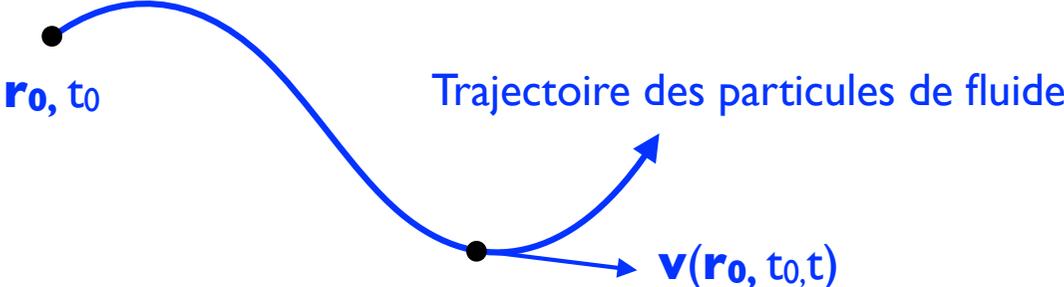
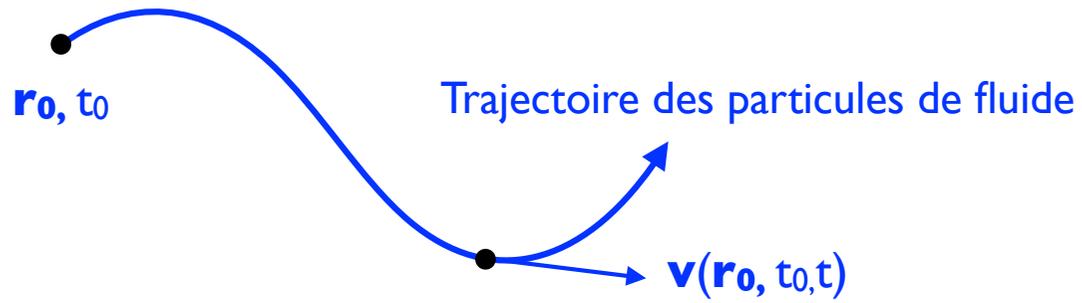


# Description des écoulements

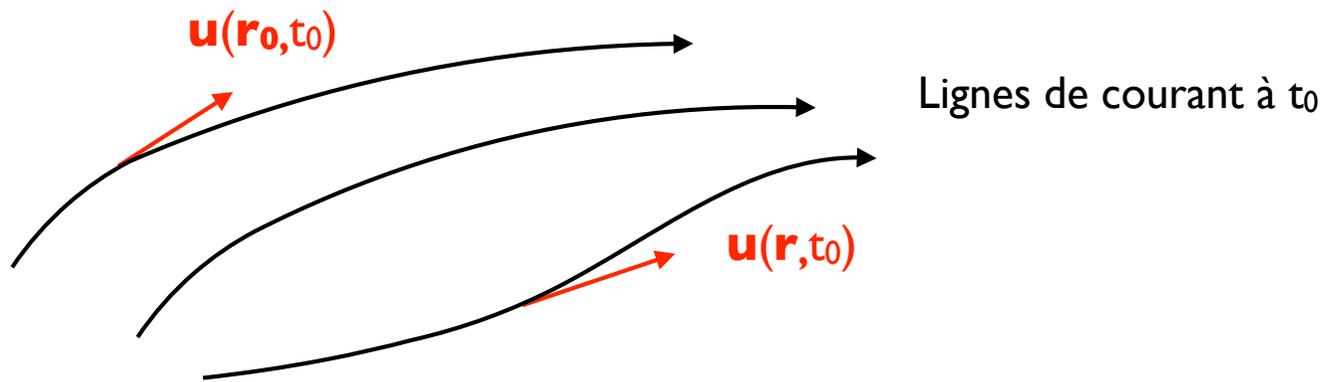


Description lagrangienne

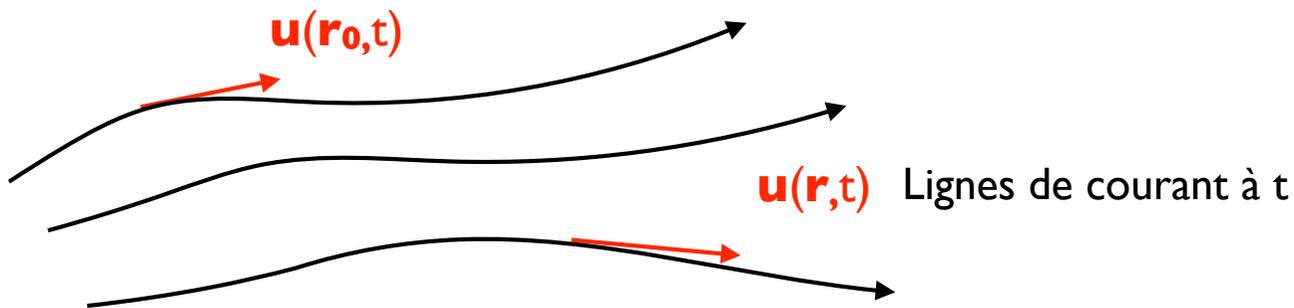
# Description des écoulements



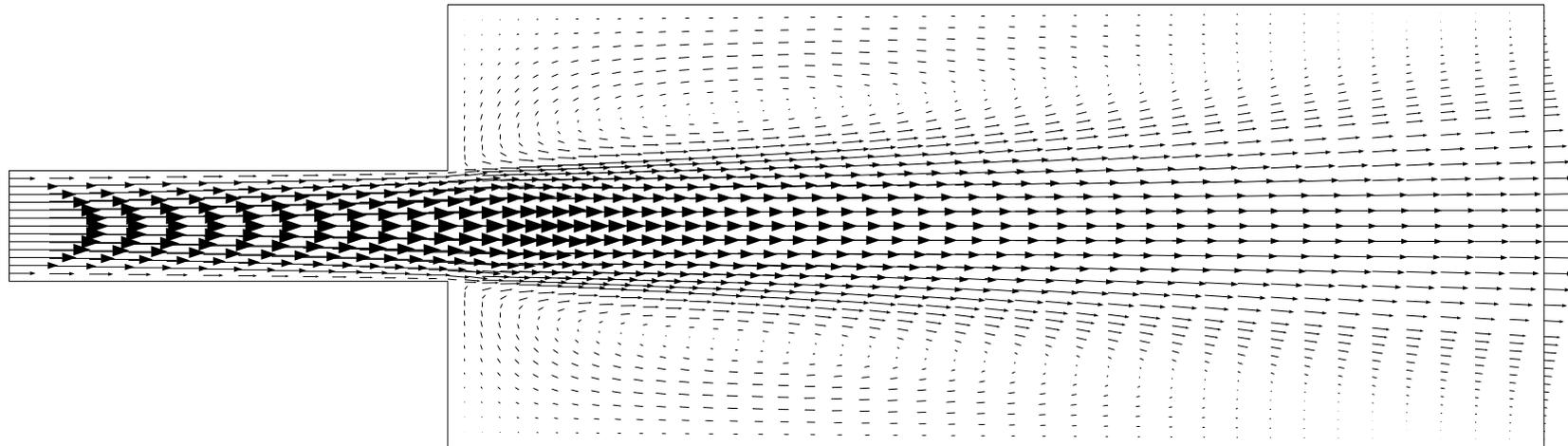
Description lagrangienne



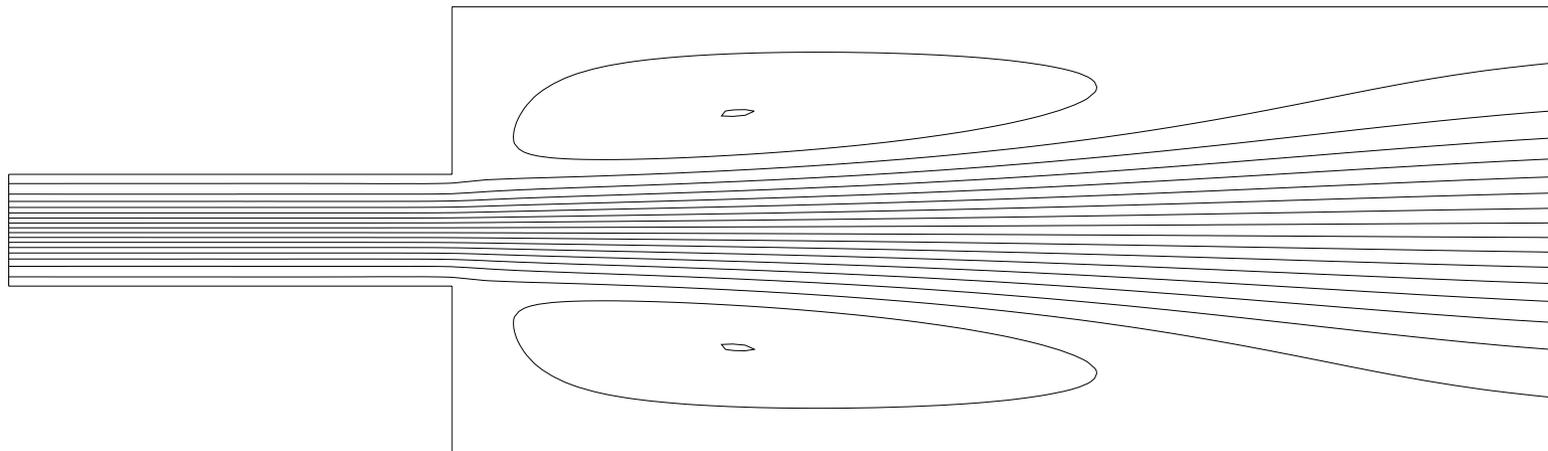
Description eulérienne



# Un exemple de champ de vitesse eulérienne

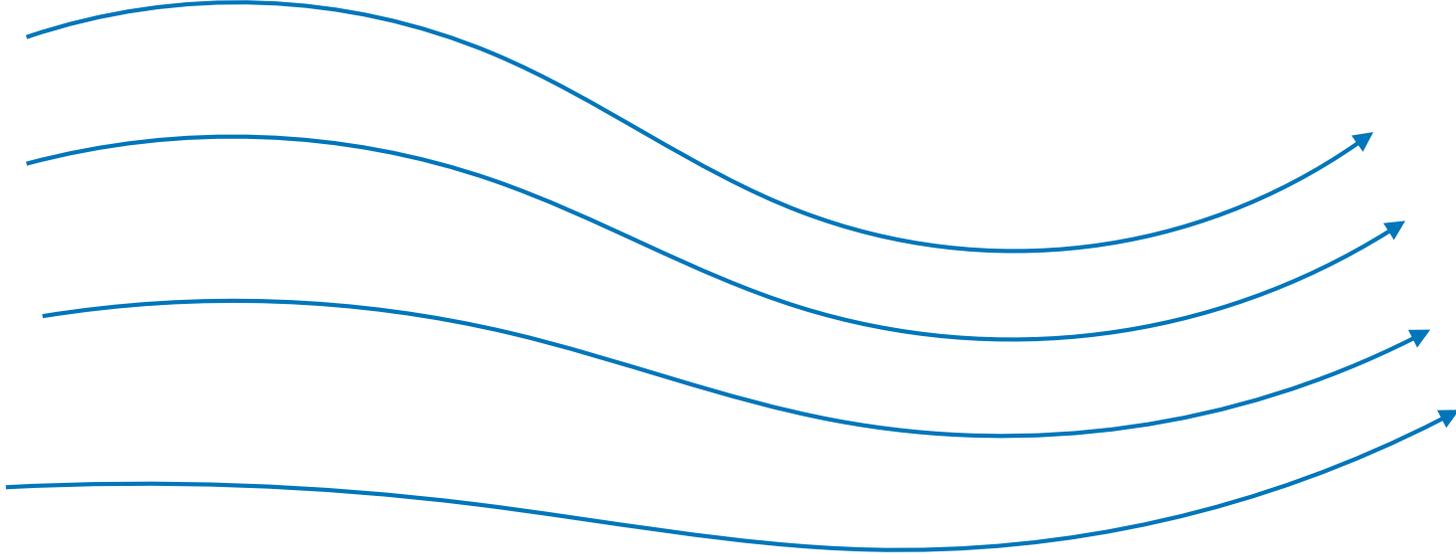


Vecteurs vitesse

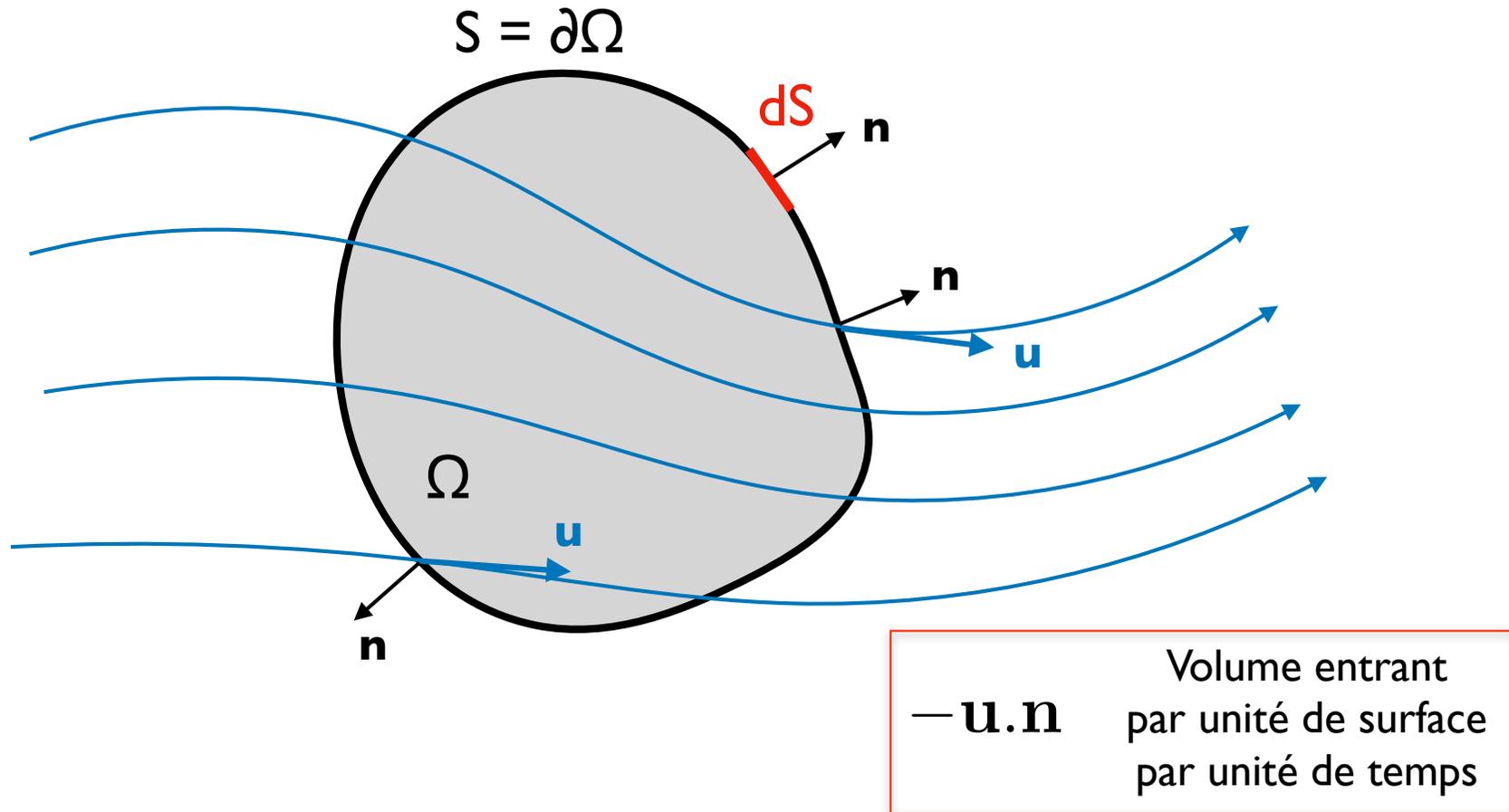


Lignes de courant

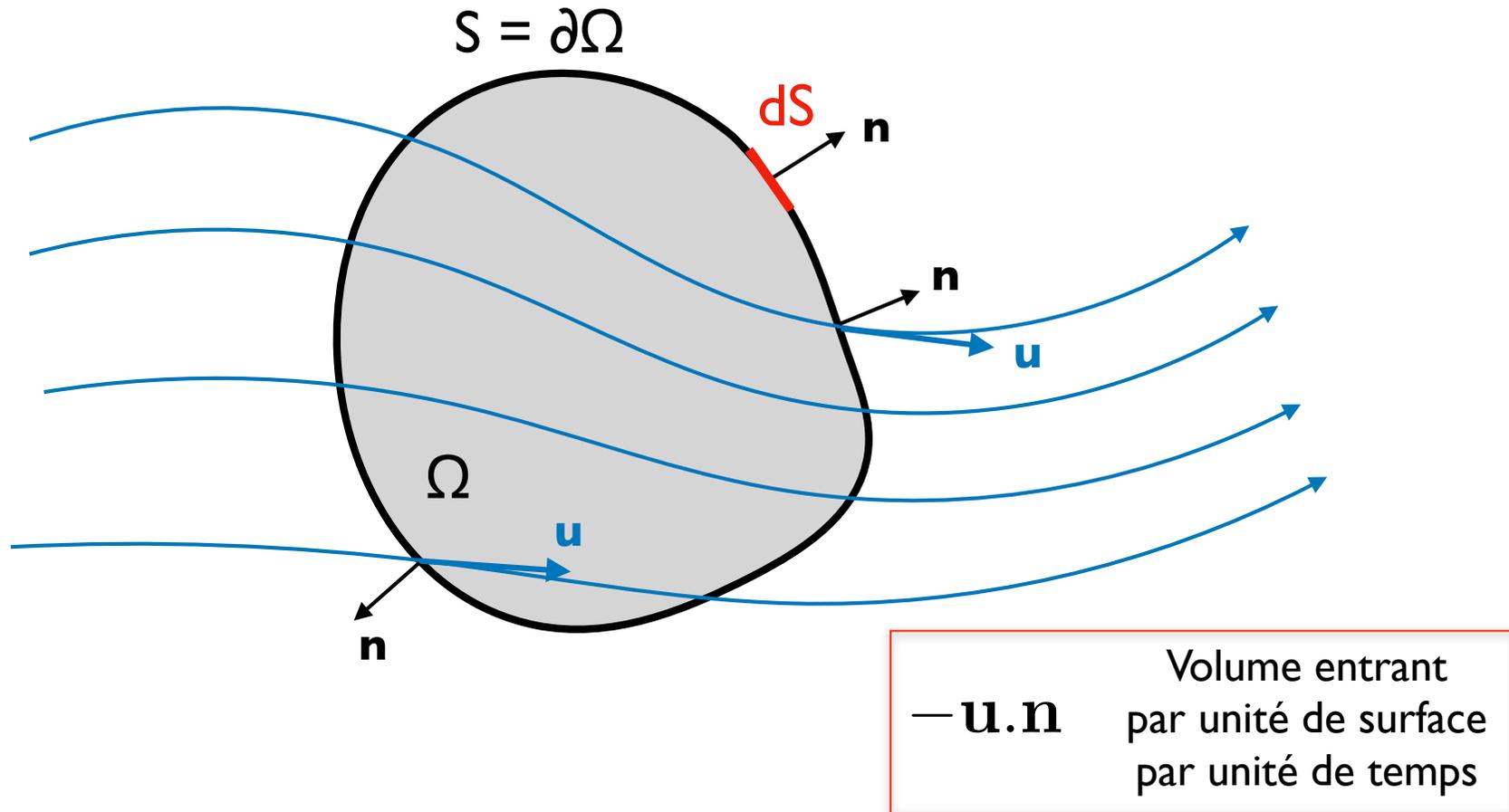
# Conservation de la masse dans un écoulement



# Conservation de la masse dans un écoulement

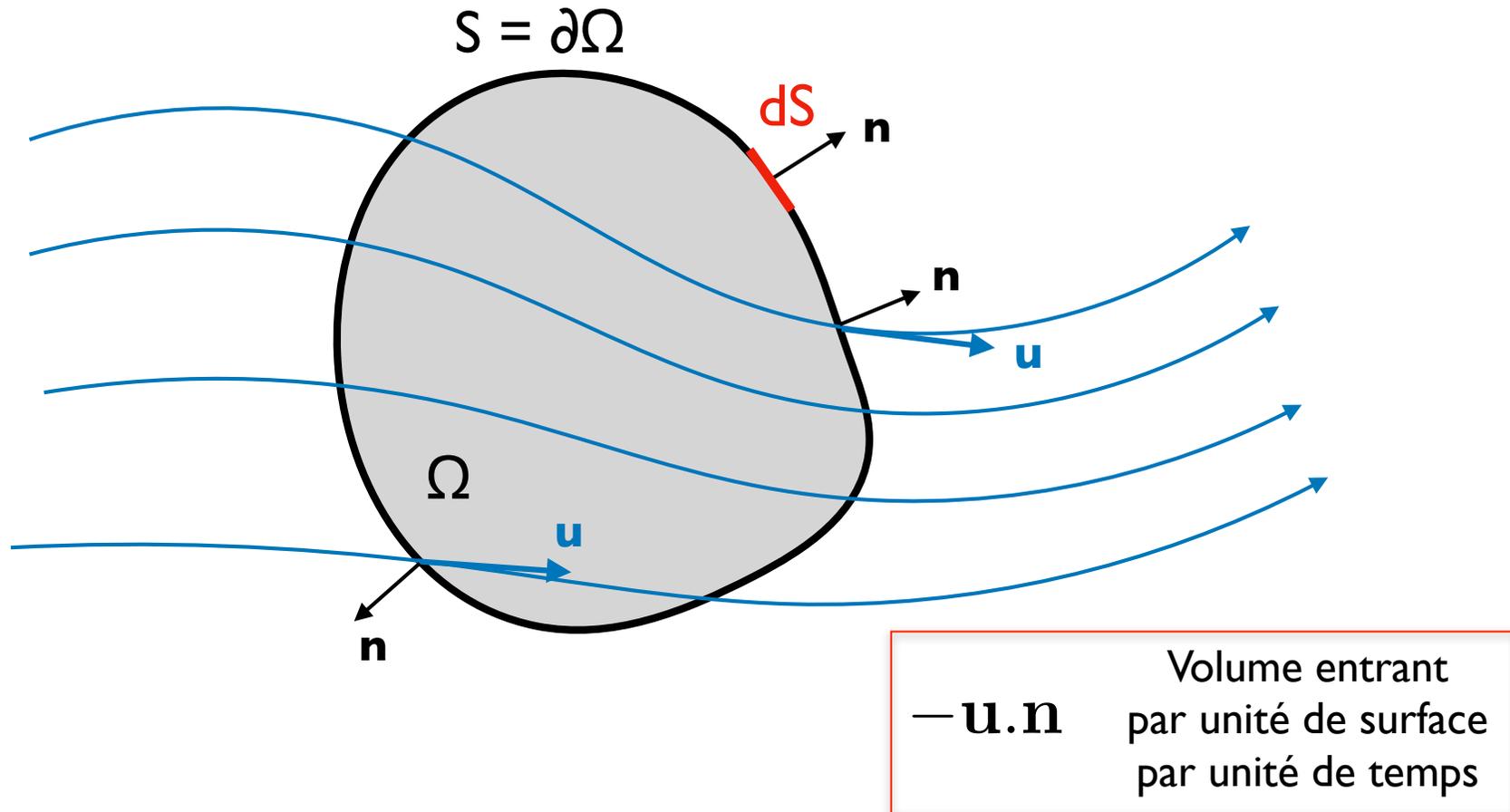


# Conservation de la masse dans un écoulement



$$-\int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} dS + Q$$

# Conservation de la masse dans un écoulement



$$\int_{\Omega} \frac{\partial \rho}{\partial t} dV = - \int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} dS + Q$$

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$$\int_{\Omega} \frac{\partial \rho}{\partial t} dV + \int_{\Omega} \nabla \cdot (\rho \mathbf{u}) dV = Q$$

$\nabla \cdot \equiv$  divergence  


$$\int_{\Omega} \frac{\partial \rho}{\partial t} dV = - \int_{\partial \Omega} \rho \mathbf{u} \cdot \mathbf{n} dS + Q$$

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En l'absence de source de masse

$\nabla \cdot \equiv$  divergence  


$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0$$

$\nabla \equiv$  gradient

$$\int_{\Omega} \frac{\partial \rho}{\partial t} dV = - \int_{\partial \Omega} \rho \mathbf{u} \cdot \mathbf{n} dS + Q$$

$$\int_{\Omega} \frac{\partial \rho}{\partial t} dV + \int_{\Omega} \nabla \cdot (\rho \mathbf{u}) dV = Q$$

En l'absence de source de masse

$\nabla \cdot \equiv$  divergence  


$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0$$

$\nabla \equiv$  gradient

Si le fluide est « incompressible », masse volumique constante

$$\nabla \cdot \mathbf{u} = 0$$

Fluide comme “incompressible” ?

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$$\frac{\delta\rho}{\rho} = \frac{1}{\rho} \frac{\partial\rho}{\partial p} \delta p$$

Fluide comme “incompressible” ?

$$\frac{\delta \rho}{\rho} = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \delta p$$

$$\frac{\delta \rho}{\rho} = \frac{1}{\rho c^2} \delta p$$

$c$  : vitesse du son

Fluide comme “incompressible” ?

$$\frac{\delta\rho}{\rho} = \frac{1}{\rho} \frac{\partial\rho}{\partial p} \delta p$$

$$\frac{\delta\rho}{\rho} = \frac{1}{\rho c^2} \delta p$$

$c$  : vitesse du son

En écoulement dominé par l'inertie du fluide:  $\delta p \sim \rho u^2$

$$\frac{\delta\rho}{\rho} \sim \frac{u^2}{c^2} = M^2$$

$M$  : Nombre de Mach

Fluide comme “incompressible” ?

$$\frac{\delta\rho}{\rho} = \frac{1}{\rho} \frac{\partial\rho}{\partial p} \delta p$$

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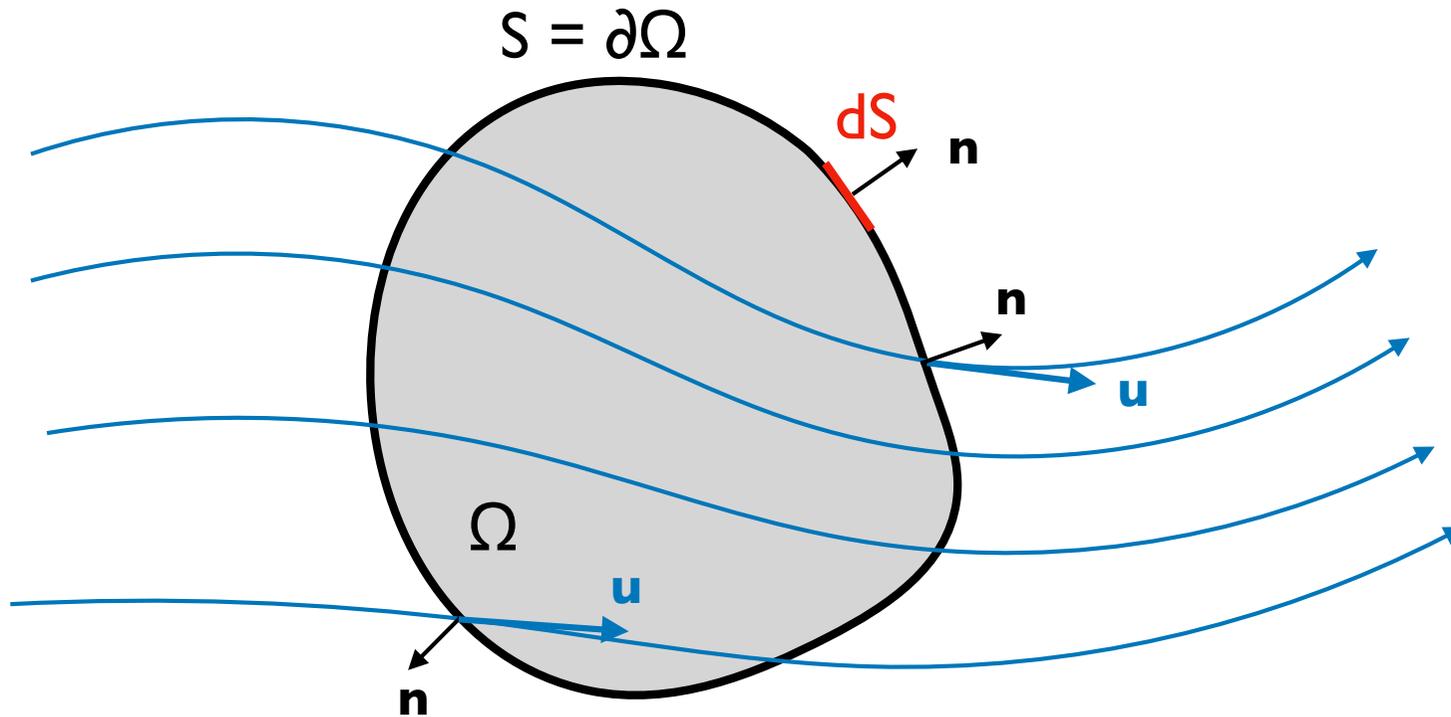
$M \ll 1$       Fluide quasi incompressible



$M \sim 1$  ou  $M > 1$

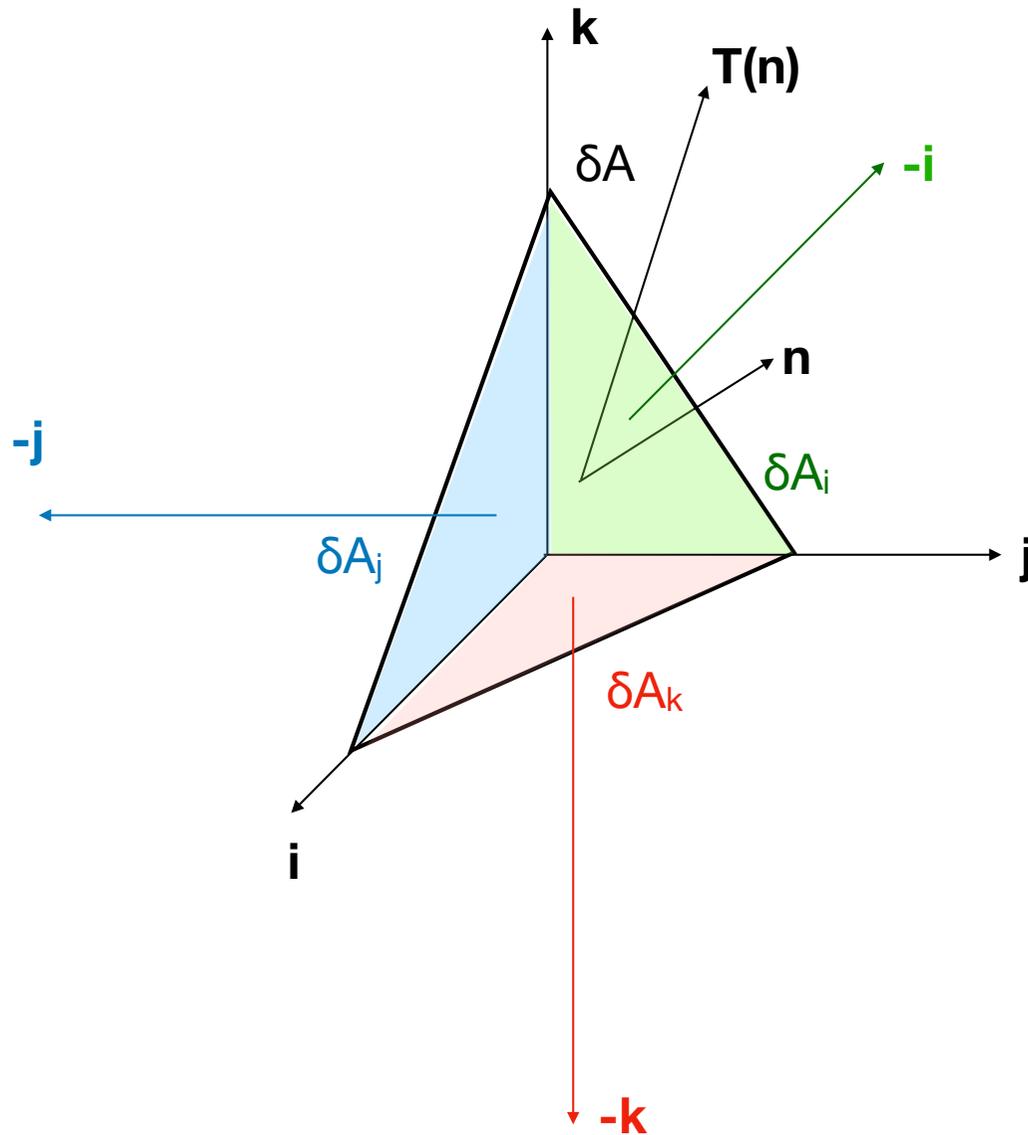
Ondes de choc

# Conservation de la quantité de mouvement



$$\int_{\Omega} \frac{\partial \rho \mathbf{u}}{\partial t} dV = \int_{\Omega} \rho \mathbf{f} dV - \int_{\partial\Omega} \rho \mathbf{u} \underbrace{\mathbf{u} \cdot \mathbf{n}}_{\text{scalaire}} dS + \int_{\partial\Omega} \underbrace{\underline{\underline{\sigma}} \cdot \mathbf{n}}_{\text{contraintes}} dS$$

$\mathbf{T}(\mathbf{n})$  : vecteur contrainte totale sur la face de normale  $\mathbf{n}$



**Forces de surface**

$$\mathbf{T}(\mathbf{n})\delta A$$

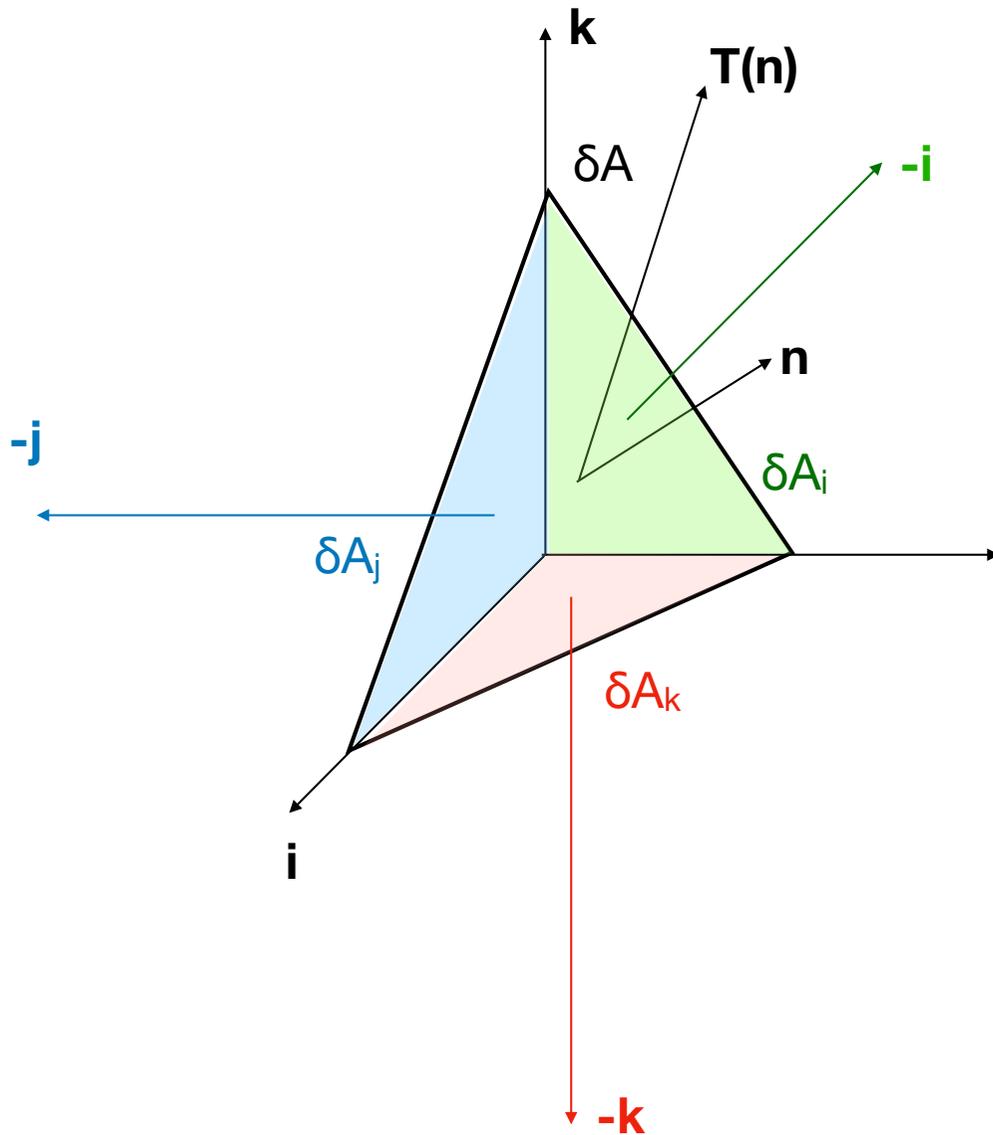
$$\mathbf{T}(-\mathbf{i})\delta A_i$$

$$\mathbf{T}(-\mathbf{j})\delta A_j$$

$$\mathbf{T}(-\mathbf{k})\delta A_k$$

$$\mathbf{T}(-\mathbf{i}) = -\mathbf{T}(\mathbf{i})$$

$$\delta A_i = \delta A \mathbf{i} \cdot \mathbf{n}$$



**Forces de surface**

$$\mathbf{T}(\mathbf{n}) \delta A$$

$$-\mathbf{T}(\mathbf{i}) \mathbf{i} \cdot \mathbf{n} \delta A$$

$$-\mathbf{T}(\mathbf{j}) \mathbf{j} \cdot \mathbf{n} \delta A$$

$$-\mathbf{T}(\mathbf{k}) \mathbf{k} \cdot \mathbf{n} \delta A$$

$$\mathbf{F}_{\text{surface}} = \sum \mathbf{T} \delta A$$

**pdf sur l'élément de volume :**

$$\rho \delta V \mathbf{a} = \rho \mathbf{f} \delta V + \sum \mathbf{T} \delta A$$

$$\rho \mathbf{a} = \rho \mathbf{f} + \sum \mathbf{T} \frac{\delta A}{\delta V}$$

$$\frac{\delta A}{\delta V} \sim \frac{1}{L} \rightarrow \infty \quad L \rightarrow 0$$

$$\sum \mathbf{T} = 0$$

$$\mathbf{T}(\mathbf{n}) = \mathbf{T}(\mathbf{i}) \mathbf{i} \cdot \mathbf{n} + \mathbf{T}(\mathbf{j}) \mathbf{j} \cdot \mathbf{n} + \mathbf{T}(\mathbf{k}) \mathbf{k} \cdot \mathbf{n}$$

$$\mathbf{T}(\mathbf{n}) = \mathbf{T}(\mathbf{i}) n_x + \mathbf{T}(\mathbf{j}) n_y + \mathbf{T}(\mathbf{k}) n_z$$

Composante x  $T_x(\mathbf{n}) = T_x(\mathbf{i}) n_x + T_x(\mathbf{j}) n_y + T_x(\mathbf{k}) n_z$

$$T_x(\mathbf{n}) = \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z$$

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

contraintes normales

contraintes tangentielles

Tenseur des contraintes

$$\begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

Vecteur contrainte  
(force/unité de  
surface)

Tenseur des  
contraintes

Normale à la  
surface considérée

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

## Conservation de la quantité de mouvement

$$\int_{\Omega} \frac{\partial \rho \mathbf{u}}{\partial t} dV = - \int_{\partial \Omega} \rho \mathbf{u} \mathbf{u} \cdot \mathbf{n} dS + \int_{\Omega} \rho \mathbf{f} dV + \int_{\partial \Omega} \boldsymbol{\sigma} \cdot \mathbf{n} dS$$

Théorème de la divergence:

$$\int_{\Omega} \frac{\partial \rho \mathbf{u}}{\partial t} dV + \int_{\Omega} \nabla \cdot \rho \mathbf{u} \mathbf{u} dV = \int_{\Omega} \rho \mathbf{f} dV + \int_{\Omega} \nabla \cdot \boldsymbol{\sigma} dV$$

$\rho \mathbf{u} \mathbf{u}$  : Flux de quantité de mouvement,

↑  
produit tensoriel  $\longrightarrow$  tenseur de composantes  $\rho u_i u_j$

Sa divergence :  $\frac{\partial}{\partial x_j} (\rho u_i u_j) = u_i \frac{\partial \rho u_j}{\partial x_j} + \rho u_j \frac{\partial u_i}{\partial x_j}$   
(somme sur  $j$ )

## Divergence des contraintes

$$\nabla \cdot \sigma = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$$

# Conservation de la quantité de mouvement

Pour un volume élémentaire de fluide :

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \frac{\partial \rho}{\partial t} + \mathbf{u} \nabla \cdot \rho \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \rho \mathbf{f} + \nabla \cdot \sigma$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \underbrace{\left( \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} \right)}_{= 0} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \rho \mathbf{f} + \nabla \cdot \sigma$$

Conservation de la masse

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \rho \mathbf{f} + \nabla \cdot \sigma$$

$$m \mathbf{a} = \mathbf{F}$$

## Accélération d'un élément de fluide

$$m\mathbf{a} = \mathbf{F}$$

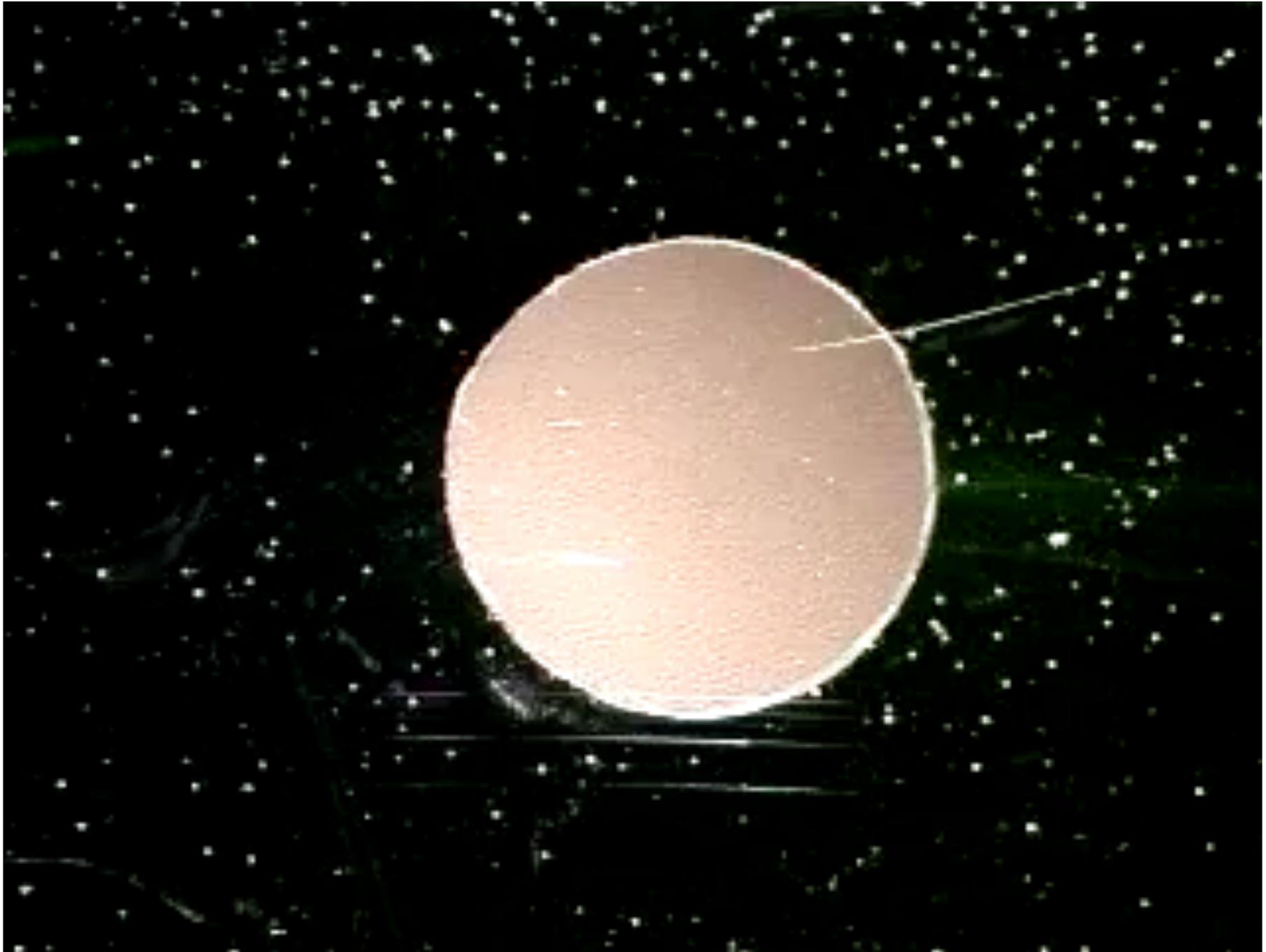
$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \rho \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$$

$$\mathbf{a} = \boxed{\frac{\partial \mathbf{u}}{\partial t}} + \boxed{\mathbf{u} \cdot \nabla \mathbf{u}}$$

instationnarité

accélération convective

un écoulement stationnaire où  $\mathbf{u} \cdot \nabla \mathbf{u} \neq 0$



## Écriture de l'accélération convective

Pour la composante  $i$  :

$$u_j \frac{\partial u_i}{\partial x_j} = \sum_{j=1,3} u_j \frac{\partial u_i}{\partial x_j}$$

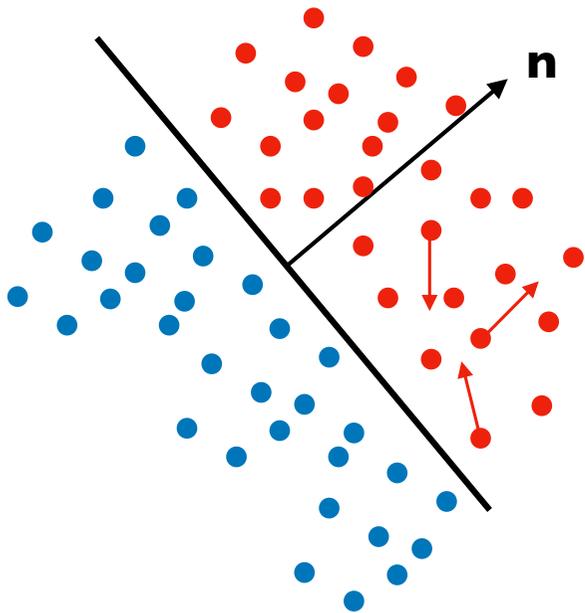
Pour la composante  $x$  :

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z}$$

Variation spatiale de  $u_x$  projetée sur la vitesse  $(u_x, u_y, u_z)$

# Les contraintes dans un fluide

Dans un fluide à l'équilibre, sans écoulement macroscopique



La résultante des interactions entre atomes ou molécules est une **pression isotrope**

$$\sigma = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

$$\sigma_{ij} = -p \delta_{ij}$$

$$\nabla \cdot \sigma = -\nabla p = \begin{pmatrix} -\frac{\partial p}{\partial x} \\ -\frac{\partial p}{\partial y} \\ -\frac{\partial p}{\partial z} \end{pmatrix}$$

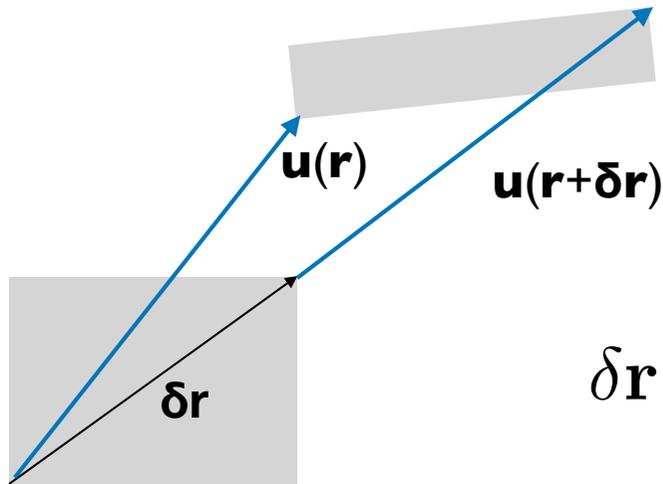
Dans un fluide à l'équilibre, sans écoulement macroscopique

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \rho \mathbf{f} + \nabla \cdot \sigma$$

$$0 = \rho \mathbf{g} - \nabla p$$

$$p = p_0 - \rho g z$$

# Les déformations d'un élément dans un fluide en écoulement



$$\delta \mathbf{r} = \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix}$$

$$u_x(\mathbf{r} + \delta \mathbf{r}) = u_x(\mathbf{r}) + \frac{\partial u_x}{\partial x} \delta r_x + \frac{\partial u_x}{\partial y} \delta r_y$$

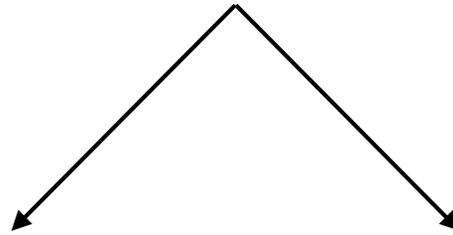
$$u_y(\mathbf{r} + \delta \mathbf{r}) = u_y(\mathbf{r}) + \frac{\partial u_y}{\partial x} \delta r_x + \frac{\partial u_y}{\partial y} \delta r_y$$

$$\begin{pmatrix} \delta u_x \\ \delta u_y \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{pmatrix} \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix}$$

$$\delta \mathbf{u} = \nabla \mathbf{u} \cdot \delta \mathbf{r}$$

# Décomposition du gradient de vitesse

$$\begin{pmatrix} \delta u_x \\ \delta u_y \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{pmatrix} \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix}$$



$$\begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\ \frac{1}{2} \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) & \frac{\partial u_y}{\partial y} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} & 0 \end{bmatrix}$$

Partie symétrique du  
gradient de vitesse

Partie antisymétrique  
du gradient de vitesse

# Décomposition de la partie symétrique

$$\begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\ \frac{1}{2} \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) & \frac{\partial u_y}{\partial y} \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} & 0 \\ 0 & \frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x} \end{bmatrix} = \frac{1}{2} \nabla \cdot \mathbf{u} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

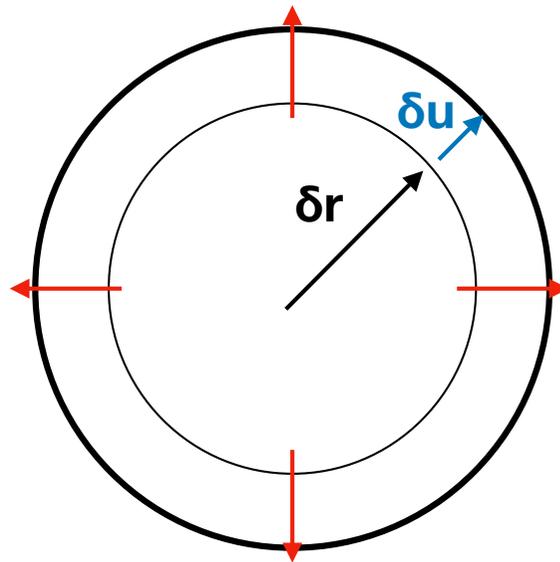
Sphérique

$$\frac{1}{2} \begin{bmatrix} \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial x} \end{bmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

Déviateur (trace nulle)

## Effet de la partie sphérique

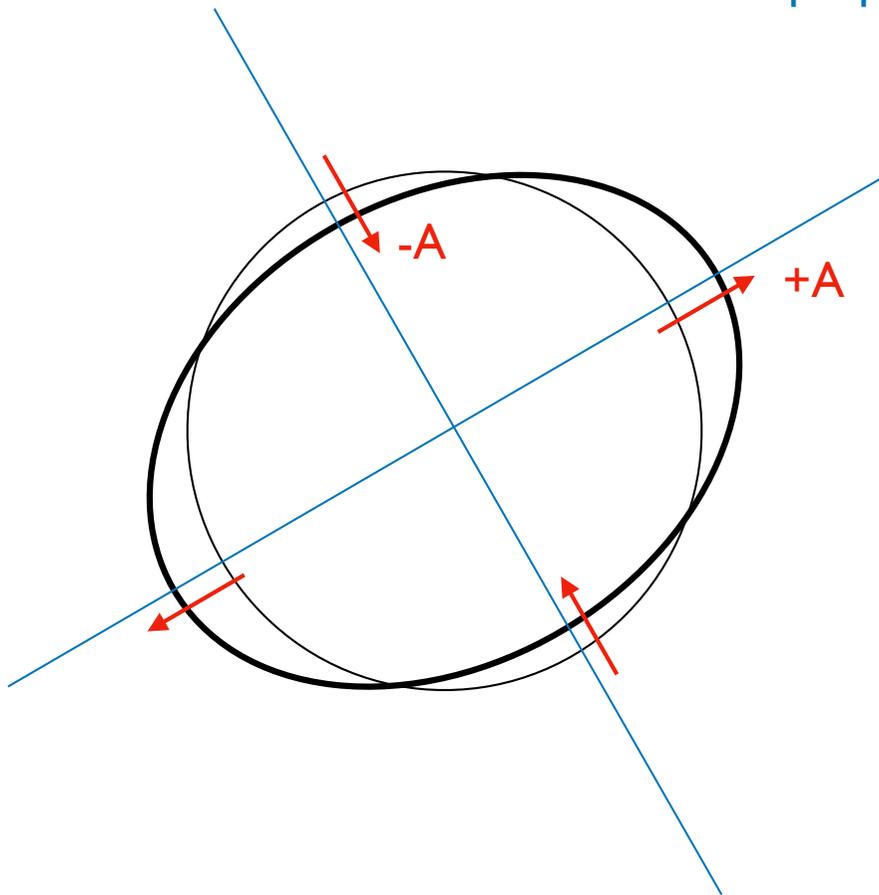
$$\begin{pmatrix} \delta u_x \\ \delta u_y \end{pmatrix} = \frac{1}{2} \nabla \cdot \mathbf{u} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix} = \frac{1}{2} \nabla \cdot \mathbf{u} \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix}$$



# Effet du déviateur

$$\begin{pmatrix} \delta u_x \\ \delta u_y \end{pmatrix} = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix} = A \begin{pmatrix} \delta r_x \\ -\delta r_y \end{pmatrix}$$

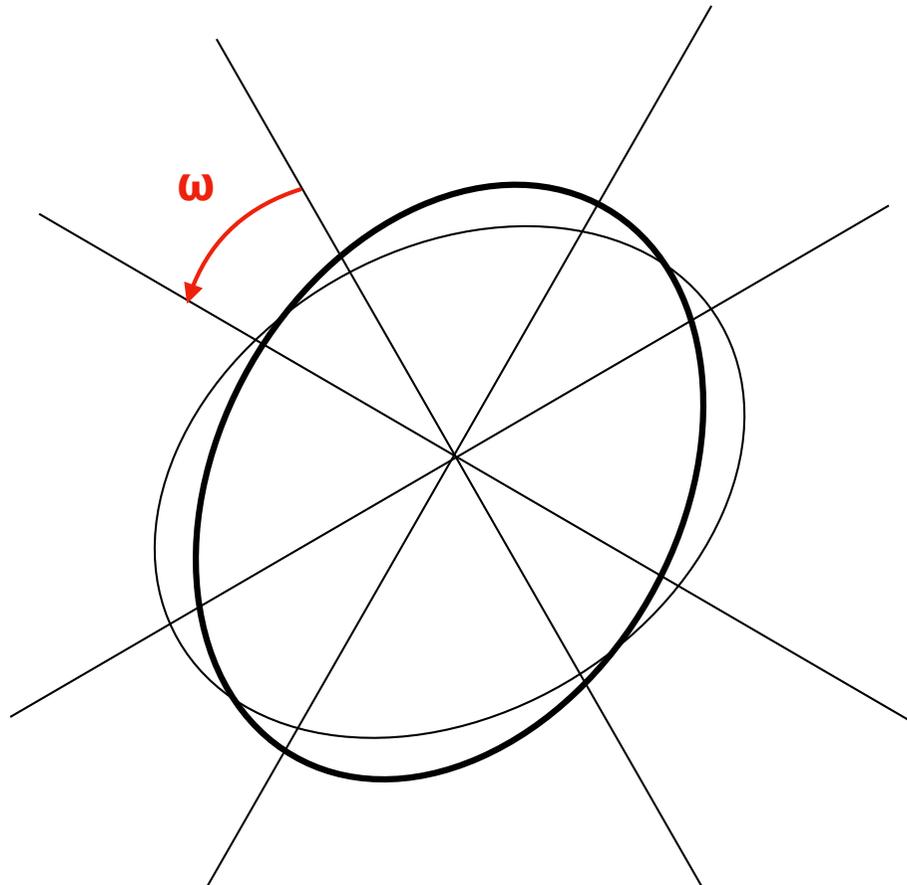
Dans les axes propres

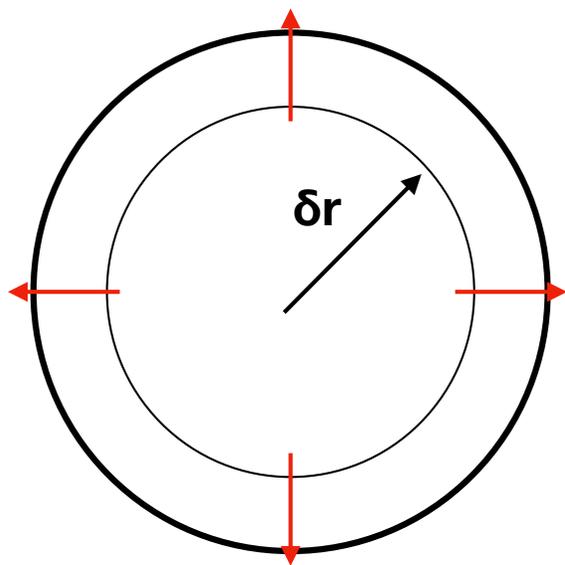


## Effet de la partie antisymétrique

$$\begin{pmatrix} \delta u_x \\ \delta u_y \end{pmatrix} = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta r_x \\ \delta r_y \end{pmatrix} = \omega \begin{pmatrix} -\delta r_y \\ \delta r_x \end{pmatrix}$$

$$\delta \mathbf{u} \perp \delta \mathbf{r}$$

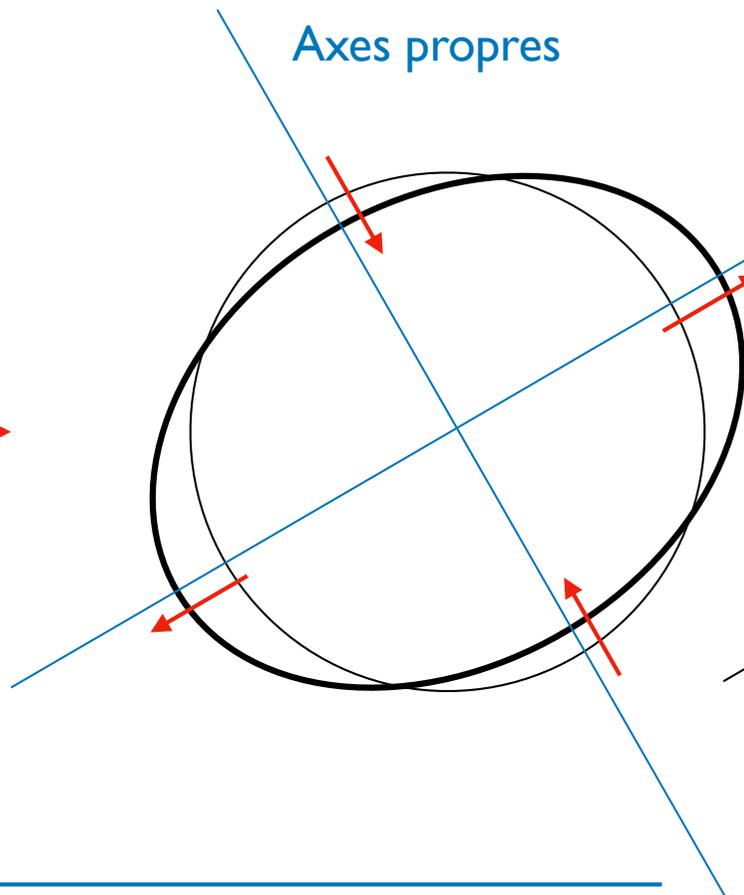




Partie symétrique du gradient de vitesse

Sphérique

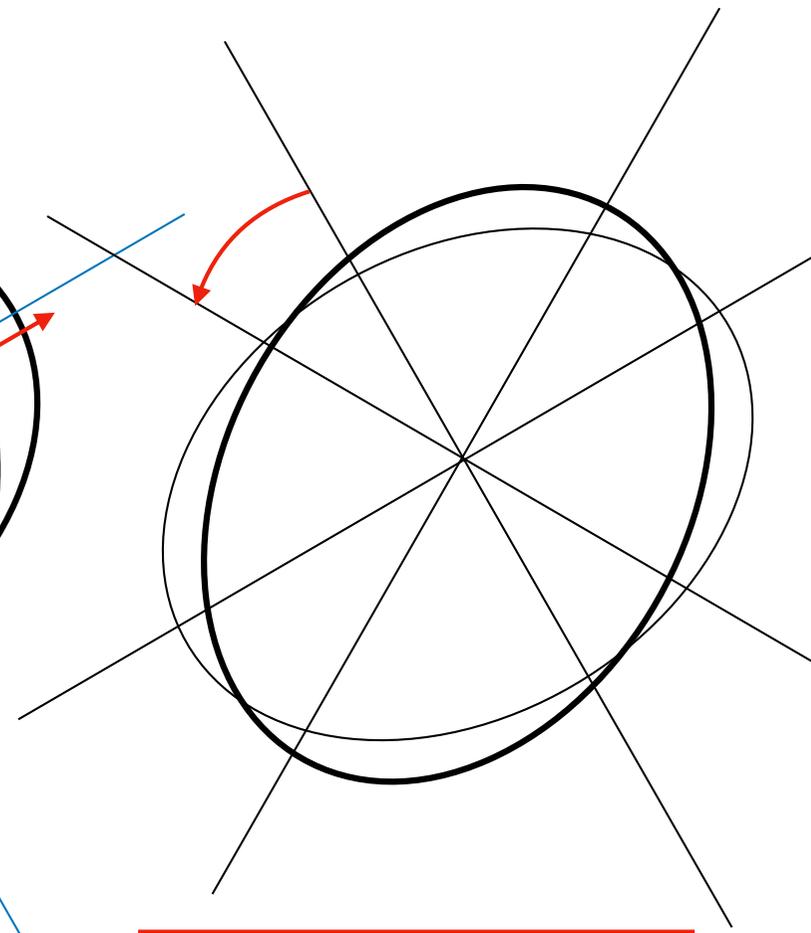
Chgt. de volume  
si  $\text{div } \mathbf{u} \neq 0$



Axes propres

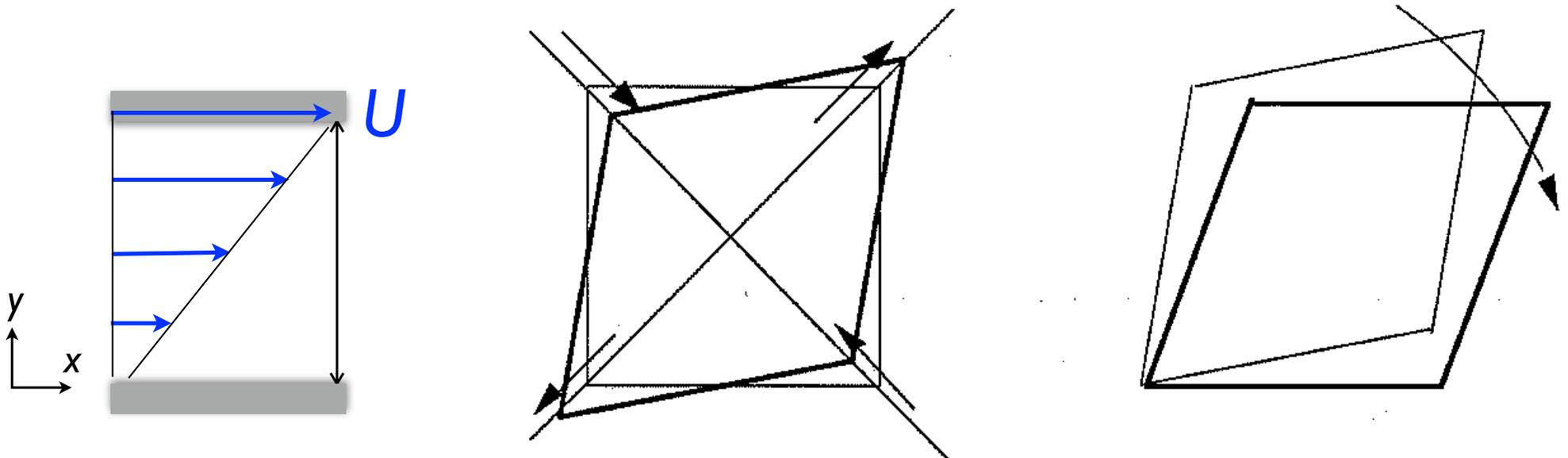
Déviateur

Déformation à volume constant



Partie antisymétrique du gradient de vitesse  
Rotationnel

Rotation sans déformation



Ecoulement de cisaillement simple  $u_x = Gy$   
 Décomposition en déformation pure suivant des axes propres et rotation

$$\nabla \mathbf{u} = \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & G \\ -G & 0 \end{pmatrix}$$

# Les contraintes dans un fluide newtonien

Comportement mécanique du fluide caractérisé par une seule quantité :

sa viscosité dynamique  $\eta$

Contraintes : fonctions linéaires de la partie symétrique du gradient de vitesse

partie symétrique du gradient de vitesse : déformation d'un élément de fluide

partie antisymétrique du gradient de vitesse : rotation en bloc d'un élément de fluide

## Les contraintes dans un fluide newtonien

$$\sigma_{ij} = -p \delta_{ij} + 2 \eta e_{ij}$$

$$\underline{\underline{e}} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

$$\sigma = \begin{pmatrix} -p + 2\eta \frac{\partial u_x}{\partial x} & \eta \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) & \eta \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\ \eta \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) & -p + 2\eta \frac{\partial u_y}{\partial y} & \eta \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \\ \eta \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) & \eta \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) & -p + 2\eta \frac{\partial u_z}{\partial z} \end{pmatrix}$$

$$\nabla \cdot \sigma = -\nabla p + \eta \Delta \mathbf{u}$$

Dans un fluide **newtonien**, avec **écoulement macroscopique**

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \rho \mathbf{f} + \nabla \cdot \sigma$$

$$\nabla \cdot \sigma = -\nabla p + \eta \Delta \mathbf{u}$$

Équation de Navier-Stokes

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \eta \Delta \mathbf{u} + \rho \mathbf{f}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} + \mathbf{f}$$

viscosité cinématique  $\nu = \eta / \rho$

## L'équation de Navier-Stokes dans toute sa splendeur (horreur ?)

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + f_x$$

$$\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right) + f_y$$

$$\frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) + f_z$$

Comment la simplifier ?

# Une analyse dimensionnelle de l'équation de Navier-Stokes

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \eta \Delta \mathbf{u} + \rho \mathbf{f}$$

$$\mathbf{u} \sim U$$

$$\nabla \sim 1/L$$

$$\Delta \sim 1/L^2$$

Échelle de longueur  
pertinente qui caractérise la  
**variation spatiale de vitesse**

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} \sim \rho \frac{U^2}{L}$$

Terme inertiel

$$\eta \Delta \mathbf{u} \sim \eta \frac{U}{L^2}$$

Terme visqueux

$$\frac{\rho \mathbf{u} \cdot \nabla \mathbf{u}}{\eta \Delta \mathbf{u}} \sim \frac{\rho U L}{\eta} = \frac{U L}{\nu}$$

Nombre de Reynolds

