How micro-flexural modes give rise to band gaps and negative refraction in metamaterials

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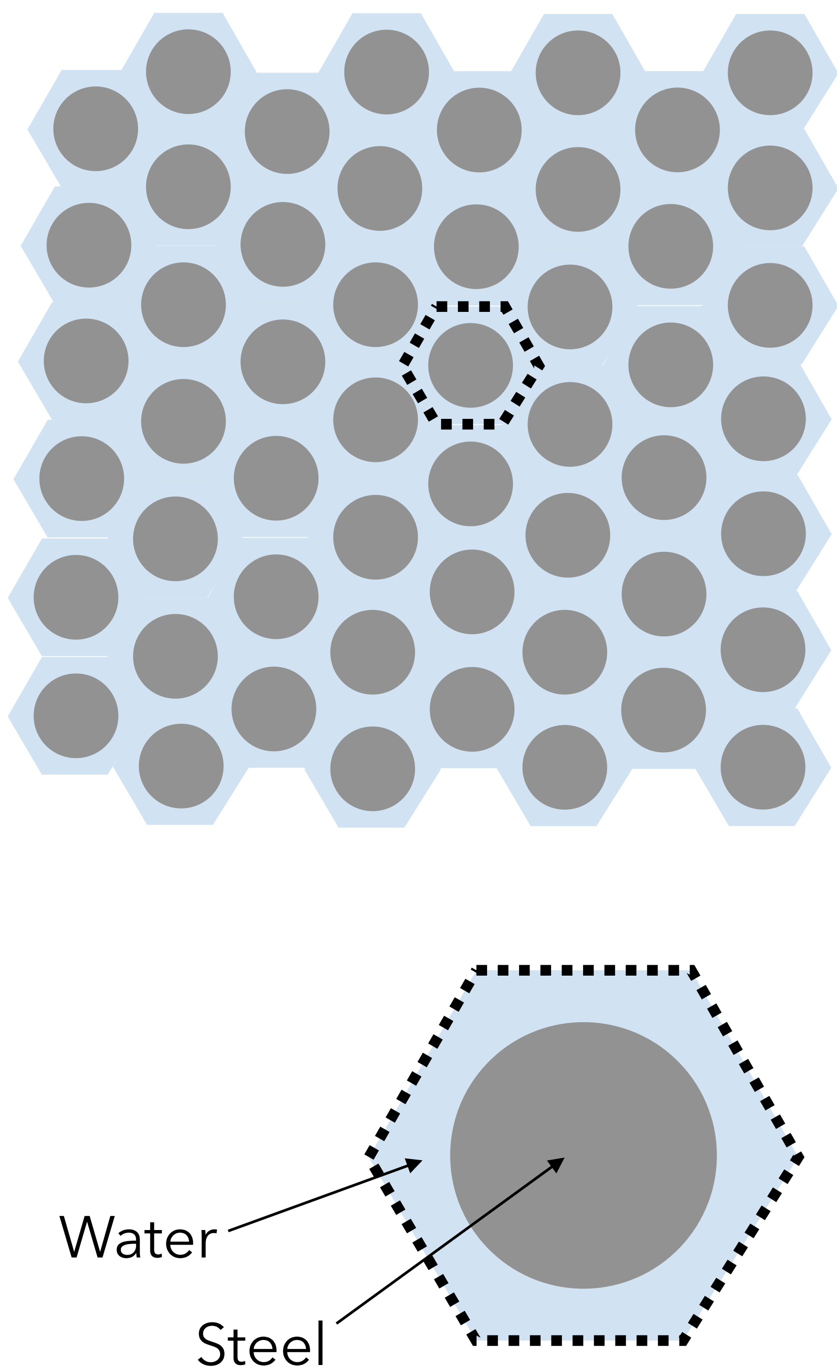
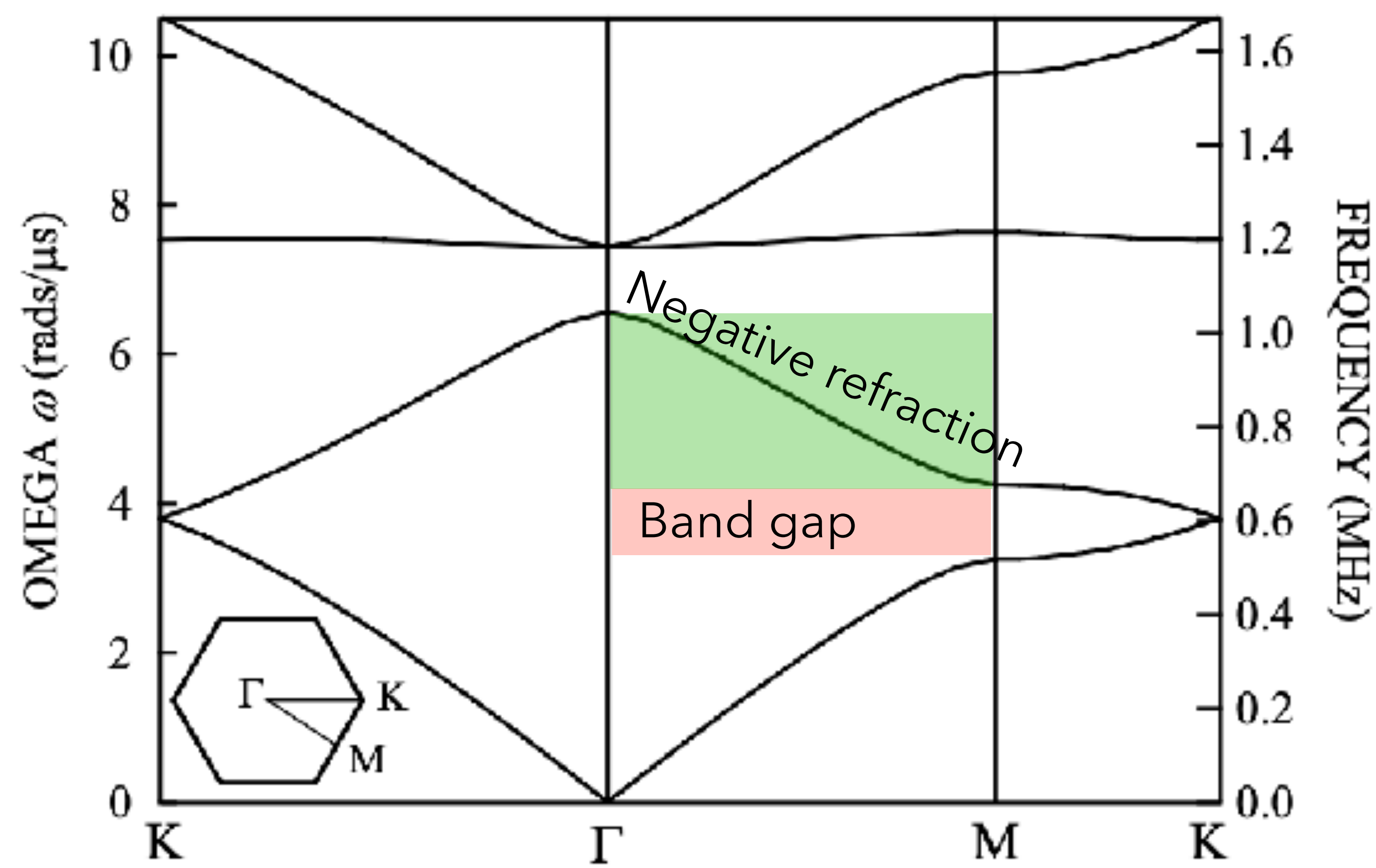
Band diagram of a phononic crystal

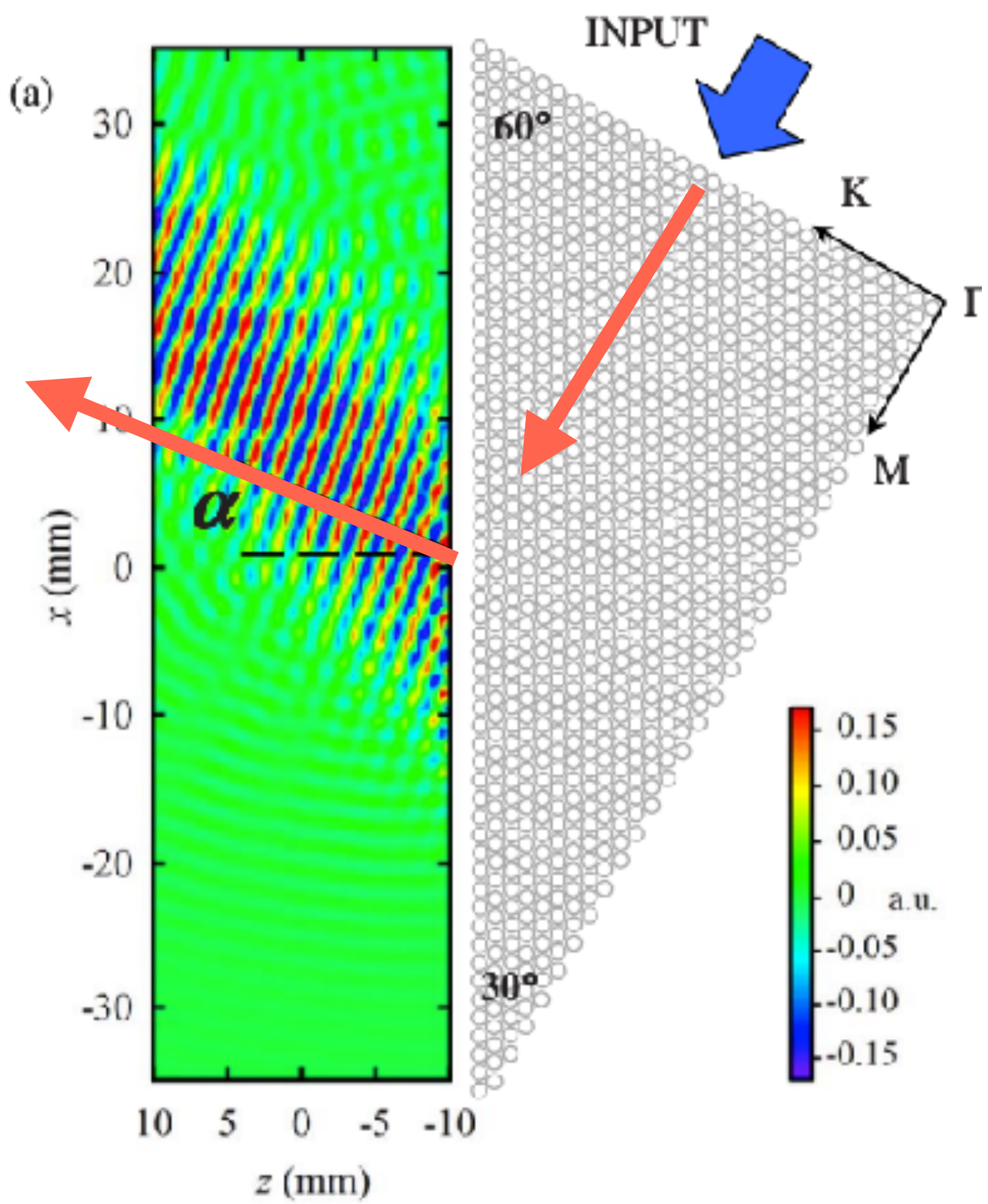


[Sukhovich et al., PRB, 2008]



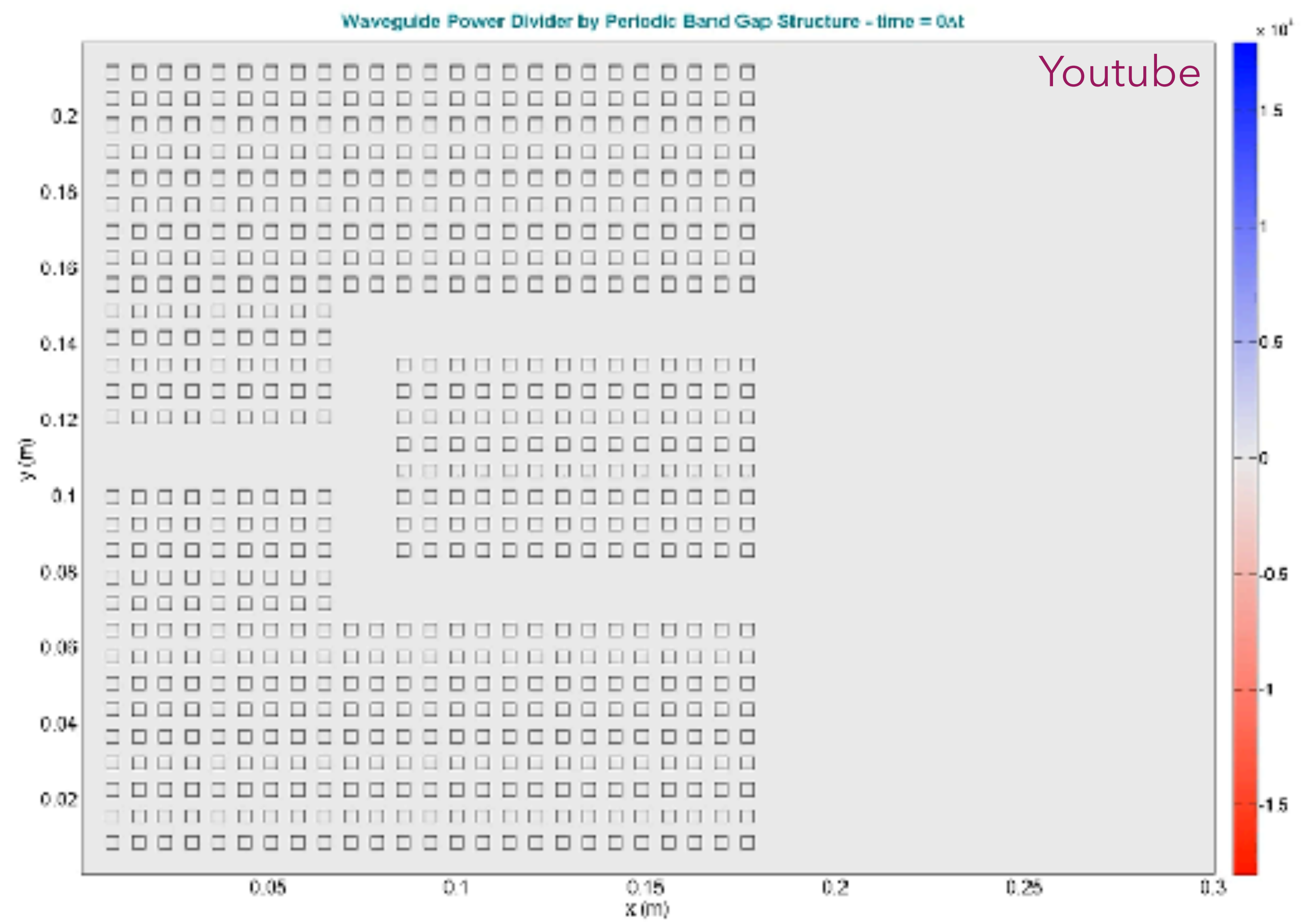
Band diagram of a phononic crystal

Negative refraction



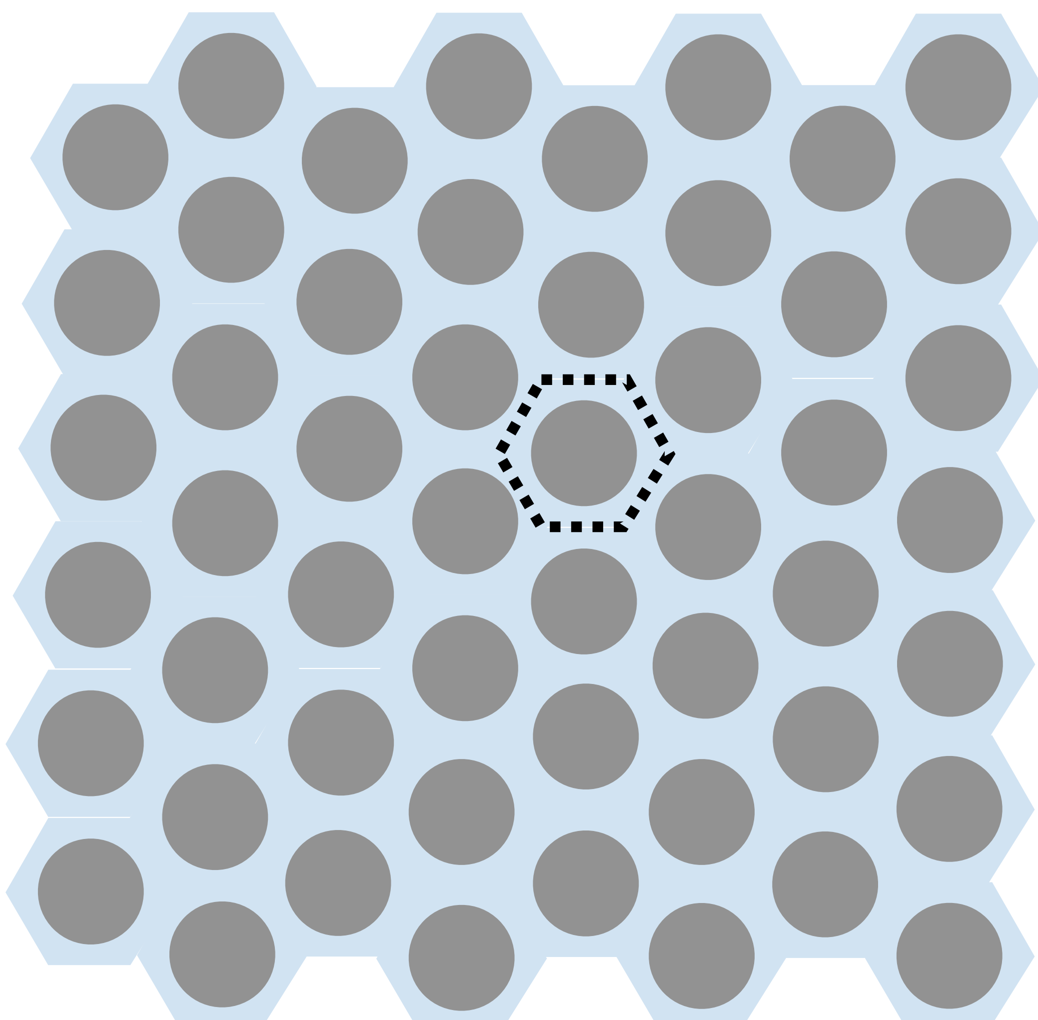
[Sukhovich et al., PRB, 2008]

Band gap



Applications : perfect flat lenses, guided waves, …

Crystal vs material



[Sukhovich et al., PRB, 2008]

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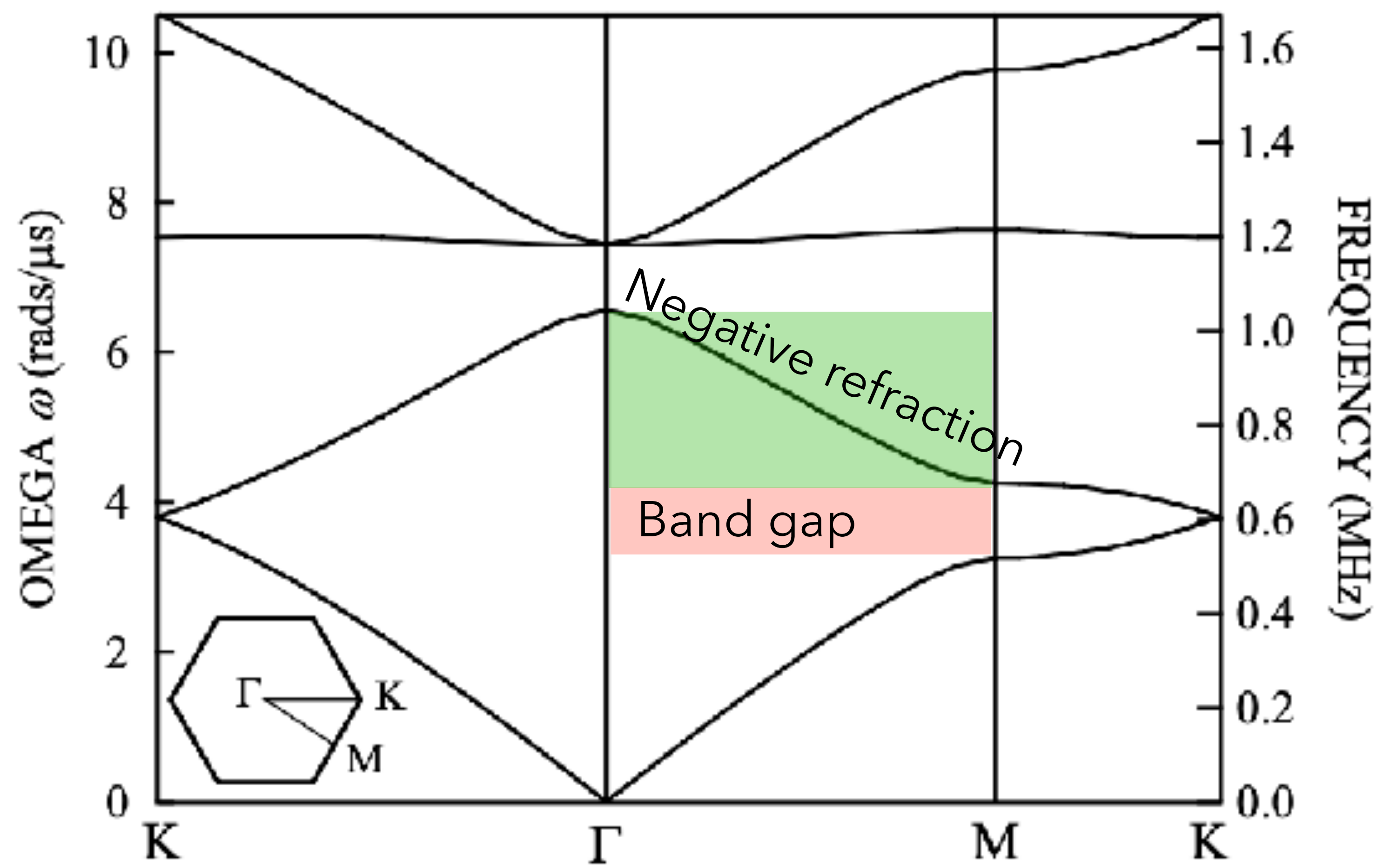

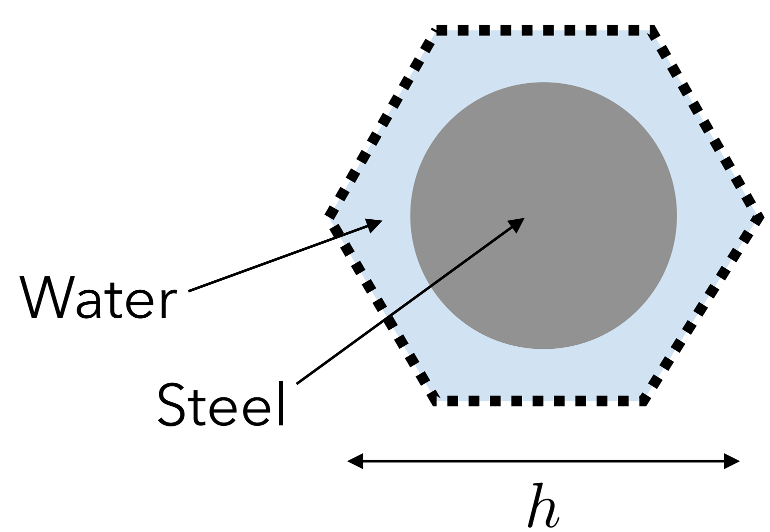

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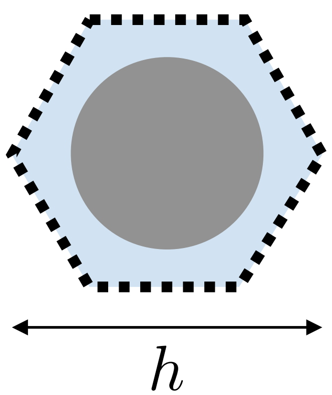
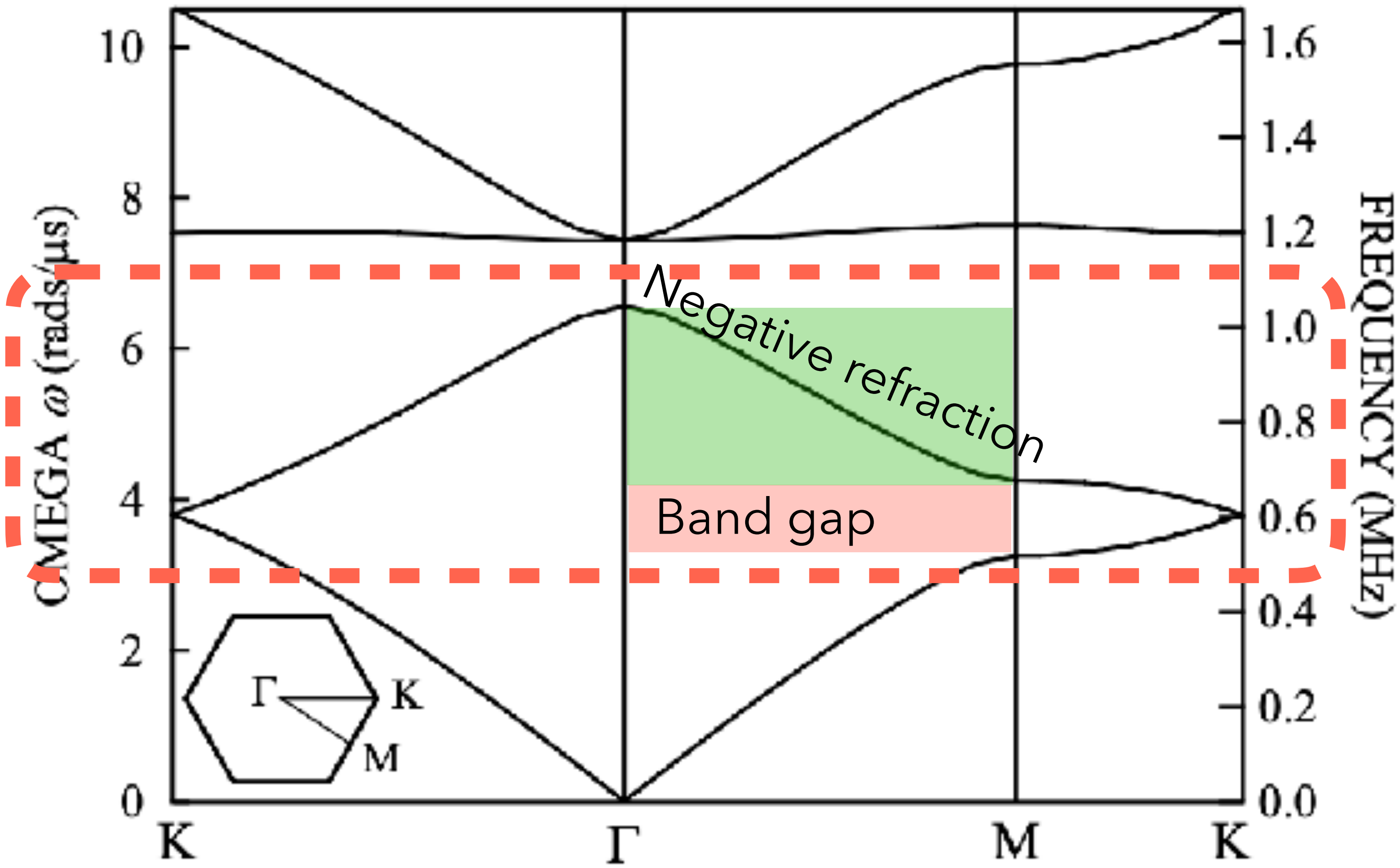


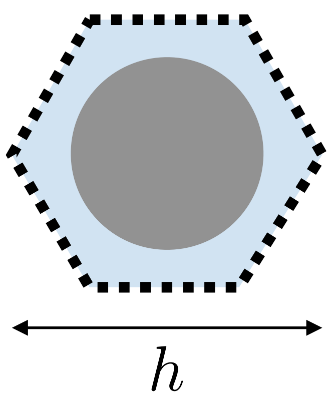
Crystal-like behavior

$$\frac{\omega h}{c} = O(1)$$

[Sukhovich et al., PRB, 2008]



Crystal vs material



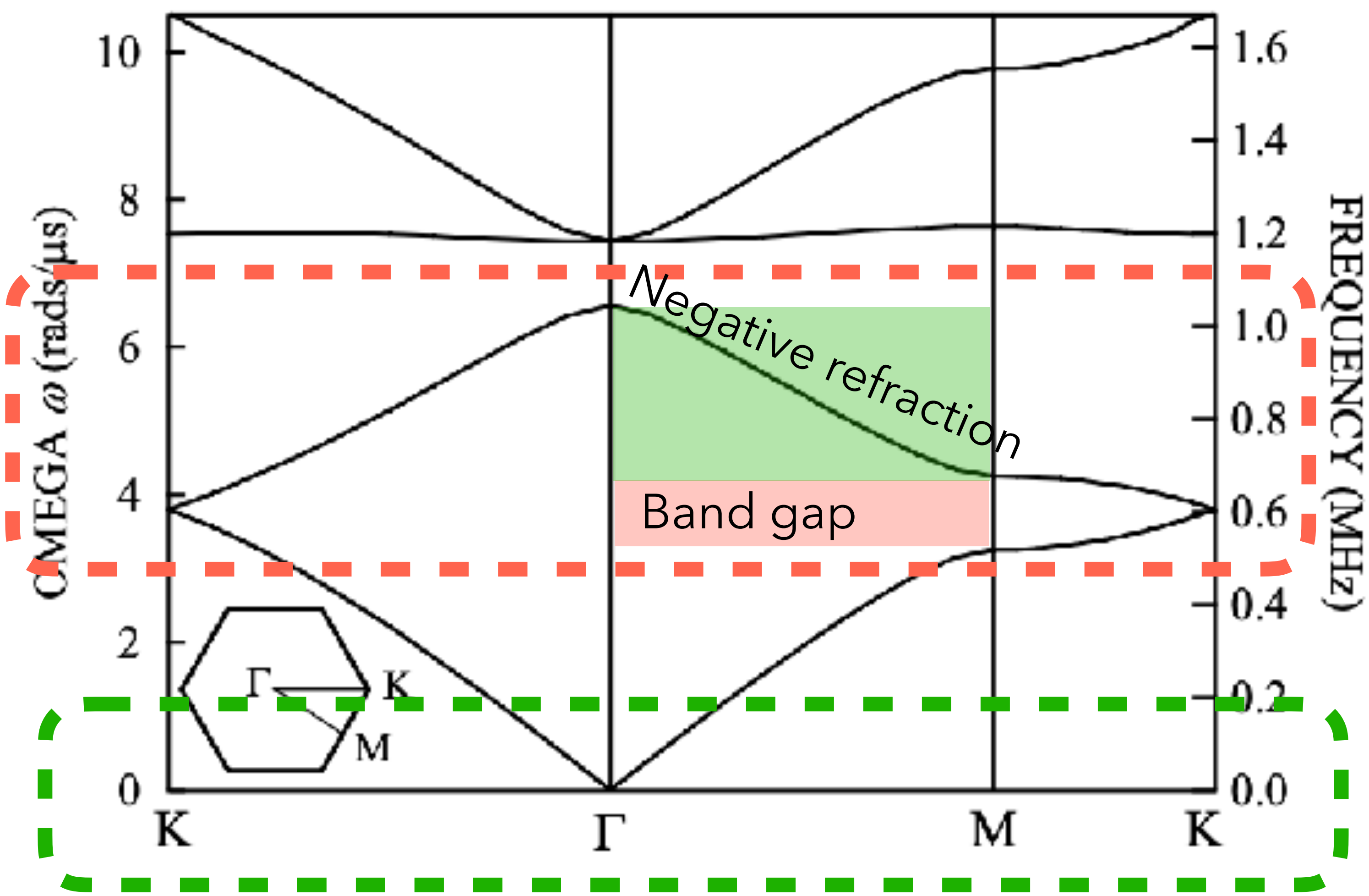
Crystal-like behavior

$$\frac{\omega h}{c} = O(1)$$

Low frequency regime

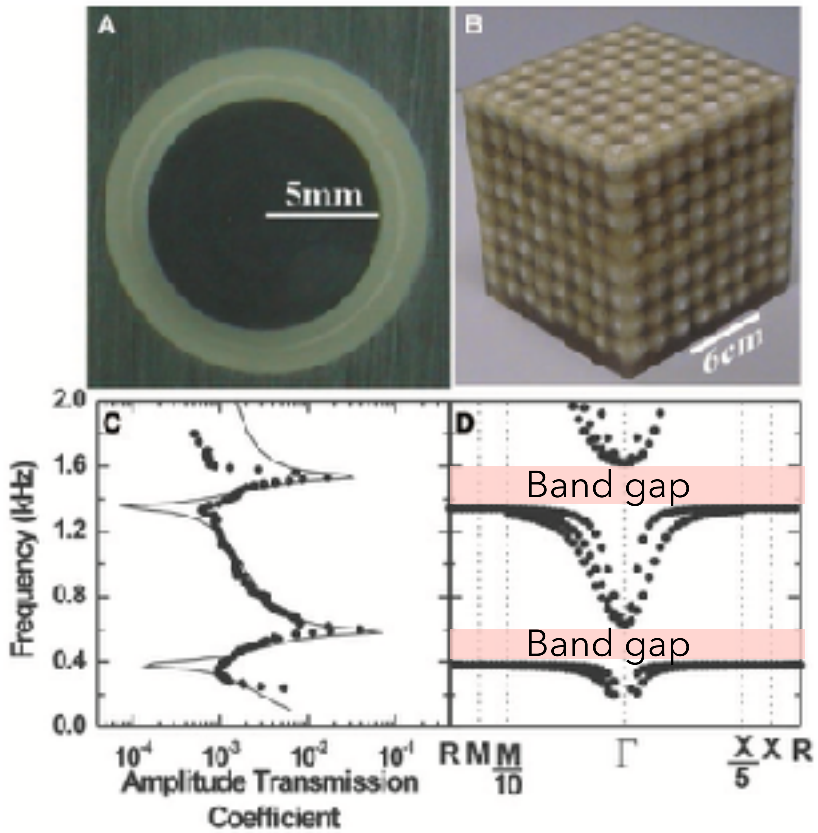
$$\frac{\omega h}{c} \ll 1$$

[Sukhovich et al., PRB, 2008]

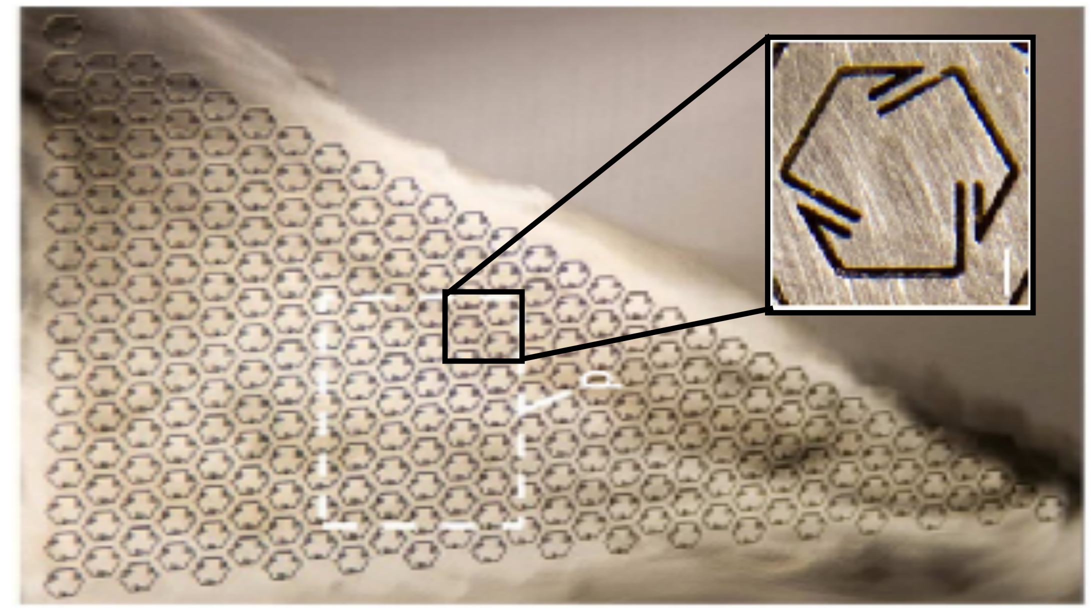


→ **No exotic behavior !**

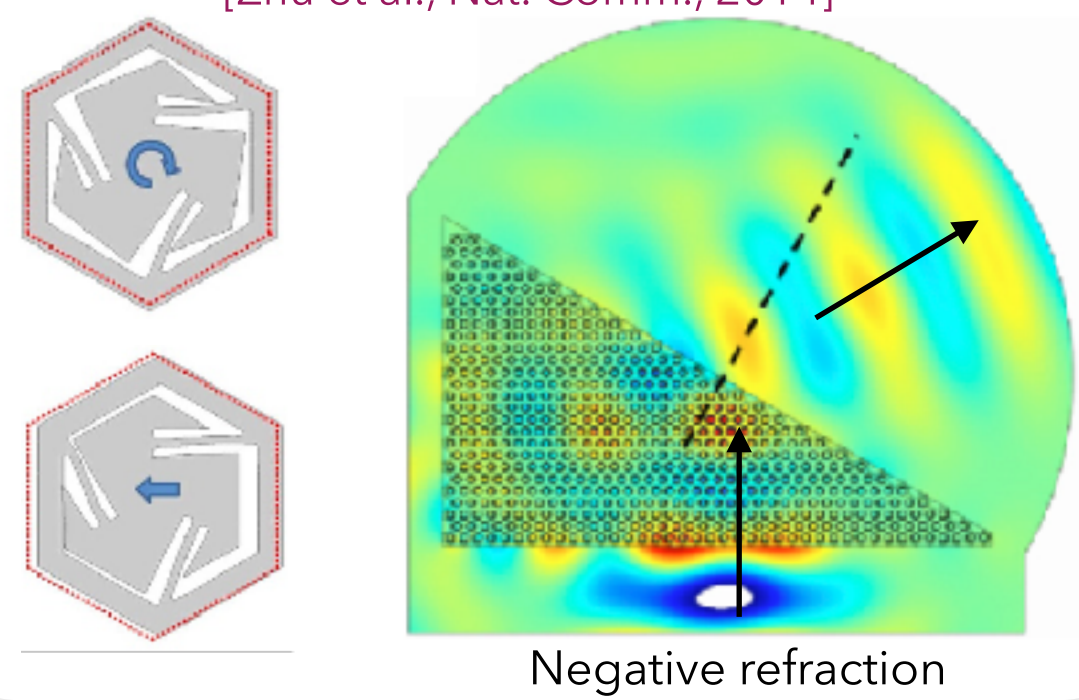
Locally resonant elastic metamaterials

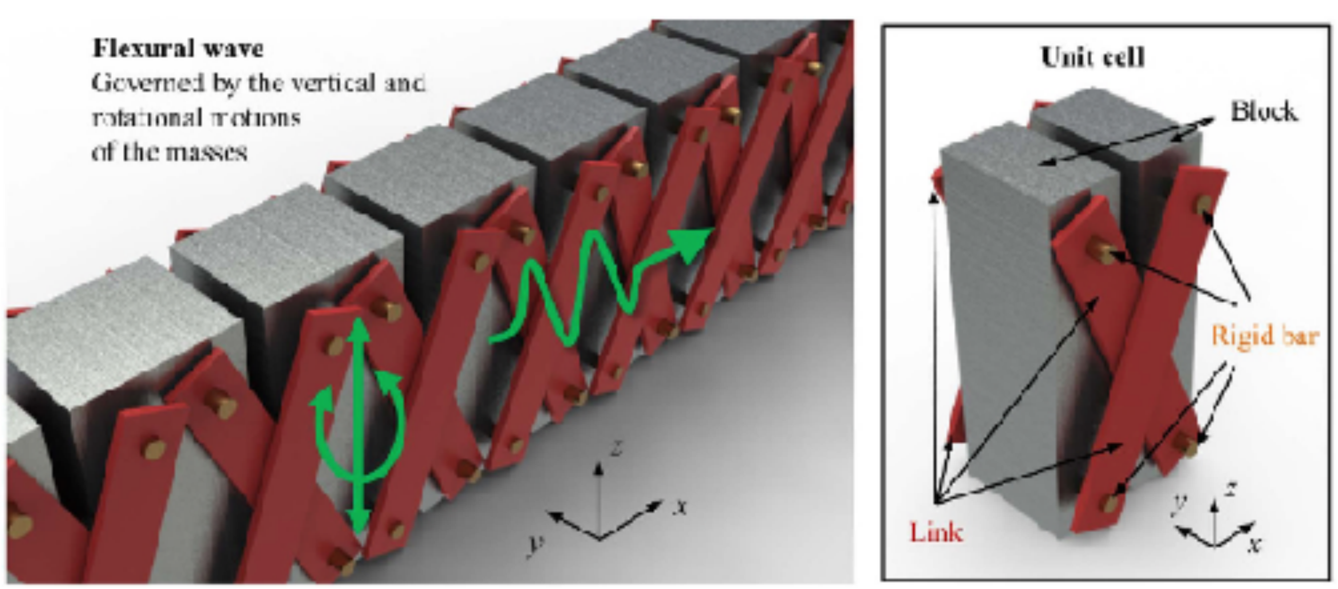


[Liu et al., Science, 2000]



[Zhu et al., Nat. Comm., 2014]





[Oh et al., Scientific Report, 2016]

- **Subwavelength microstructure**
- **Resonant mechanism**

Effective models

$$\mathrm{div}\boldsymbol{\sigma} + \omega^2 p = 0, \quad \boldsymbol{\sigma} = \mu_{\mathrm{eff}} \nabla u, \quad p = \rho_{\mathrm{eff}} u$$

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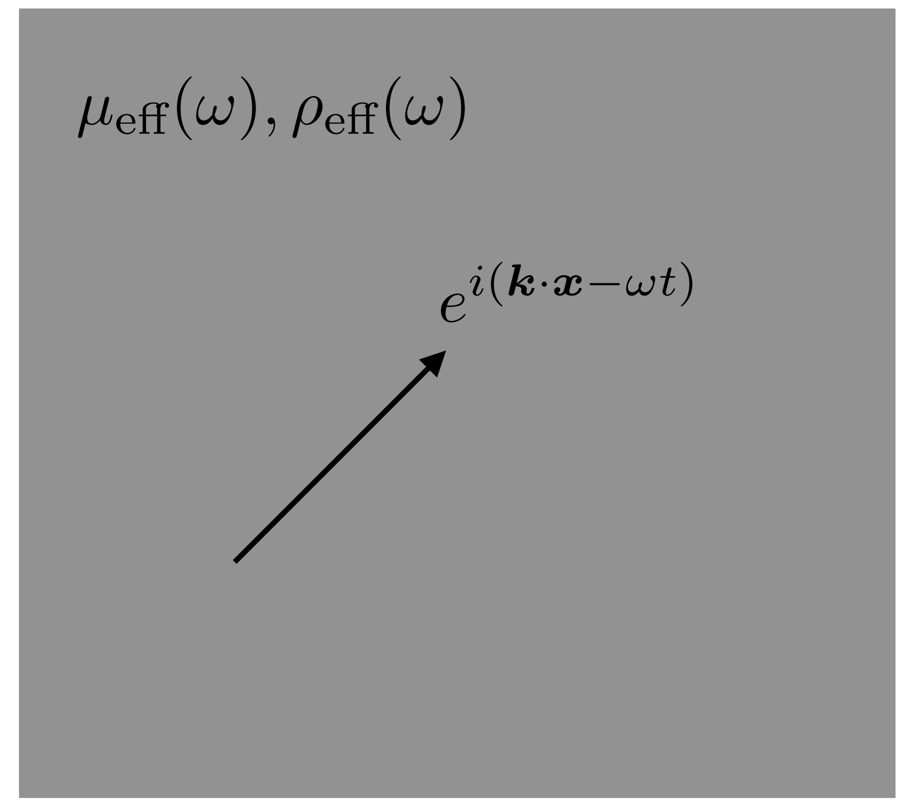
- **Band gap**

$$\mu_{\mathrm{eff}}(\omega) < 0 \quad \text{or} \quad \rho_{\mathrm{eff}}(\omega) < 0$$

$$|\boldsymbol{k}|^2 = \frac{\rho_{\mathrm{eff}}(\omega)}{\mu_{\mathrm{eff}}(\omega)} \omega^2$$



Effective models

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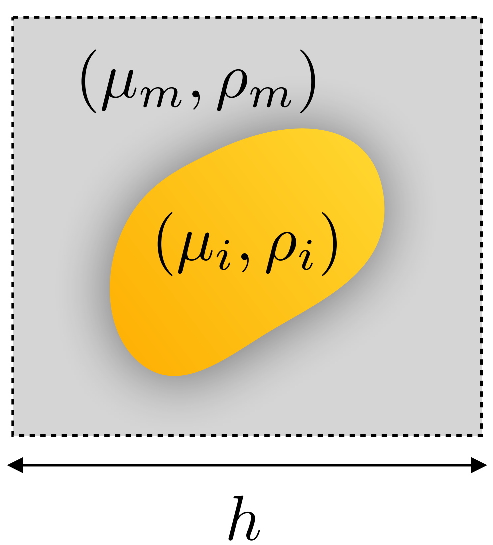
- **Band gap**

$\rho_{\text{eff}}(\omega) < 0$ on a range of frequencies



$\mu_i \ll \mu_m$

$\rho_i \sim \rho_m$

$kh \ll 1$

$\longrightarrow$ Mie resonances

Effective models

$$\mathrm{div}\boldsymbol{\sigma} + \omega^2 p = 0, \quad \boldsymbol{\sigma} = \mu_{\mathrm{eff}} \nabla u, \quad p = \rho_{\mathrm{eff}} u$$

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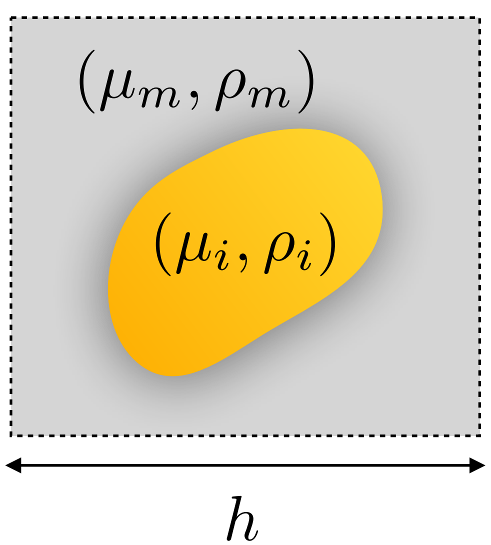
$$\rho_{\mathrm{eff}}(\omega) < 0 \quad \text{on a range of frequencies}$$



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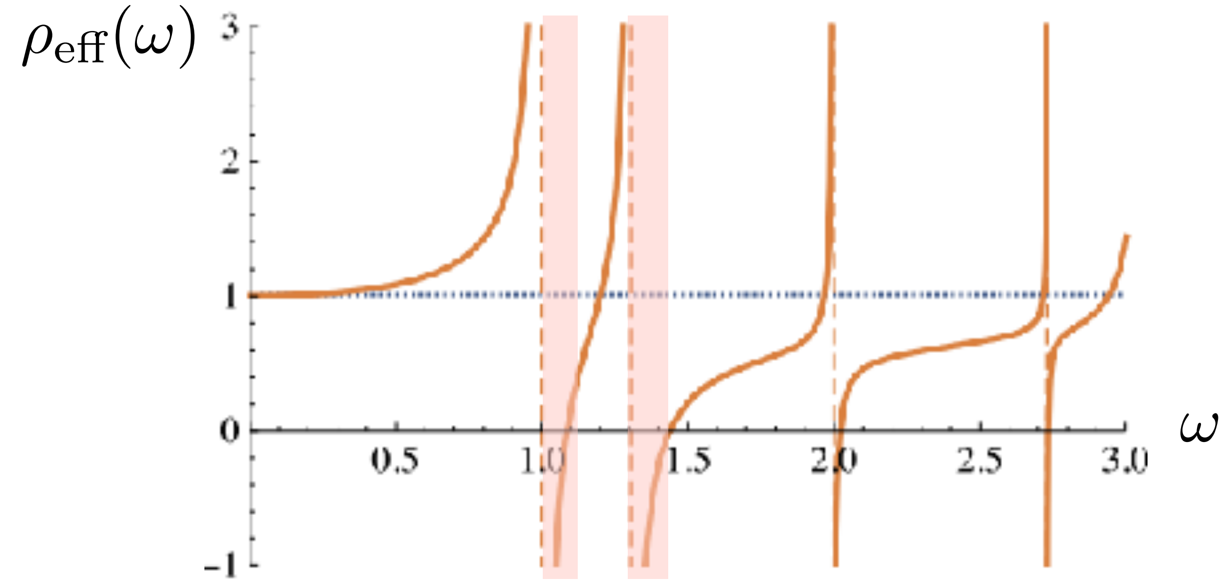

Band gaps (eigenfrequencies)

(with $\mu_{\mathrm{eff}} > 0$)

- Homogenization approach [Boutin et al, Bouchitte et al., Marigo et al, Pham et al…]
- Discrete approach [Milton and Willis, …]
- Retrieval methods [Liu et al., …]

Effective models

- **Negative refraction**

[Veselago, 1967] $\mu_{\text{eff}}(\omega) < 0, \quad \rho_{\text{eff}}(\omega) < 0$ Two resonant mechanisms are needed

Usual method : Band diagram ⟶ Retrieval methods ⟶ $(\mu_{\text{eff}}(\omega), \rho_{\text{eff}}(\omega))$

Retrieval methods are not predictive (provide a posteriori interpretations)

Miss the physics of the resonant mechanism and the link with the microstructure

Lack of homogenization results (too costly)

Effective models

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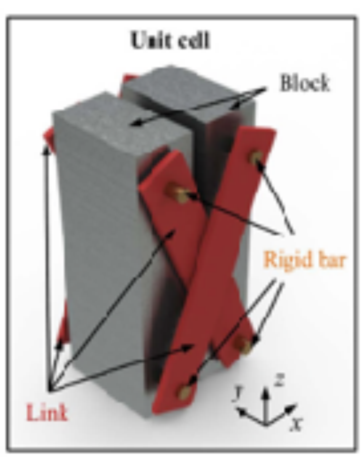
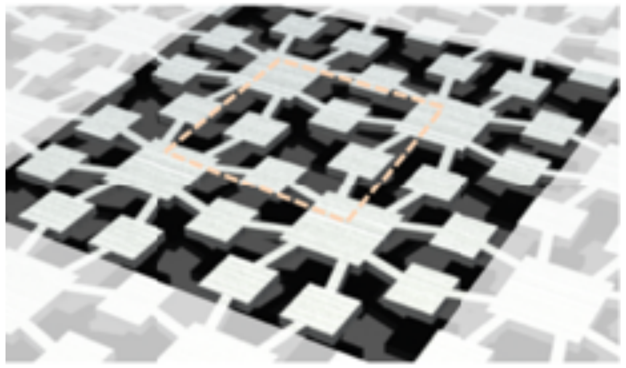
- **Elastic metamaterials showing negative refraction due to flexural resonances**



[Oh et al., 16]

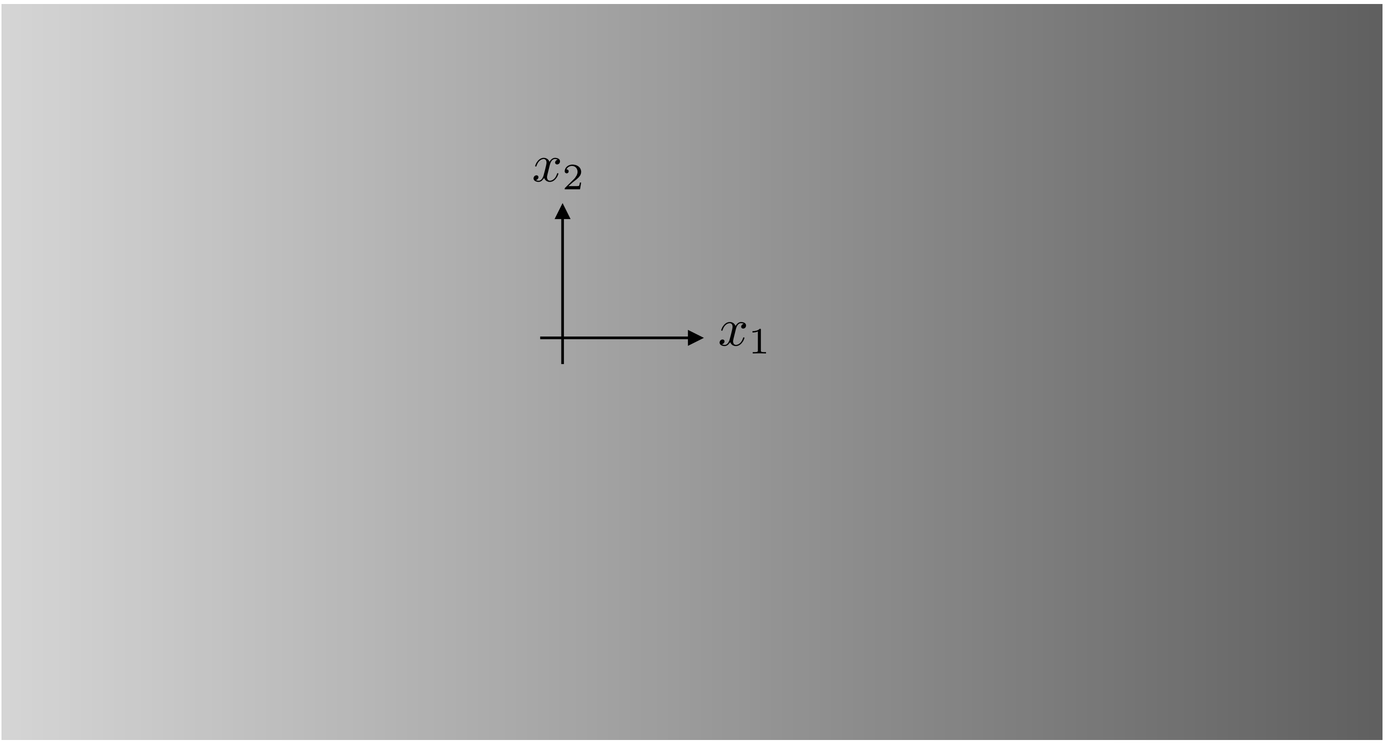
[Zhu et al., 14]



[Oh et al., 16]

Simpler microstructure ?

Setting of the problem



$$\mathrm{div}\boldsymbol{\sigma} = -\rho\omega^2\boldsymbol{u}, \quad \boldsymbol{\sigma} = E\boldsymbol{\varepsilon}(\boldsymbol{u}), \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + {}^{\mathrm{t}}\nabla\boldsymbol{u}\right) \qquad (\nu = 0)$$

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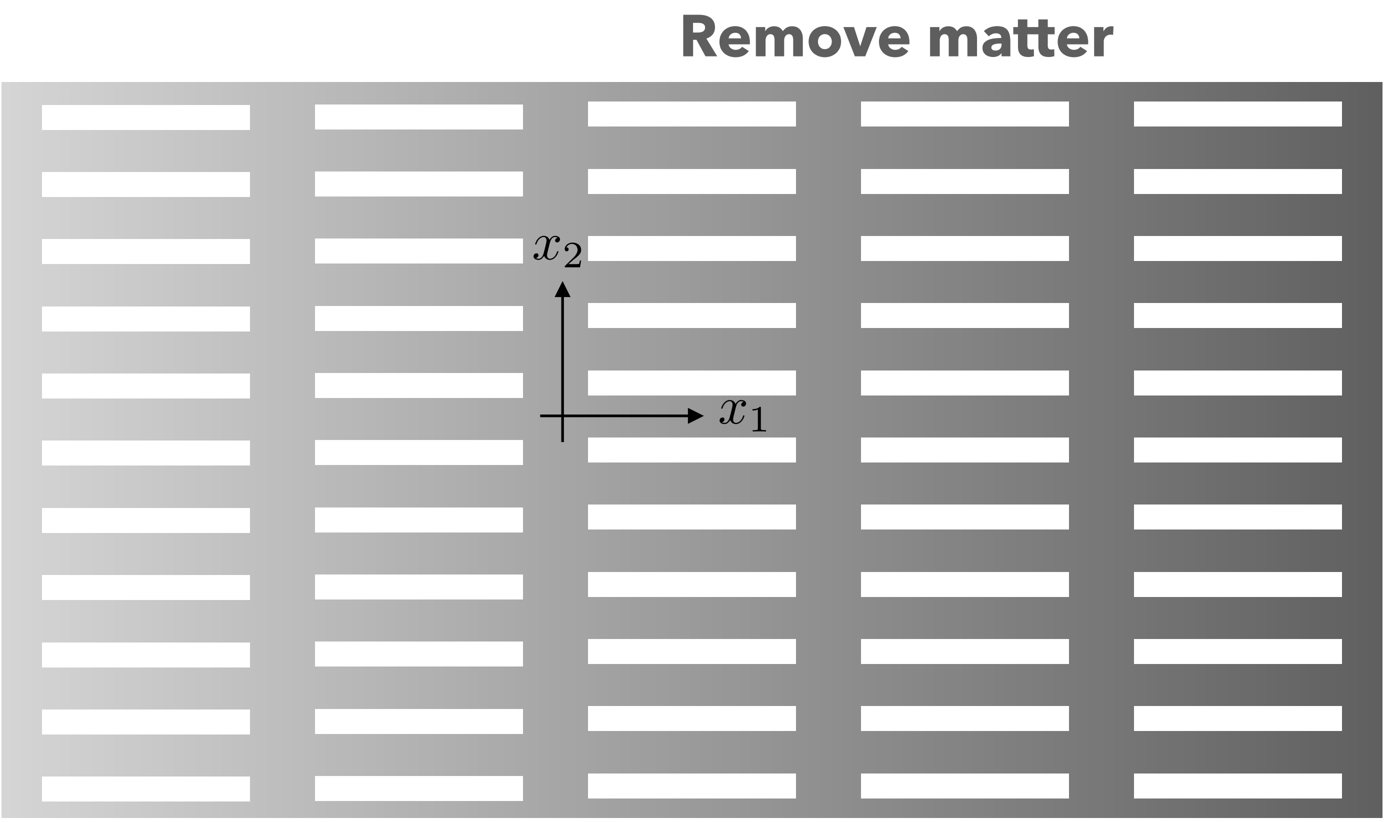

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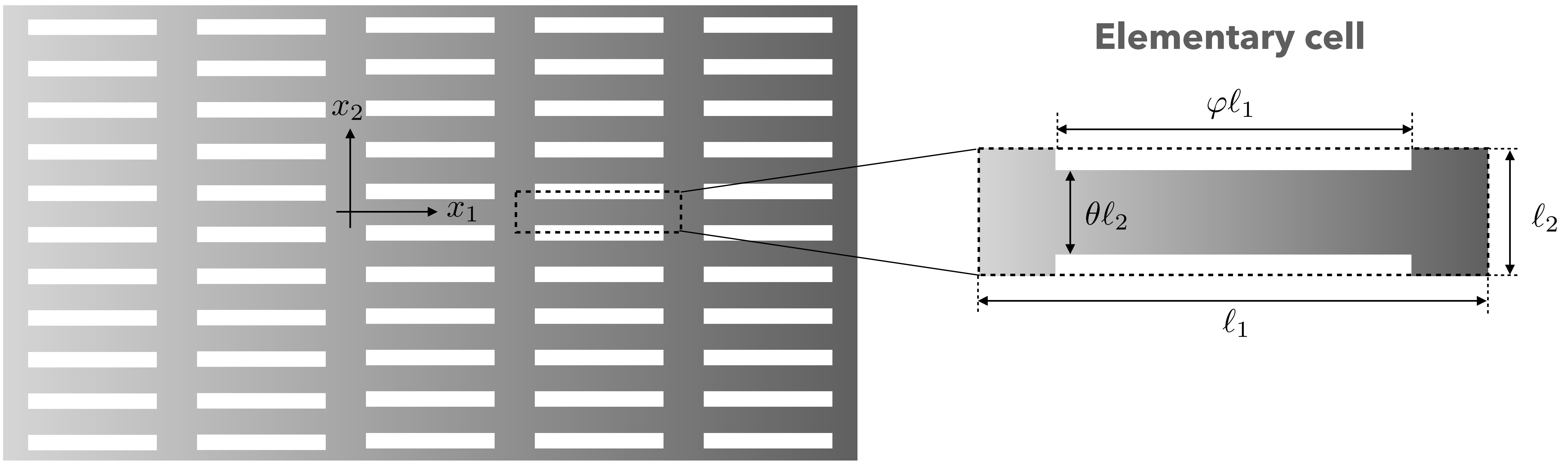

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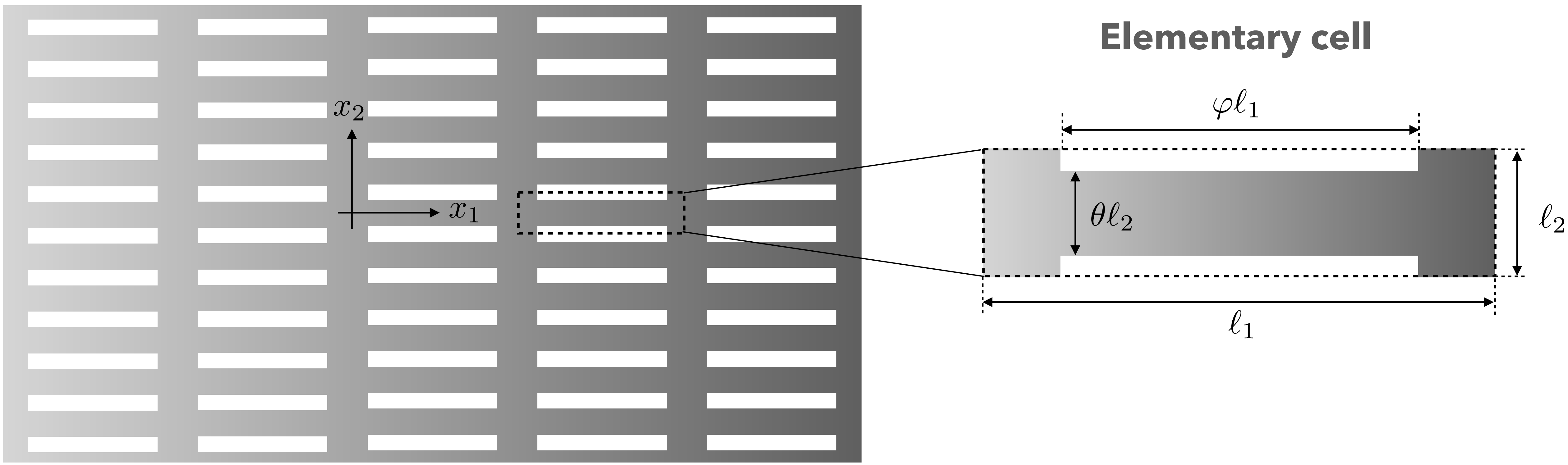
$$\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{0} \quad \text{on free edges}$$

Setting of the problem



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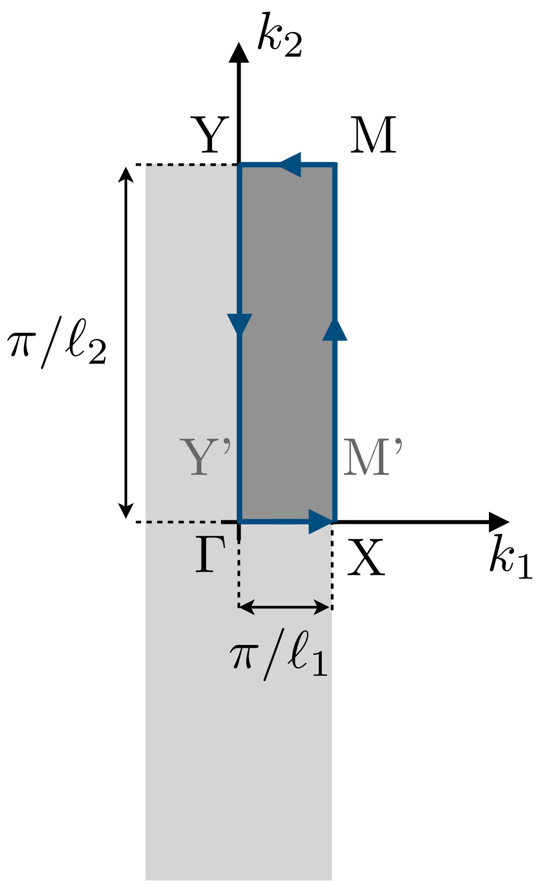
Conditions for the propagation of waves ?

Band diagram

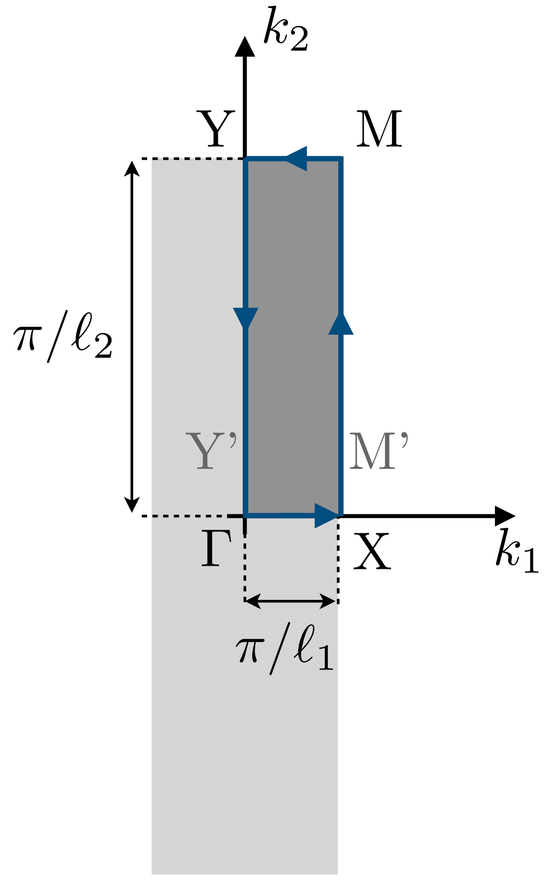
Reciprocal space



$\ell_2 \ll \ell_1$

Band diagram

Reciprocal space



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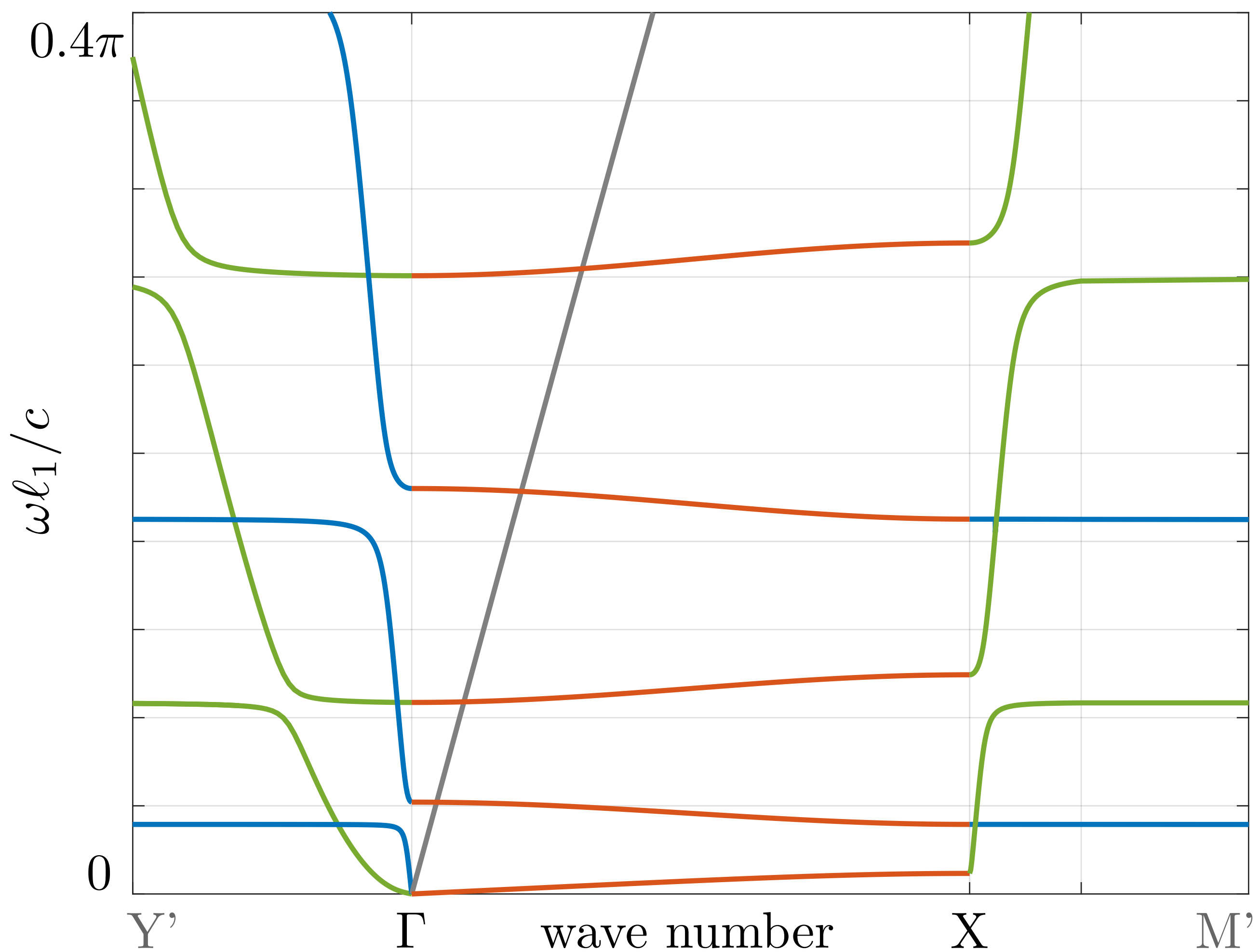

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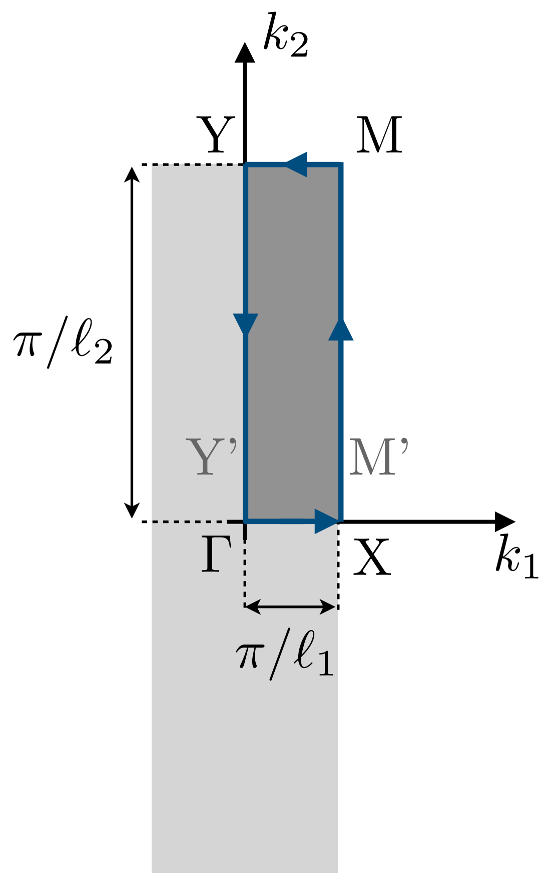

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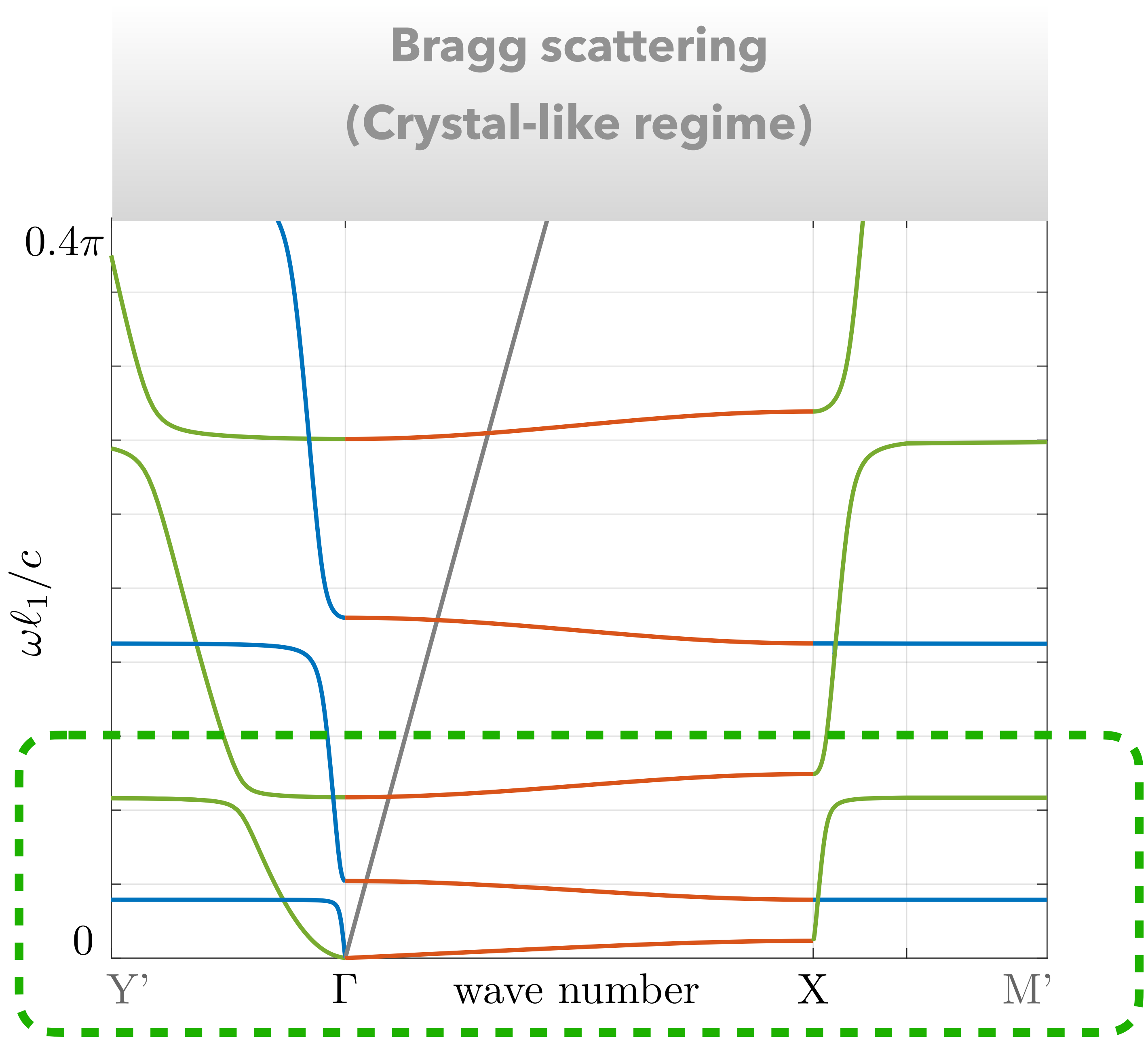
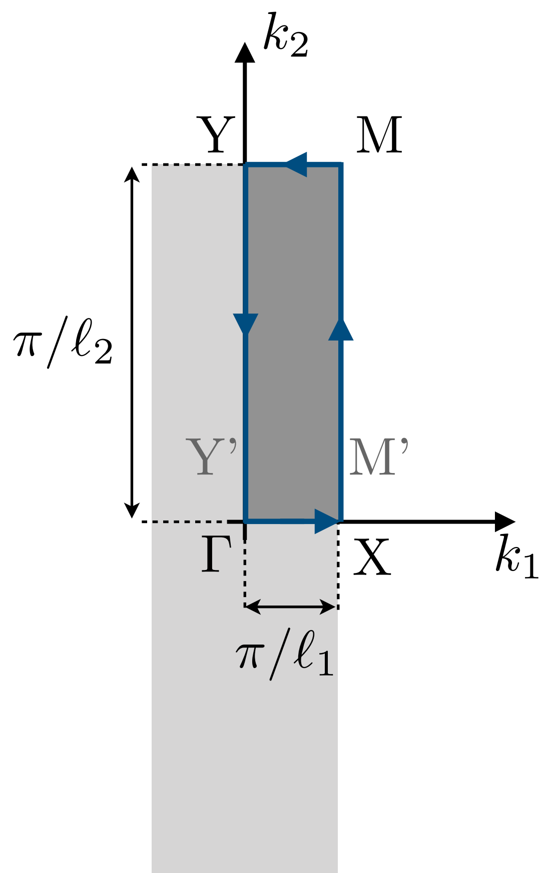


Exotic behavior at low frequency regime

Band diagram

Reciprocal space



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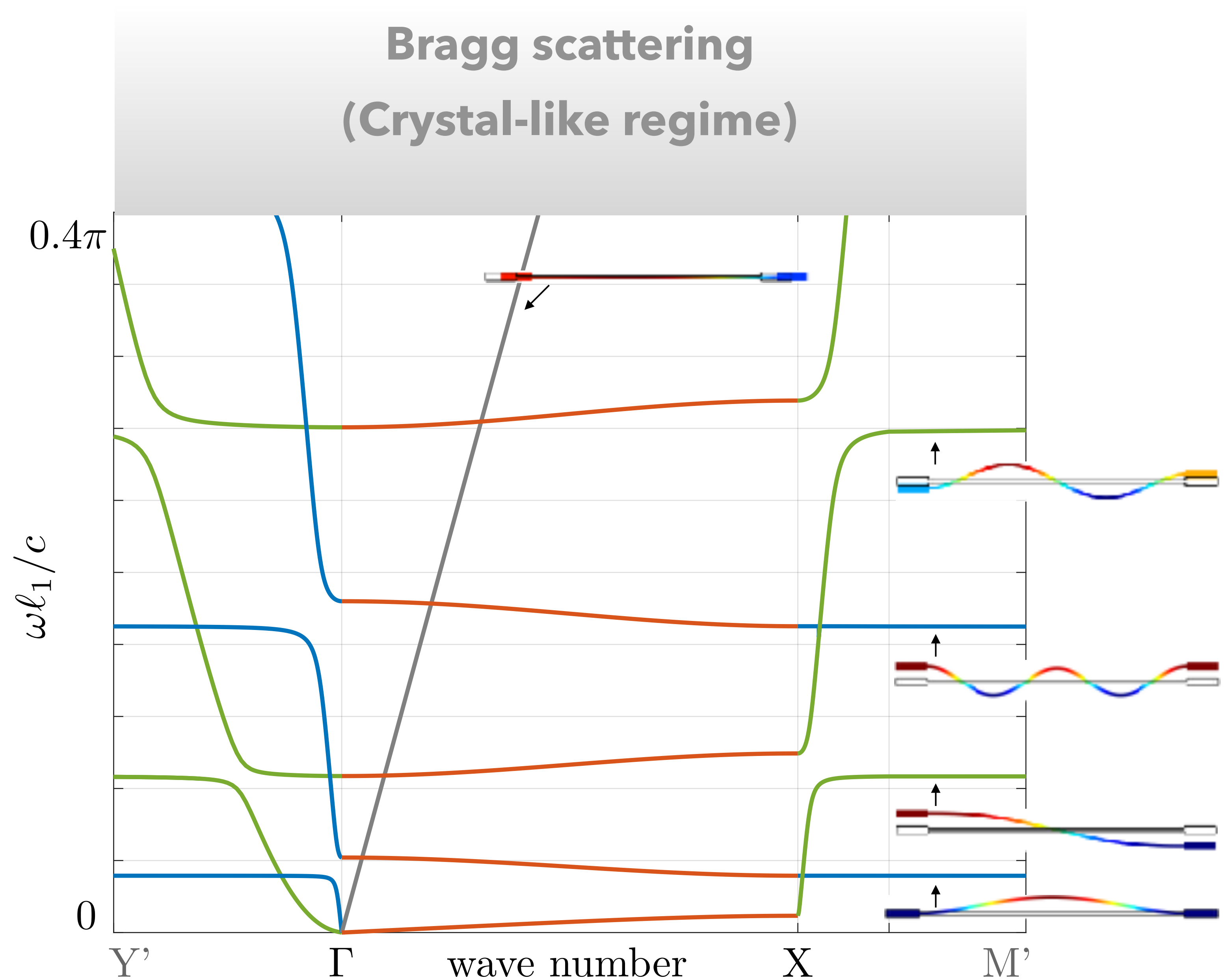
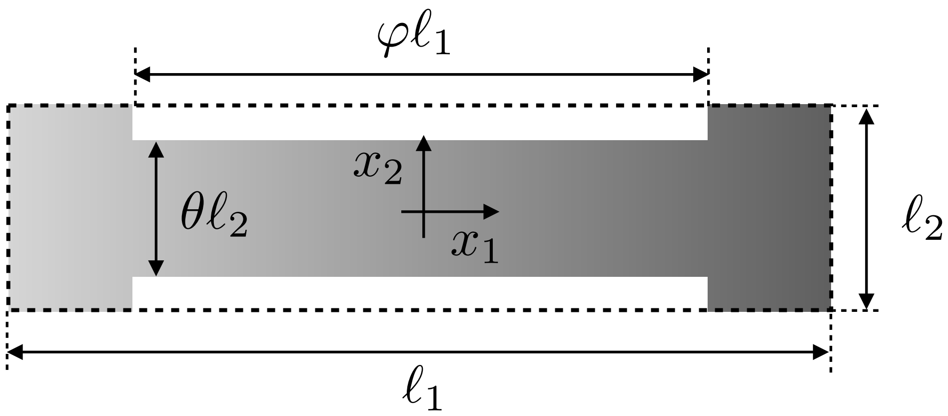


Dispersion curves dictated by flexural resonances

Asymptotic Bloch-Floquet analysis



$$\eta = \frac{\ell_2}{\ell_1} \ll 1$$

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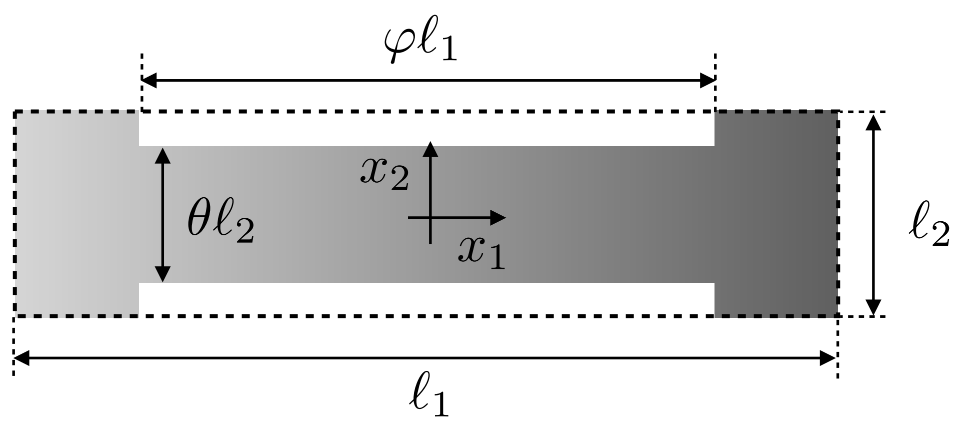

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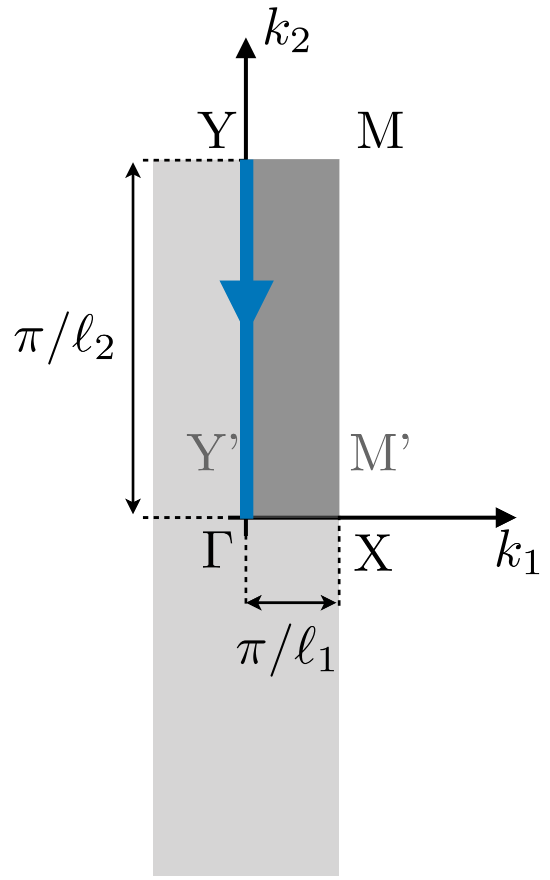
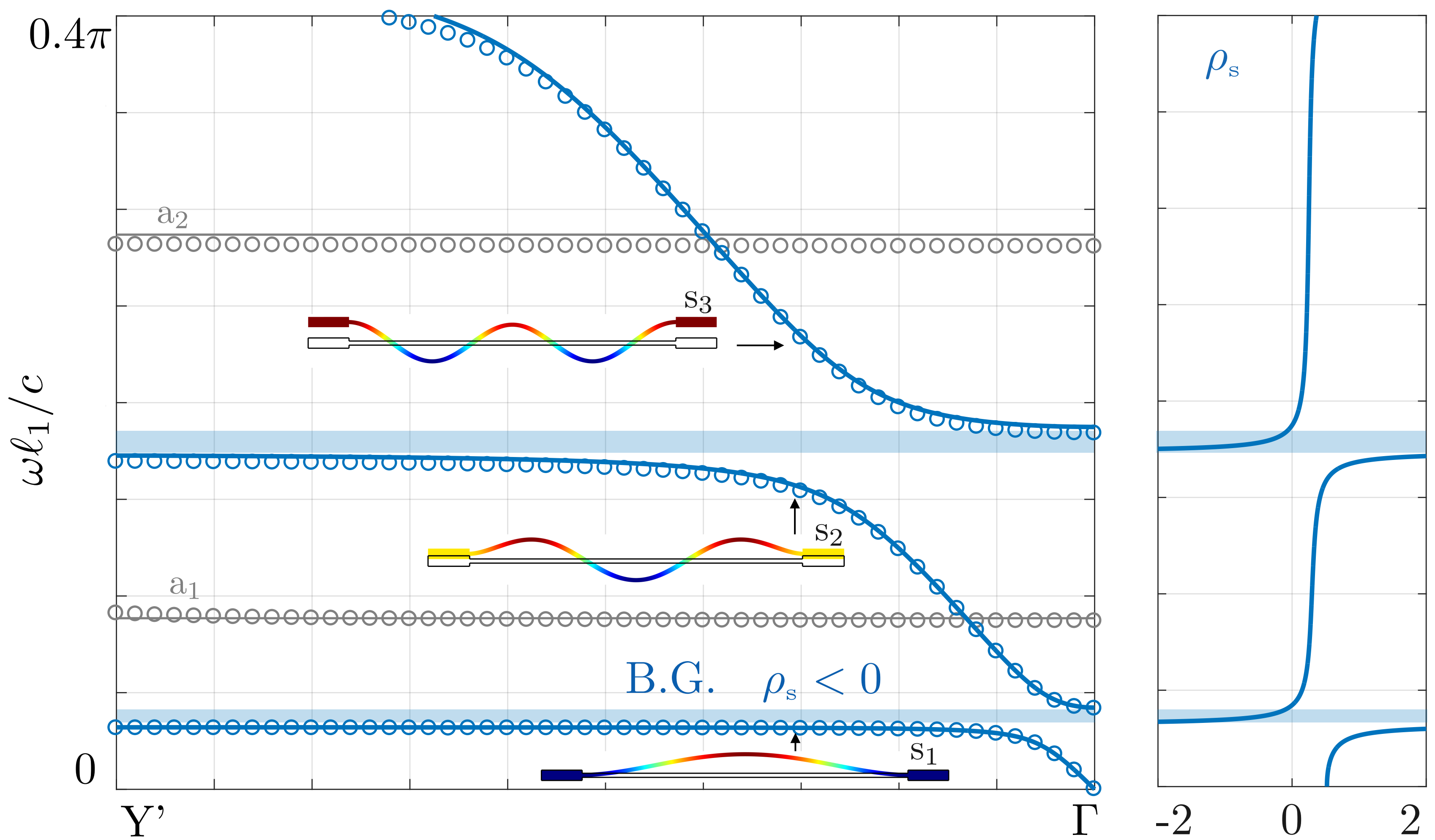
Bloch-Floquet Decomposition

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^p(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{\sigma}^p(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

$$(\boldsymbol{u}^p(\boldsymbol{x}), \boldsymbol{\sigma}^p(\boldsymbol{x})) \quad \text{periodic}$$

Dispersion relations

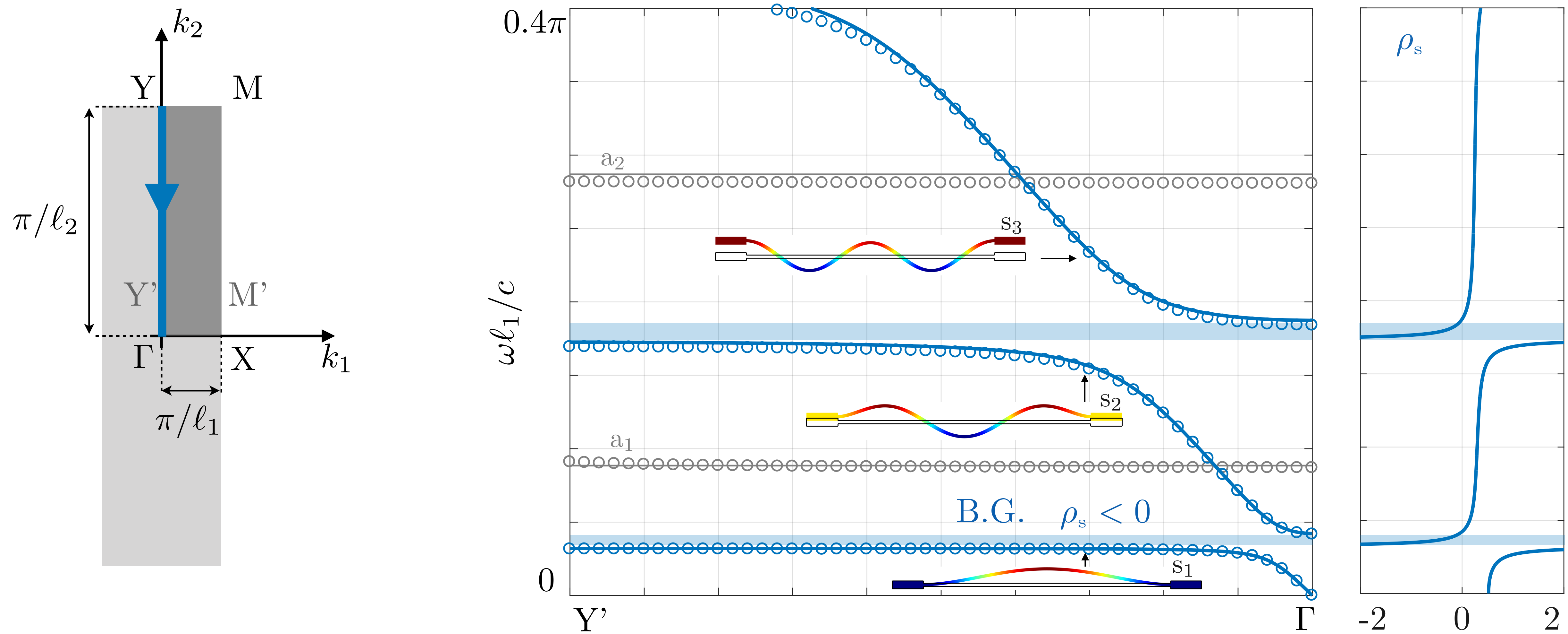




$$\left((1-\varphi)k_2^2 - \rho_s\omega^2\right)\hat{U}_2^0 = 0 \longrightarrow$$

$$k_2 = \sqrt{\frac{\rho_s(\omega)}{(1-\varphi)E}}\;\omega,$$

Dispersion relations

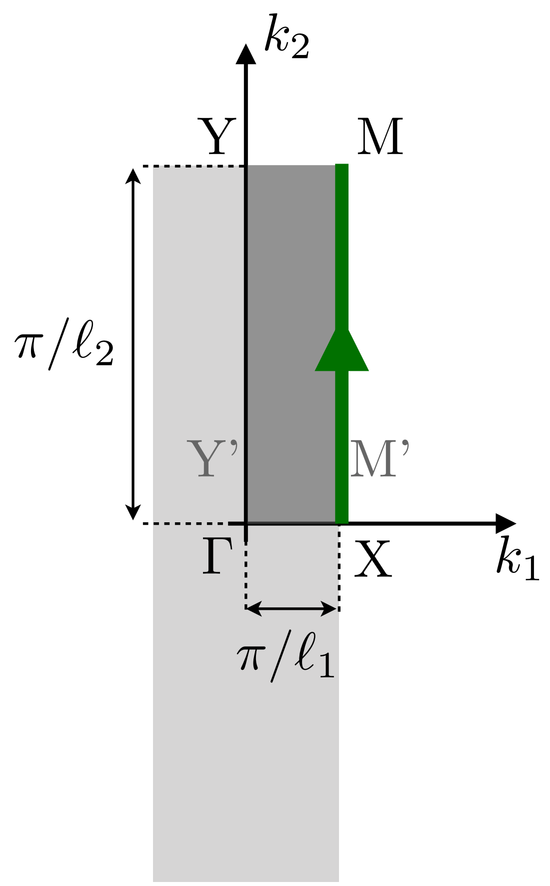


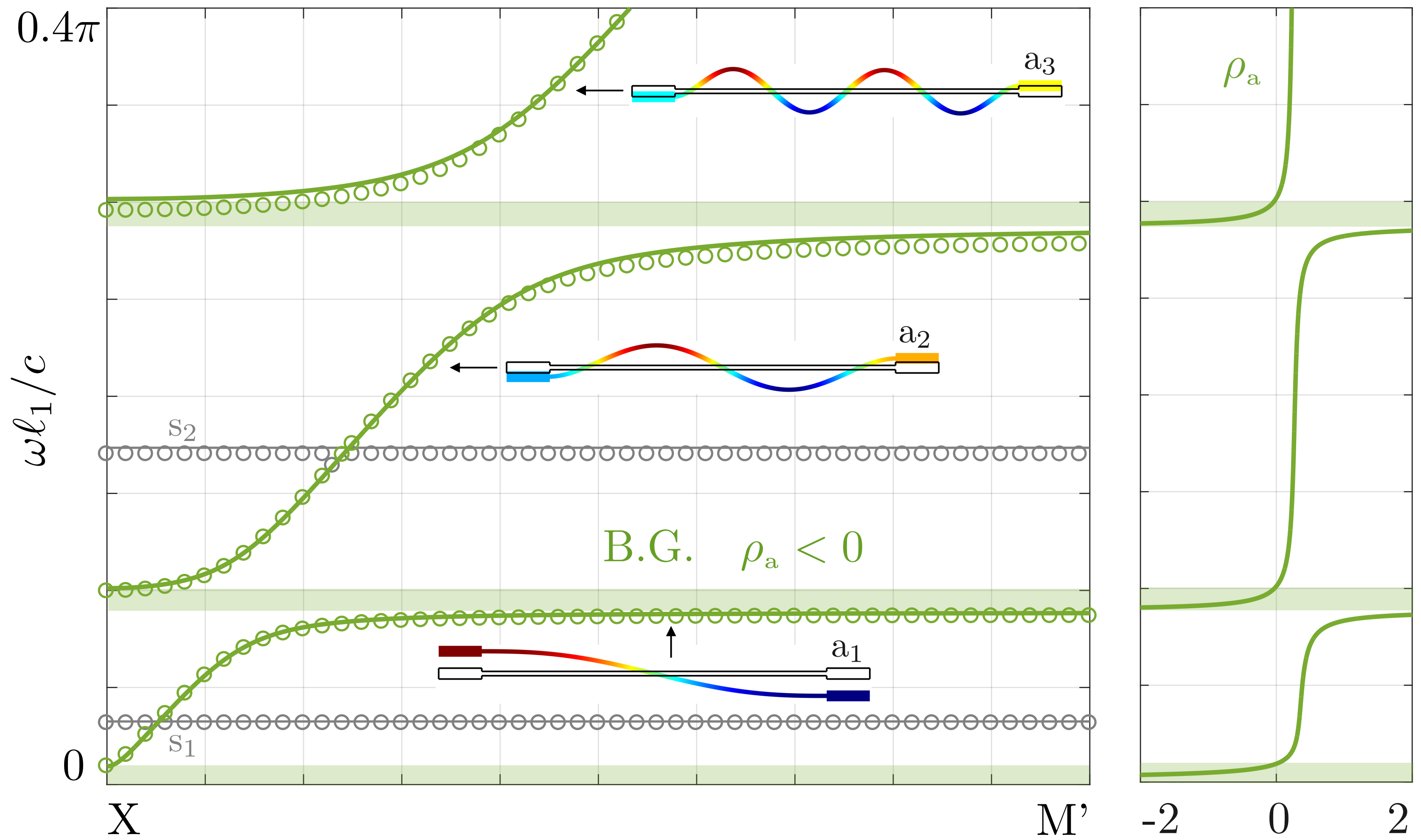
$$\left((1-\varphi)k_2^2 - \rho_s\omega^2\right)\hat{U}_2^0 = 0 \longrightarrow k_2 = \sqrt{\frac{\rho_s(\omega)}{(1-\varphi)E}}\;\omega,$$

$$\frac{\rho_s(\omega)}{\rho} = 1-\varphi+\frac{2\varphi\theta}{\Lambda(\omega)}\frac{\tan\Lambda(\omega)\tanh\Lambda(\omega)}{\tanh\Lambda(\omega)+\tan\Lambda(\omega)}, \qquad \Lambda(\omega) = \frac{\kappa(\omega)\varphi\ell_1}{2}, \qquad \kappa(\omega) = \left(\frac{12\rho\omega^2}{E\theta^2\ell_2^2}\right)^{1/4},$$

Even flexural modes

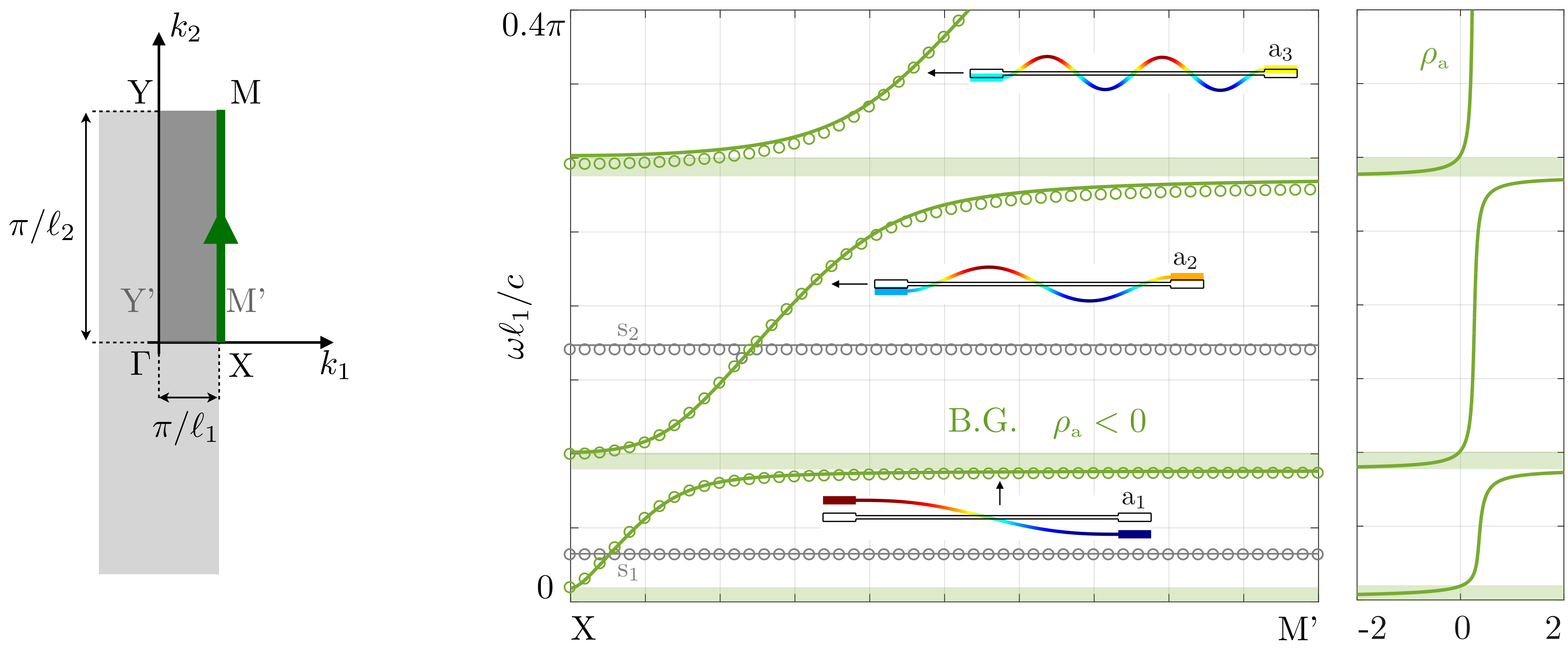
Dispersion relations





$$\left((1-\varphi)k_2^2 - \rho_a\omega^2\right)\hat{U}_2^0 = 0 \longrightarrow \quad k_2 = \sqrt{\frac{\rho_a(\omega)}{(1-\varphi)E}}\;\omega,$$

Dispersion relations



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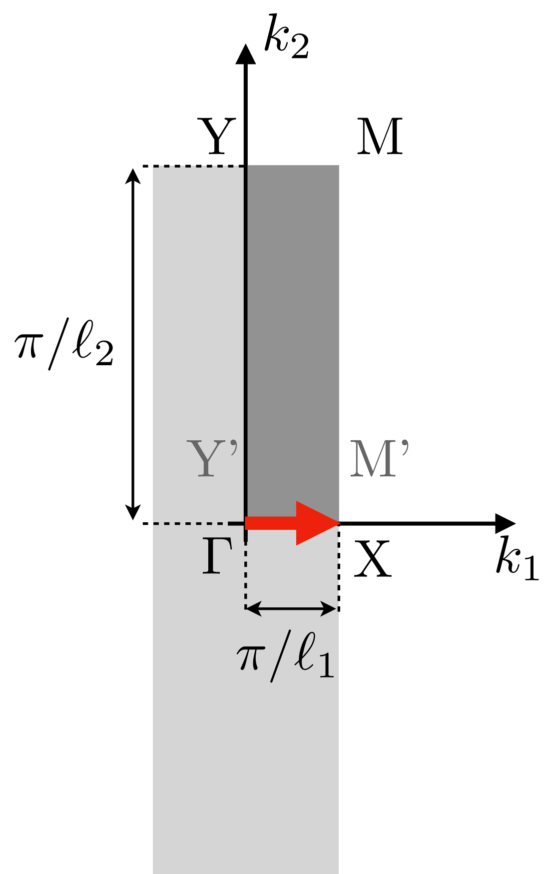
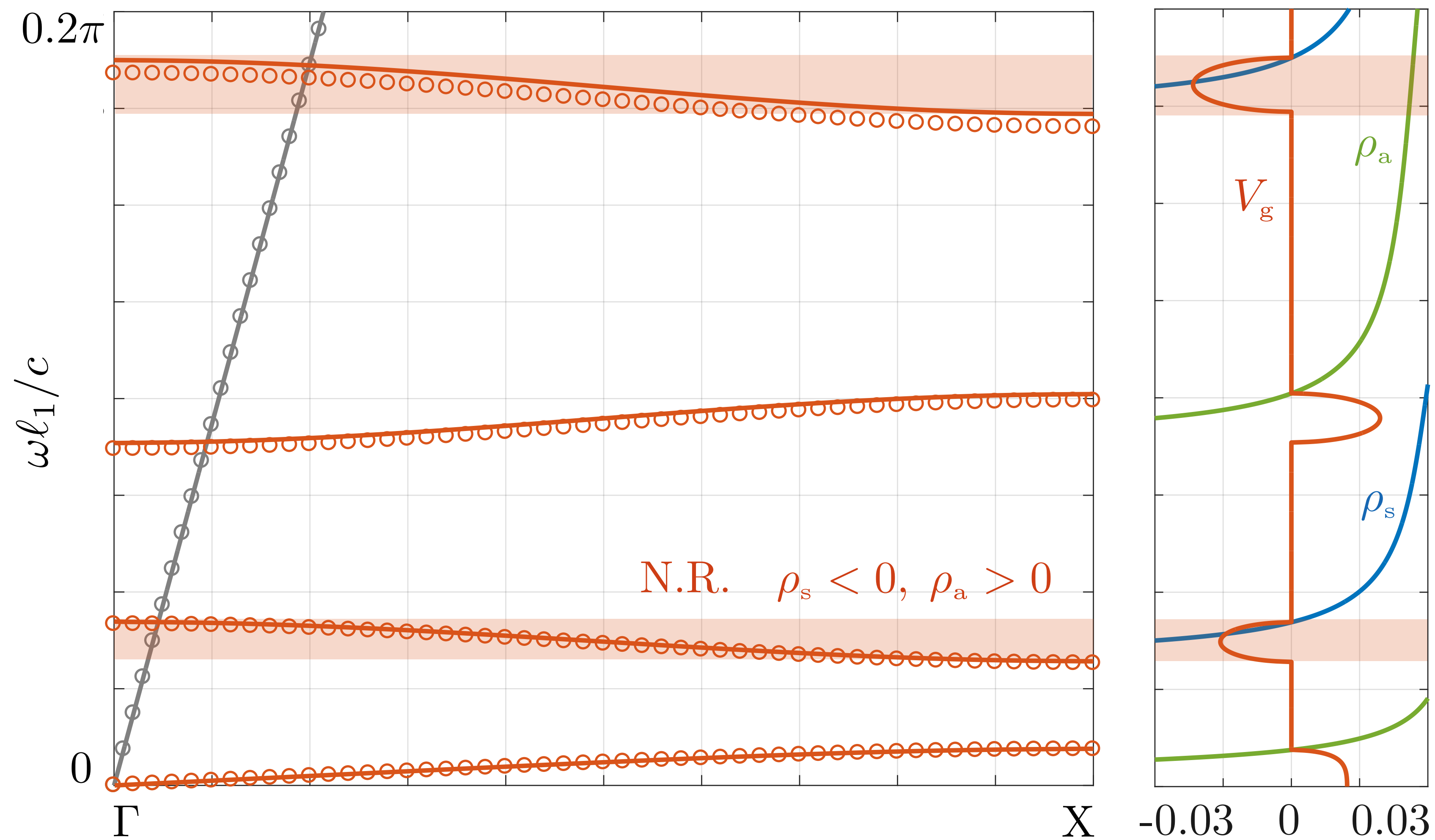
$$\frac{\rho_a(\omega)}{\rho} = 1-\varphi+\frac{2\varphi\theta}{\Lambda(\omega)}\frac{1}{\tanh\Lambda(\omega)-\tan\Lambda(\omega)},\qquad \Lambda(\omega)=\frac{\kappa(\omega)\varphi\ell_1}{2},\qquad \kappa(\omega)=\left(\frac{12\rho\omega^2}{E\theta^2\ell_2^2}\right)^{1/4},$$

Odd flexural modes

Dispersion relations



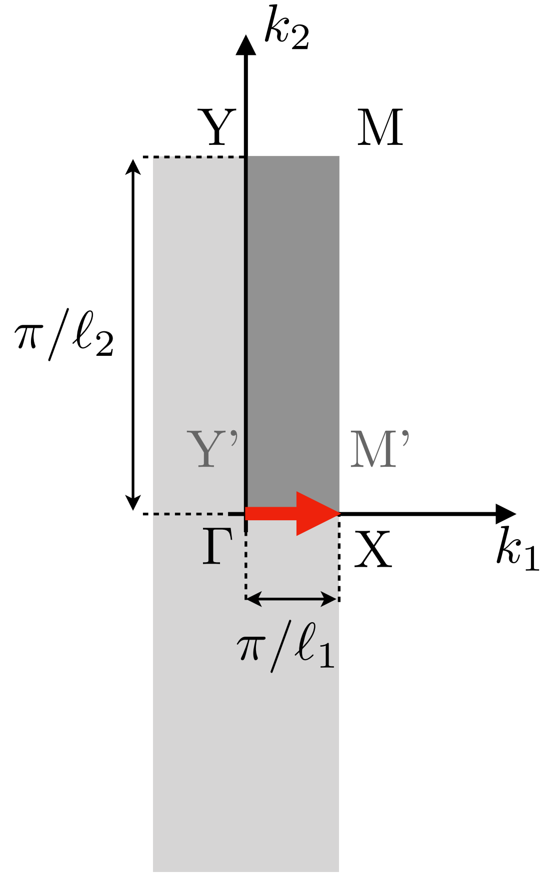


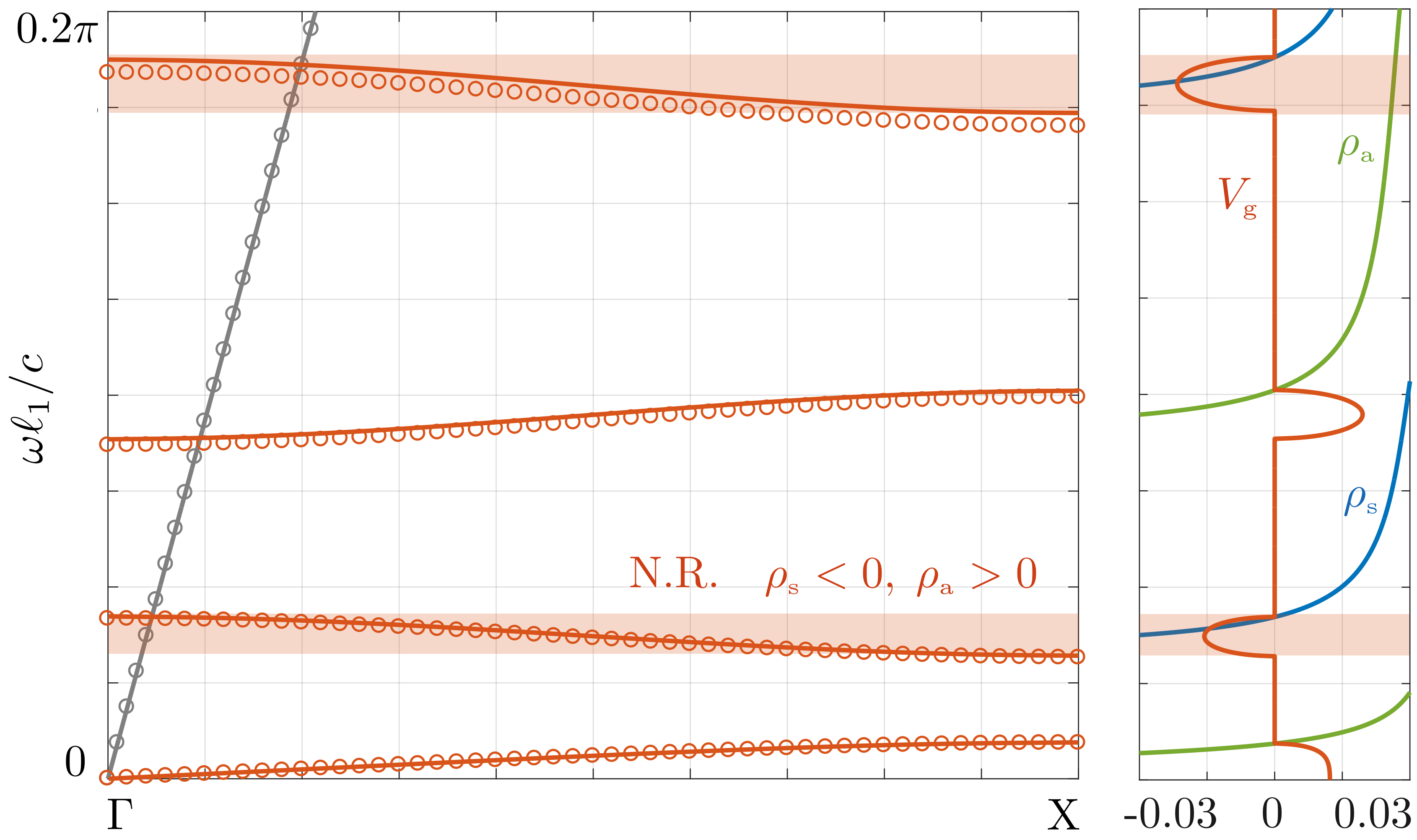
$$k_1 = \sqrt{\frac{\bar{\rho}}{\bar{E}}}\;\omega$$

$$\bar{E} = \frac{E\theta}{\varphi + \theta(1-\varphi)}, \quad \bar{\rho} = \rho(\varphi\theta + 1 - \varphi)$$

Classical homogenization formula

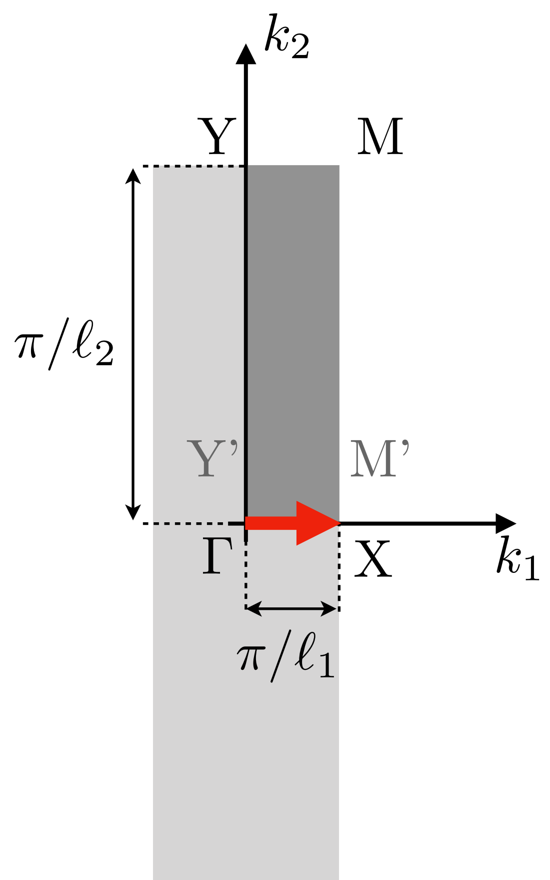
Dispersion relations

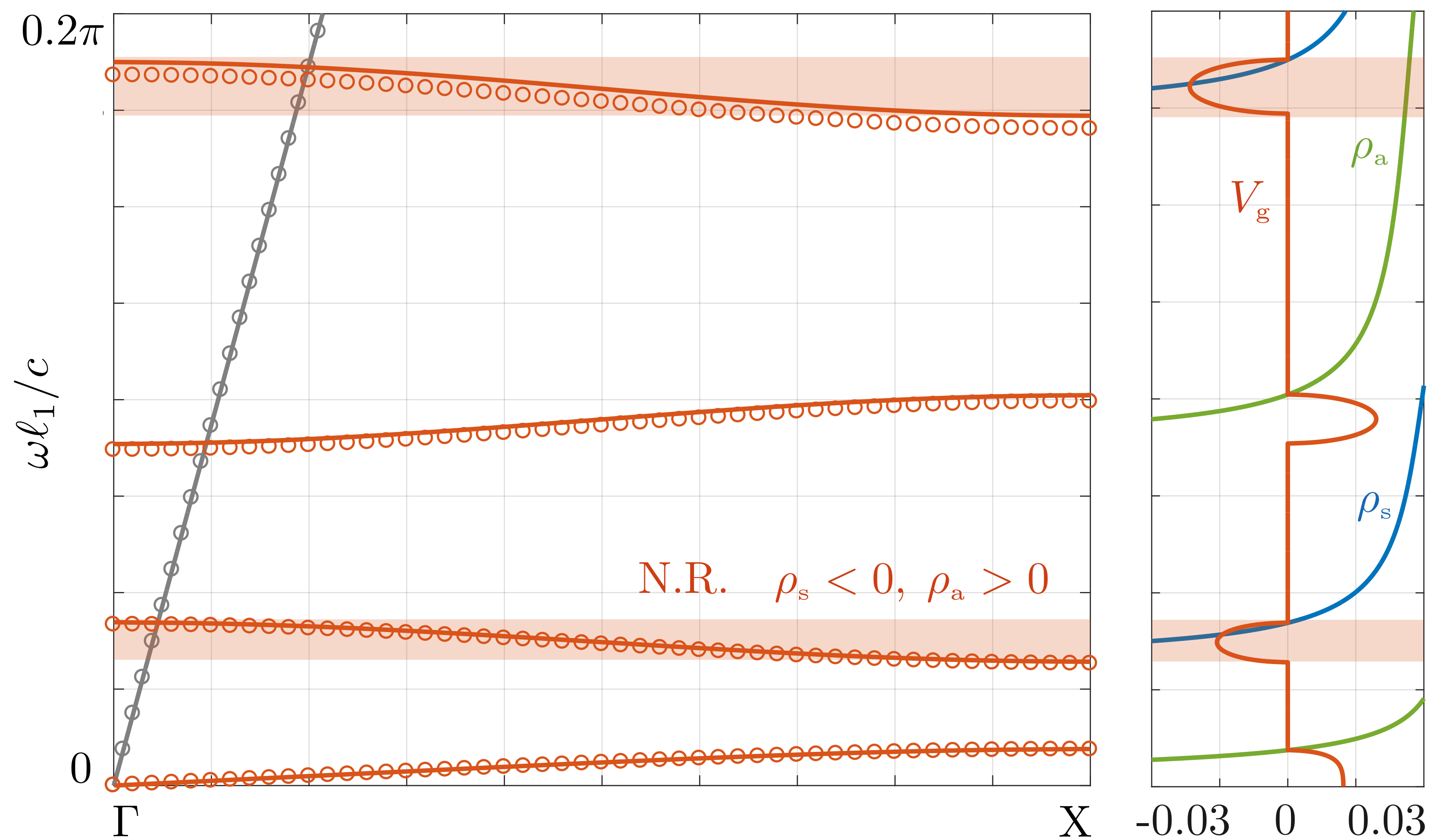




$$\cos(k_1\ell_1) = \frac{\rho_a(\omega) + \rho_s(\omega)}{\rho_a(\omega) - \rho_s(\omega)},$$

Dispersion relations





$$\cos(k_1\ell_1) = \frac{\rho_a(\omega) + \rho_s(\omega)}{\rho_a(\omega) - \rho_s(\omega)},$$

Band gap : $\rho_s(\omega)\varrho_a(\omega) > 0$

Negative refraction : $\rho_s(\omega) < 0, \quad \rho_a(\omega) > 0$

$$V_g \simeq \frac{\beta^2}{\tanh^2\Lambda - \tan^2\Lambda}\left(\frac{1}{\tanh\Lambda + \tan\Lambda} - \frac{\tanh\Lambda\tan\Lambda}{\tanh\Lambda - \tan\Lambda}\right)$$

$V_g < 0$ when negative refraction

[Marigo, Maurel et Pham, submitted, 2022]