THE AVERAGE TRANSPORT PATH LENGTH IN SCATTERING MEDIA*

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Abstract. For connected regions of arbitrary shape containing a distribution of scattering centers that need not be homogeneous, the average path length for particles that traverse the medium is shown to be unaffected by the presence of the scattering centers, provided that the incidence of the particles is uniform and isotropic, that no absorption or particle reproduction occurs in the medium, and that the scattering probabilities satisfy detailed balance.

1. Introduction. The study of the number of reactions that occur in a volume of material subjected to a uniform and isotropic inflow of particles played a significant role in the early development of neutron transport theory [1]–[4]. While involved in such studies Dirac [5] derived, by means of simple physical arguments, an elegant geometrical theorem that the average chord length for a connected region of volume V and surface area S is equal to $4V/S$. For lumps of absorbing material in which little scattering occurs, such as fuel elements in a fast reactor, this result can be used to estimate the probability of absorption of neutrons which enter the volume V from outside. Also, by means of a reciprocity theorem, the escape probability for neutrons which are created in the volume V can be estimated.

In this note we present a generalization of Dirac's result to allow for scattering. We show that the average path length for particles traversing a region of volume V in which they are deflected by an arbitrary array of scattering centers is also equal to $4V/S$ in a wide range of situations, provided that the particle inflow is uniform and isotropic. The physics underlying this generalization is essentially unchanged, however the mathematical statement of this theorem becomes more complex since we must define an ensemble of random paths rather than a set of chords.

2. The average-chord-length theorem. Consider a connected region of volume V and total surface area S. For simplicity of discussion we will assume that the region is convex with a single surrounding surface, but these restrictions are not necessary for our results. We will denote a point on the surface by A and a surface element by dS. Let $l(A, \boldsymbol{\Omega})$ denote the length of the chord which extends in V from A in the direction $\boldsymbol{\Omega}$, and let $\mathbf{n}$ denote the direction of the inward normal at A. (Note that $l = 0$ unless $\boldsymbol{\Omega} \cdot \mathbf{n} > 0$). Then the average chord length is defined to be

$$(1) \qquad \bar{l} = \frac{\int dS \int d\boldsymbol{\Omega}(\mathbf{n} \cdot \boldsymbol{\Omega})l(A, \boldsymbol{\Omega})}{\int dS \int d\boldsymbol{\Omega}(\mathbf{n} \cdot \boldsymbol{\Omega})} \equiv \frac{N}{D}.$$

The weighting factor $\mathbf{n} \cdot \boldsymbol{\Omega}$ implies that the spacing between the chords with a given direction $\boldsymbol{\Omega}$ is independent of the orientation of the surface elements that are intersected by the chords. To avoid double counting and ensure a nonnegative weighting function, the integration over $\boldsymbol{\Omega}$ is restricted to those directions for which $\boldsymbol{\Omega} \cdot \mathbf{n} > 0$.

The numerator in (1) can be evaluated by noting that the integrand represents the volume of a column within the volume V erected on the base dS with height l (see Fig. 1). Thus

$$(2) \qquad \int dS\, (\mathbf{n} \cdot \boldsymbol{\Omega})l(A, \boldsymbol{\Omega}) = V$$
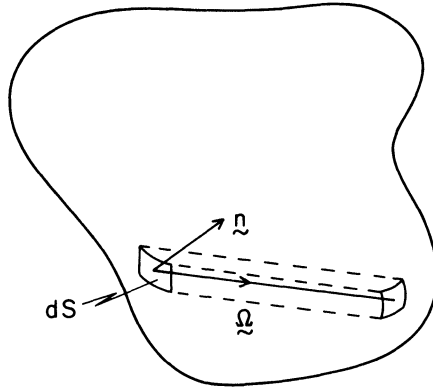
FIG. 1. *Each element of surface dS and chord, drawn along the direction* $\mathbf{\Omega}$, *defines a column with the volume V. The vector* **n** *is perpendicular to the surface.*

for all $\mathbf{\Omega}$, and $N = 4\pi V$. In the denominator,

$$(3) \qquad \int d\mathbf{\Omega}\,(\mathbf{n} \cdot \mathbf{\Omega}) = \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \cos\theta = \pi,$$

so that

$$D = \pi \int dS = \pi S$$

and

$$(4) \qquad \bar{l} = \frac{4V}{S}.$$

This geometric result had been anticipated by Cauchy [6] and Czuber [7]. However, the significance of the result was not fully appreciated until it was shown that the average defined above is appropriate for a uniform isotropic flux of particles entering the volume V across the surface S. For example, if the particles may take part in a reaction for which the cross section σ_R is small ($\bar{l}\sigma_R \ll 1$), then the probability that a particle entering the region undergoes this reaction is approximately given by

$$(5) \qquad P_R = \bar{l}\sigma_R = \frac{4V}{S}\,\sigma_R.$$

For an incident flux of Q particles/sec, the reaction rate is then $4QV\,\sigma_R/S$.

3. The average-path-length theorem. Let us now replace the straight chords across the volume V by a set of random walks. As shown in Fig. 2, we consider each path to consist of a number of straight line segments of length l_i, with a total path length of

$$(6) \qquad L(A, \mathbf{\Omega}) = \sum_{i=1}^{k} l_i.$$

The starting point and the direction of the first element are defined by A and $\mathbf{\Omega}$. The length of each segment and the direction of all further segments are determined stochastically. For example, suppose that, after travelling a distance x in the direction of $\mathbf{\Omega}_i$ along the ith segment, we reach the point $\mathbf{r}_i$. Then the probability that this segment is terminated with a length between x and $x + dx$ is given by $\sigma_T(\mathbf{\Omega}_i, \mathbf{r}_i)\, dx$. The probability that the direction of the track is changed from $\mathbf{\Omega}_i$ to $\mathbf{\Omega}_{i+1}$ is given by $\sigma(\mathbf{\Omega}_i, \mathbf{\Omega}_{i+1}; \mathbf{r}_i)\, dx$.
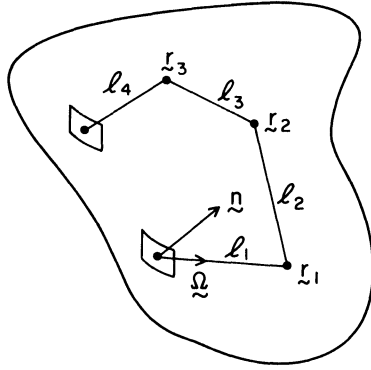
FIG. 2. *A typical track across the volume V, with three scattering events.*

These functions may be completely arbitrary, except that they should be consistent and satisfy detailed balance,

(7)
$$\int d\mathbf{\Omega}' \, \sigma(\mathbf{\Omega}, \mathbf{\Omega}'; \mathbf{r}) = \sigma_T(\mathbf{\Omega}, \mathbf{r}),$$

and

(8)
$$\sigma(\mathbf{\Omega}, \mathbf{\Omega}'; \mathbf{r}) = \sigma(-\mathbf{\Omega}', -\mathbf{\Omega}; \mathbf{r}).$$

If $\bar{L}(A, \mathbf{\Omega})$ is the average length of all such tracks beginning at the surface point A in the direction $\mathbf{\Omega}$, then we wish to prove that

(9)
$$\bar{L} = \frac{\int dS \int d\mathbf{\Omega}(\mathbf{n} \cdot \mathbf{\Omega}) \bar{L}(\mathbf{A}, \mathbf{\Omega})}{\int dS \int d\mathbf{\Omega}(\mathbf{n} \cdot \mathbf{\Omega})} = \frac{4\pi V}{S}.$$

4. Application of transport theory. Let us suppose that region I discussed above is surrounded by a region II which contains a distribution of sources that leads to a uniform isotropic flux of particles $\phi_0/4\pi$ across the boundary from II to I. The number of particles flowing across a surface element dS in the direction $\mathbf{\Omega}$ per unit time is $(\phi_0/4\pi)(\mathbf{n} \cdot \mathbf{\Omega}) \, dS$, and the total incoming flux is $\phi_0 S/4$ (from (3)).

Let us suppose that in the interior of region I the particles undergo scattering collisions with cross sections $\sigma(\mathbf{\Omega}, \mathbf{\Omega}'; \mathbf{r})$ and $\sigma_T(\mathbf{\Omega}, \mathbf{r})$, as defined above, and that there is no absorption and no source of particles inside I. The particle flux $\phi(\mathbf{r}, \mathbf{\Omega})$ satisfies the transport equation

(10)
$$\mathbf{\Omega} \cdot \nabla\phi(\mathbf{r}, \mathbf{\Omega}) + \sigma_T(\mathbf{\Omega}, \mathbf{r})\phi(\mathbf{r}, \mathbf{\Omega}) = \int d\mathbf{\Omega}'\sigma(\mathbf{\Omega}', \mathbf{\Omega}; \mathbf{r})\phi(\mathbf{r}, \mathbf{\Omega}').$$

For all choices of σ, consistent with (7) and (8), this equation, with the boundary condition specified above, has the simple solution

(11)
$$\phi(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{4\pi} \phi_0.$$

Thus the flow inside region I is uniform and isotropic, and so is unaffected by the scattering.

To derive the average path length from the transport theory, let us suppose that while traversing the region I the particles may undergo some reaction, for which the cross section σ_R is constant. This reaction should not change the particle flux, and so we will assume that the reaction does not destroy the particle or lead to any modification of

its path across the volume V. The rate at which reactions occur is

$$(12) \qquad R = \int_V d\mathbf{r} \int d\mathbf{\Omega}\, \sigma_R \phi(\mathbf{r}, \mathbf{\Omega}) = \sigma_R \phi_0 V.$$

For each particle that enters the volume V the probability of reaction is equal to $\sigma_R L$, where L is the total path length. Hence R must be equal to the product of σ_R, the average path length $\bar{\bar{L}}$, and the total current of incoming particles,

$$(13) \qquad R = \sigma_R \bar{\bar{L}} \int d\mathbf{\Omega} \int dS (\mathbf{\Omega} \cdot \mathbf{n}) \frac{\phi_0}{4\pi} = \sigma_R \bar{\bar{L}} \frac{\phi_0}{4V} S.$$

Here again the integration over $\mathbf{\Omega}$ is restricted to those angles for which $\mathbf{\Omega} \cdot \mathbf{n} > 0$. From (12) and (13) we obtain

$$(14) \qquad \bar{\bar{L}} = \frac{4S}{V}.$$

This result represents the physical generalization of Dirac's Theorem.

In order to estimate the average path length by Monte Carlo techniques one would generate a large number of random trajectories following the stochastic rules specified in § 3. The use of such simulation procedures to estimate reaction rates is a classic problem in neutron physics. For example, if a typical trajectory has k segments with lengths l_i then

$$(15) \qquad \eta = \sum_{i=1}^{k} \sigma_R l_i$$

provides an unbiased estimate of the reaction probability for particles entering the region I [3, p. 73]. If the nth trial results in a value of η_n, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \eta_n = \langle \eta \rangle = \frac{4\sigma_R V}{S}.$$

The right side of this equation represents the total reaction rate divided by incoming current, as calculated above. Hence, setting $\sigma_R = 1$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{i=1}^{k} l_i = \frac{4V}{S}.$$

5. Discussion. We have shown that for particles flowing uniformly and isotropically into a region of volume V, the average length of the path followed by the particles in V is not changed by scattering, provided that there is no absorption inside V and that the scattering cross sections satisfy detailed balance (7). We have not assumed that the scattering centers are distributed uniformly, or that the scattering is isotropic.

In defining the scattering probabilities we did not introduce the speed of the particles. The theorem cannot be applied when the scattering probabilities depend on properties changed in previous collisions. Hence, if cross sections are velocity dependent and the speed of the particles is changed by the collisions, the result is in general not true. However if the whole system is in thermal equilibrium so that the velocity distribution is unchanged by collisions, then our assumptions are valid and the thermally averaged cross sections should be used in (7) and (8).

Although we have considered a single surface S surrounding the volume V, this is unnecessary. The theorem can be applied to a volume of material with holes, provided the area S includes all bounding surfaces and the volume of the holes is not included in

V. Since the result depends crucially on the homogeneity of the flux inside V, if we cover part of the surface by a perfect reflecting material the theorem still holds with S measuring the uncovered surface area.

The practical value of this theorem lies in the estimation of reaction rates for processes that occur within a finite volume with sufficiently low probability that the reaction does not destroy a significant fraction of the ensemble of particles. It has been used by one of us (J.N.B.) in the calculation of ionic recombination rates in gases, by giving an estimate of the length of time for which the separation between two gas molecules remains less than some critical value [8]. Although we cannot allow for absorption in the derivation of the theorem, the result can be used to estimate the absorption rate in situations where elastic scattering is more likely than absorption. In strongly absorbing media we believe, but have not yet proved, that the absorption rate is always reduced by scattering under the flow conditions assumed above.

Analogous results can be obtained in any dimension. For example, consider particles moving along the line $O \leq x \leq l$ with a probability $\sigma(x)$ of being backscattered. In the appendix we will show, for constant $\sigma(x)$, that scattering leads to increased path length for the transmitted particles, but to no increase in the average path length for all entering particles.

The one-dimensional theorem can be extended to treat random walks along a discrete mesh. Consider a lattice, as shown in Fig. 3, which contains a set of line
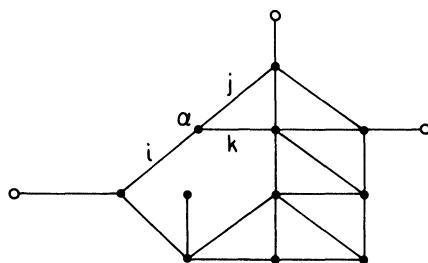

FIG. 3. *A lattice with 3 boundary nodes (○), 11 internal nodes (●) and 21 connecting lines. The average number of steps in a random walk across this lattice is 14.*

segments i of length l_i $(i = 1, \cdots N)$ with nodes $\alpha, \beta, \gamma \cdots$, which are classified either as internal or boundary nodes. The boundary nodes are attached to only a single line, whereas each internal node can be attached to an arbitrary positive number of lines. The mesh must be constructed so that any two nodes are connected by at least one chain of lines. A random walk begins with equal probability at any boundary node, and is terminated when a boundary node is again encountered. At each internal node α a set of probabilities $P(\alpha; i \to j)$ is specified such that

(16)
$$0 \leq P(\alpha; i \to j) \leq 1,$$
$$P(\alpha; i \to j) = P(\alpha; j \to i),$$

and

$$\sum_j P(\alpha; i \to j) = 1.$$

We also require that $P(\alpha; i \to j)$ be nonzero if and only if the lines i and j intersect at α.

Under these conditions the average number of steps in each random walk is $2N/n_e$, where N is the total number of line segments and n_e is the number of boundary nodes.

The average distance traveled is

(17) $$\bar{L} = \frac{2}{n_e} \sum_{i=1}^{N} l_i.$$

These results follow directly from the fact that the flux along each line in each direction is uniform.

Appendix. To demonstrate the validity of this result in one dimension, consider a strip of length l, for which the average path length must be equal to l in the absence of scattering. For a homogeneous strip a complete solution of the scattering problem is possible. Suppose that each element of length dx leads to a probability σdx that the particle's direction of motion is reversed. Let $R(l)$ and $T(l)$ be the reflection and transmission probabilities for the whole strip, and let $\lambda_R(l)$ and $\lambda_T(l)$ be the average path length for the reflected and transmitted particles.

Consideration of the effects of an increase in the length of the strip from l to $l + \delta l$ leads to the following differential equations,

$$\frac{dR}{dl} = \sigma T^2 = -\frac{dT}{dl},$$

$$\frac{d\lambda_T}{dl} = 1 + \sigma R \lambda_R,$$

$$\frac{d\lambda_R}{dl} = \frac{\sigma T^2}{R}(2\lambda_T - \lambda_R).$$

These equations have the solution

$$T = \frac{1}{1+\sigma l}, \qquad R = \frac{\sigma l}{1+\sigma l},$$

$$\lambda_T = \frac{l}{1+\sigma l}(1 + \sigma l + \tfrac{1}{3}\sigma^2 l^2),$$

$$\lambda_R = \frac{l}{1+\sigma l}(1 + \tfrac{2}{3}\sigma l),$$

from which it can be confirmed that the average path length for all particles is independent of σ, since

$$\bar{\lambda} \equiv T\lambda_T + R\lambda_R = l.$$

As the scattering probability is increased, the larger path length for transmitted particles is offset by a reduction in the average path length of reflected particles.

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