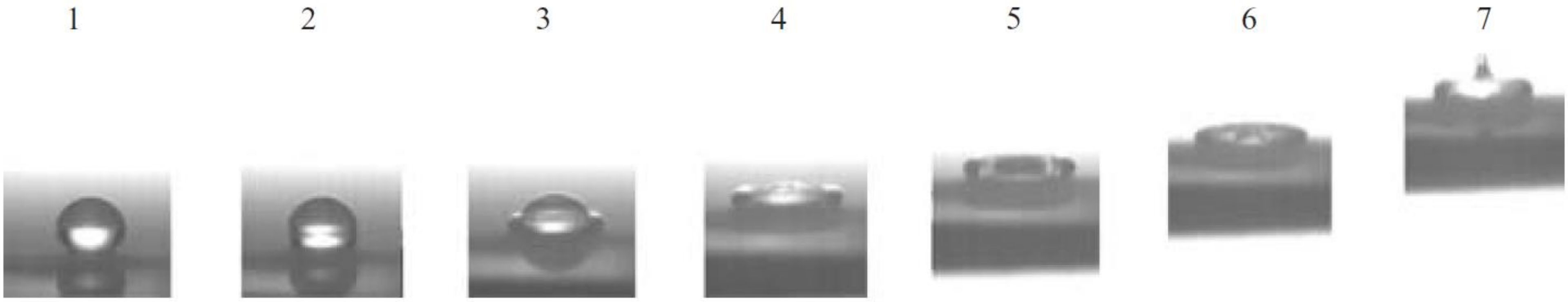
IMPACT ON A WATER DROP



Clanet et al., J. Fluid Mech (2004) 517, 199