Dynamics of a bistable Ericksen bar: an extended Lagrangian approach

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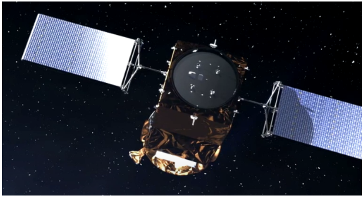
Part I

Introduction

A space tape spring

Thales Alenia Space (collab LMA : S. Bourgeois, B. Cochelin)

- motivation:
  - ✓ photovoltaic cell array folded and unfolded according to the energy needs
  - ✓ up to 25 m
- difficulties:
  - ✗ extreme conditions (motor avoided)
  - ✗ constant uncoiling speed (no shocks)
  - ✗ control by an extremity (boundary conditions)
  - $\rightsquigarrow$ usual tape spring inadequate

A good candidate: bistable tape spring

- carbo/epoxy fiber composite, with 2 stable equilibrium positions:

  - unfolded (curved cross section)

  - coiled (flat cross section)

- slow variation of the boundary conditions:

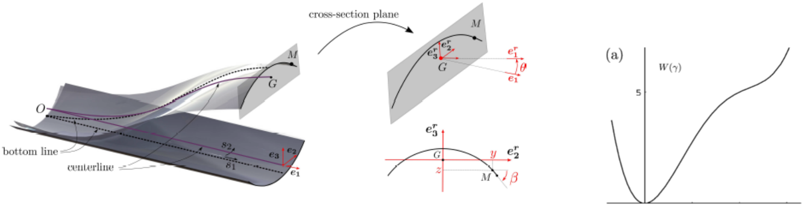
  ✓ fast deployment of the tape

  ✓ 3 zones: unfolded / transition area / coiled up

Families of models

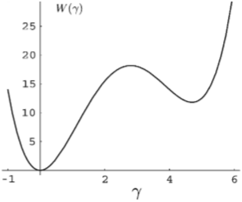
- many models: 3D, shells, flexible section beams
  - Picault-Bourgeois-Cochelin-Guinot (2016): 7 dof (3D)
  - Martin-Bourgeois-Cochelin-Guinot (2020): 2 dof

    flexion of the section θ, opening angle of the section β



- statics: analogy with a regularized Ericksen bar
  - ✓ local non-convexity of energy: formation of folds
  - ✓ regularization: transition zones, finite number of folds
  - ✓ boundary conditions on derivatives: variation of the cross-section shape

Objective of the study



- a step further for the analogy tape spring / Ericksen bar
  - introduction of dynamics and bistability
  - elementary model: 1 dof
- capture the main features observed
  - ✓ switching from one stable state via a higher-order boundary condition
  - ✓ propagation of a transition zone at constant speed (travelling wave)

Sketch of the study

1. bistable regularized Ericksen bar

   master Lagrangian, PDE, dispersion, exact solution (kink wave)

2. extended Lagrangian

   PDE, boundary conditions, dispersion, hyperbolic system

3. numerical experiments

   Riemann problem, variable boundary conditions, properties of the front

4. conclusion and prospects

Part II

Bistable regularized Ericksen bar

Master Lagrangian

- displacement u, velocity $v = \partial_t u$, strain $\varepsilon = \partial_x u$

  kinetic energy $\mathcal{T}$, potential energy $\mathcal{V}$, Lagrangian $\mathcal{L}$

$$\mathcal{T} = \frac{\rho}{2}\,(\partial_t u)^2, \qquad \mathcal{V} = W(\varepsilon) + \frac{\alpha}{2}\,(\partial_x \varepsilon)^2, \qquad \mathcal{L} = \mathcal{T} - \mathcal{V}$$

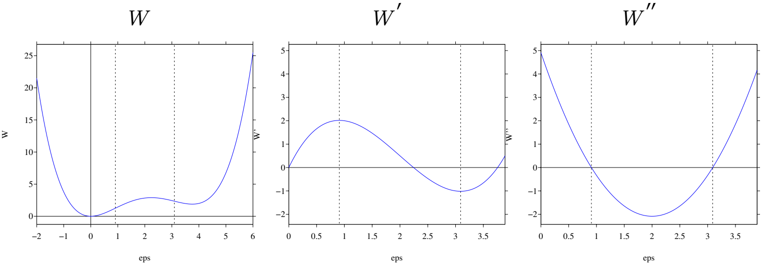
- Ericksen energy: W double-well potential

$$W(\varepsilon) = a_2\,\varepsilon^2 + a_3\,\varepsilon^3 + a_4\,\varepsilon^4, \qquad a_2 = 59/24,\; a_3 = -7/6,\; \text{and } a_4 = 7/48$$

- properties:

  ✓ W concave on $]\varepsilon_1, \varepsilon_2[$, with $\varepsilon_1 \approx 0.90$ and $\varepsilon_2 \approx 3.09$

  ✓ $W' = 0$ at 0 (stable) $< \varepsilon_1^0 \approx 2.21$ (unstable) $< \varepsilon_2^0 \approx 3.75$ (stable)



Evolution equations

- Euler-Lagrange on the master Lagrangian

$$\begin{cases} \partial_t \varepsilon - \partial_x v = 0 \\ \rho\, \partial_t v - \partial_x \sigma = 0, \qquad \sigma = W'(\varepsilon) - \alpha\, \partial^2_{xx} \varepsilon \end{cases}$$

- energy balance: $\dfrac{d}{dt}\mathcal{E} = \mathcal{P}$

$$\mathcal{E} = \int_0^L \left(\frac{\rho}{2} v^2 + W(\varepsilon) + \frac{\alpha}{2} (\partial_x \varepsilon)^2 \right) dx, \qquad \mathcal{P} = [v\, \sigma + \alpha\, \partial_t \varepsilon\, \partial_x \varepsilon]_0^L$$

- admissible boundary conditions:

$$\begin{cases} v(0,t) = 0 \\ \partial_x \varepsilon(0,t) = 0 \\ \varepsilon(L,t) = f_\varepsilon(t) \longleftarrow \text{ variable B.C. on derivative of kinematics} \\ \sigma(L,t) = 0 \end{cases}$$

✓ steady forcing $f_\varepsilon = 0 \Rightarrow \dfrac{d}{dt}\mathcal{E} = 0$

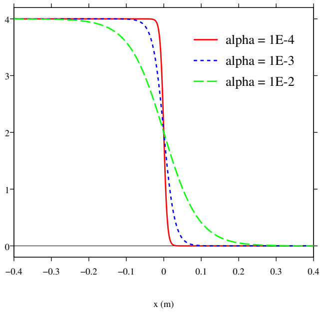
Exact solution

- speed of propagation V, thickness of transition D

  dimensional analysis $V \propto \sqrt{\frac{a_i}{\rho}}$, $D \propto \sqrt{\frac{\alpha}{a_i}}$
- travelling wave $\varepsilon(\xi)$, with $\xi = x - Vt$

  kink wave between 2 states $\varepsilon_A = 0 < \varepsilon_B = -\dfrac{a_3}{2\,a_4} \equiv 4$

$$V = \sqrt{\frac{2}{\rho}\left(a_2 - \frac{a_3^2}{4\,a_4}\right)} \equiv 0.5, \qquad \varepsilon(\xi) = \frac{\varepsilon_B}{1 + \exp\left(\varepsilon_B \sqrt{\dfrac{2\,a_4}{\alpha}}\,\xi\right)}$$



Dispersion analysis

- linearization $\varepsilon = \overline{\varepsilon} + \zeta\, e^{i(\omega t - kx)}$ with $0 < \zeta \ll 1$: dispersion relation

$$\omega(\overline{\varepsilon}, k) = k\sqrt{\frac{W''(\overline{\varepsilon}) + \alpha\, k^2}{\rho}}, \qquad c_p(\overline{\varepsilon}, k) = \frac{\omega(\overline{\varepsilon}, k)}{k} = \sqrt{\frac{W''(\overline{\varepsilon}) + \alpha\, k^2}{\rho}}$$

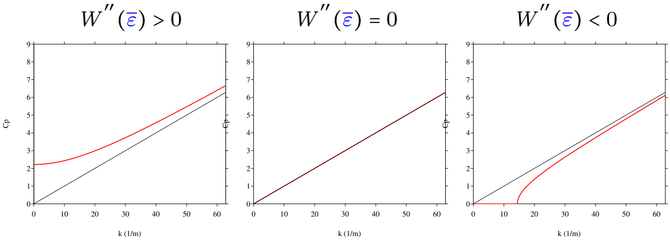
- $\overline{\varepsilon} \notin]\varepsilon_1, \varepsilon_2[$: $W''(\overline{\varepsilon}) > 0$

  ✓ $\forall k$: propagation

- $\overline{\varepsilon} \in [\varepsilon_1, \varepsilon_2]$: $W''(\overline{\varepsilon}) \leq 0 \Rightarrow$ critical wavenumber $k_c = \sqrt{-W''(\overline{\varepsilon})/\alpha}$

  ✓ $k > k_c$: propagation

  ✗ $k < k_c$: exponential growth of $e^{i(\omega t - kx)} \Rightarrow \varepsilon \notin]\varepsilon_1, \varepsilon_2[$ stabilization



straight line: asymptote $\sqrt{\alpha/\rho}k$

Dispersive system

- 1^{st} order in time

$$\partial_t \mathbf{U} + \partial_x \mathbf{f}(\mathbf{U}) = \mathbf{S}\, \partial^3_{xxx} \mathbf{U}$$

$$\mathbf{U} = \begin{pmatrix} \varepsilon \\ v \end{pmatrix}, \qquad \mathbf{f}(\mathbf{U}) = \begin{pmatrix} -v \\ -\dfrac{1}{\rho} W'(\varepsilon) \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}$$

- eigenvalues of the Jacobian $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{U}}$

$$\Lambda^{(1,2)} = \pm \sqrt{\frac{W''(\varepsilon)}{\rho}}$$

$\varepsilon \in]\varepsilon_1, \varepsilon_2[\Rightarrow W''(\varepsilon) < 0$: imaginary sound speed

- difficulties:

✗ inversion of an elliptic operator at each time step (computational cost)

✗ adhoc numerical methods for nonconvex f and dispersive term (convergence ?)

$\rightsquigarrow$ hyperbolisation (explicit scheme)

Part III

Extended Lagrangian for a bistable regularized Ericksen bar

Approach: extended Lagrangian method

- principle: introduction of penalized auxiliary variables in the Lagrangian
  - ✓ real sound speed (unconditional hyperbolicity)
  - ✓ large penalization: extended Lagrangian → master Lagrangian
  - ✓ similar dispersion properties
- recent works :
  - Favrie-Gavrilyuk (Nonlinearity 2017): Serre-Green-Naghdi
  - Dhaouadi-Favrie-Gavrilyuk (SAP 2019): nonlinear Schrödinger
  - Duchêne (Nonlinearity 2019): mathematical analysis

Extended Lagrangian

- master Lagrangian $\mathcal{L}$ (reminder): kinetic energy $\mathcal{T}$, potential energy $\mathcal{V}$

$$\mathcal{T} = \frac{\rho}{2}(\partial_t u)^2, \qquad \mathcal{V} = W(\varepsilon) + \frac{\alpha}{2}(\partial_x \varepsilon)^2, \qquad \mathcal{L} = \mathcal{T} - \mathcal{V}$$

- extended Lagrangian $\mathcal{L}_e$: field η, kinetic energy $\mathcal{T}_e$, potential energy $\mathcal{V}_e$

$$\mathcal{T}_e = \frac{\rho}{2}(\partial_t u)^2 + \frac{\beta}{2}(\partial_t \eta)^2, \qquad \mathcal{V}_e = W(\varepsilon) + \frac{\alpha}{2}(\partial_x \eta)^2 + \frac{\lambda}{2}(\varepsilon - \eta)^2, \qquad \mathcal{L}_e = \mathcal{T}_e - \mathcal{V}_e$$

- parameters:

  penalization λ $\curvearrowright$ real sound speed (hyperbolicity)

  micro-inertia β $\curvearrowright$ equation for η (e.g. : Poisson effect)

- property (to be proven):

  ✓ $\eta = \varepsilon + \mathcal{O}(\lambda^{-1}) + \mathcal{O}(\beta^n), \ n \geq 0$

Evolution equations

- Euler-Lagrange on the extended Lagrangian: $w = \partial_t \eta$, $p = \partial_x \eta$

$$\begin{cases} \partial_t \varepsilon - \partial_x v = 0 \\ \rho\, \partial_t v - \partial_x \sigma_e = 0, \qquad \sigma_e = W'(\varepsilon) + \lambda\, (\varepsilon - \eta) \\ \partial_t \eta = w \\ \partial_t p - \partial_x w = 0 \\ \beta\, \partial_t w - \partial_x (\alpha\, p) = \lambda(\varepsilon - \eta) \end{cases}$$

- energy balance: $\frac{d}{dt} \mathcal{E}_e = \mathcal{P}_e$

$$\mathcal{E}_e = \int_0^L \left(\frac{\rho}{2} v^2 + \frac{\beta}{2} w^2 + W(\varepsilon) + \frac{\alpha}{2} p^2 + \frac{\lambda}{2} (\varepsilon - \eta)^2 \right) dx, \quad \mathcal{P}_e = \left[v\, \sigma_e + \alpha\, p\, w \right]_0^L$$

- admissible boundary conditions:

$$\begin{cases} v(0,t) = 0 \\ p(0,t) = 0 \\ \eta(L,t) = f_\eta(t) \longleftarrow \text{ variable B.C. on derivative of kinematics} \\ \varepsilon(L,t) = s^{-1}(\lambda\, f_\eta(t)) \equiv f_\varepsilon(t) \\ w(L,t) = f_\eta'(t) \equiv f_w(t) \end{cases}$$

Dispersion analysis

- linearization $\varepsilon = \overline{\varepsilon} + \zeta\, e^{i(\omega t - kx)}$ with $0 < \zeta \ll 1$: quartic equation

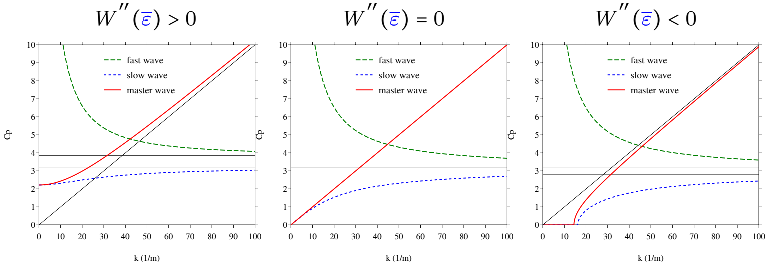
$$A_4\,\omega^4 + B(k,\varepsilon)\,\omega^2 + C(k,\omega) = 0$$

$c_p^+(\overline{\varepsilon}, k) = \frac{\omega^+(\overline{\varepsilon},k)}{k}$ fast wave, $c_p^-(\overline{\varepsilon}, k) = \frac{\omega^-(\overline{\varepsilon},k)}{k}$ slow wave

✓ fast wave ≡ spurious acoustic wave: $c_p^+ \in \mathbb{R}^+ \quad \forall k > 0, \quad c_p^+(\overline{\varepsilon}, k) \underset{0}{\sim} \frac{1}{k} \to +\infty$

✓ slow wave ≡ physical wave ⇝ critical wavenumber k_c and stabilization

$$c_p^- \in \mathbb{R}^+ \text{ if } \left(W''(\overline{\varepsilon}) > 0 \text{ or } k > k_c^e = \sqrt{-\frac{W''(\overline{\varepsilon})\lambda}{\alpha(W''(\overline{\varepsilon}) + \lambda)}} \right), \qquad \text{else } c_p^- \in \mathbb{C}$$



horizontal lines: $\Lambda^{(2)}(\overline{\varepsilon})$ and $\Lambda^{(4)}$, straight line: asymptote $\sqrt{\alpha/\rho}\,k$

Hyperbolic system

- 1^{st} order in space and time

$$\partial_t \mathbf{U}_e + \partial_x \mathbf{f}_e(\mathbf{U}_e) = \mathbf{S}_e\,\mathbf{U}_e$$

$$\mathbf{U}_e = (\varepsilon, v, \eta, p, w)^\top, \quad \mathbf{f}_e(\mathbf{U}_e) = \left(-v, -\frac{1}{\rho}\sigma_e(\varepsilon,\eta), 0, -w, -\frac{\alpha}{\beta}\,p\right)^\top$$

- sound speed: eigenvalues $\Lambda_e^{(i)}$ of the Jacobian $\mathbf{A}_e = \frac{\partial \mathbf{f}_e}{\partial \mathbf{U}_e}$

$$\Lambda_e^{(1)} = 0, \quad \Lambda_e^{(2,3)}(\varepsilon) = \pm\sqrt{\frac{W''(\varepsilon) + \lambda}{\rho}}, \qquad \Lambda_e^{(4,5)} = \pm\sqrt{\frac{\alpha}{\beta}}$$

$$\varepsilon \in [\varepsilon_1, \varepsilon_2] \;\Rightarrow\; W''_{\min} \equiv -\tfrac{25}{12} \le W''(\varepsilon) \le 0$$

$$\lambda > \lambda_{\min} \equiv \tfrac{25}{12} \Rightarrow W''(\varepsilon) + \lambda > 0\text{: unconditional hyperbolicity}$$

- standard numerical methods for hyperbolic systems with relaxation

Numerical methods

- nonlinear hyperbolic system with linear source term

$$\boxed{\frac{\partial}{\partial t}\mathbf{U}_e + \frac{\partial}{\partial x}\mathbf{f}_e(\mathbf{U}_e) = \mathbf{S}_e\,\mathbf{U}_e}$$

Strang splitting (propagation and relaxation)

$$\frac{\partial}{\partial t}\mathbf{U}_e + \frac{\partial}{\partial x}\mathbf{f}_e(\mathbf{U}_e) = \mathbf{0} \qquad (\mathbf{H}_p)$$

$$\frac{\partial}{\partial t}\mathbf{U}_e = \mathbf{S}_e\,\mathbf{U}_e \qquad (\mathbf{H}_r)$$

- propagation step: finite-volume scheme with flux limiters

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}\right), \qquad \Delta t \leq \frac{\Delta x}{c_{\max}}$$

boundary conditions: ghost cells (📚 LeVeque 2002)

- relaxation step: exact solution

$$\mathbf{H}_r(\tau)\,\mathbf{U}_i = \exp(\mathbf{S}_e\tau)\,\mathbf{U}_i$$

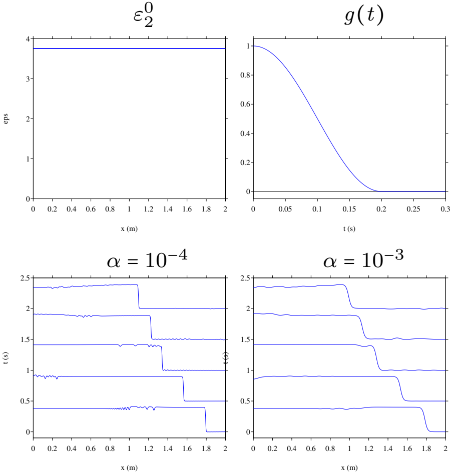
Part IV

Numerical experiments

Variable boundary conditions

✓ $\eta(L,t) = \varepsilon_2^0\, g(t)$ with g decreasing quasi-statically

✓ propagation of a front: D increase with α

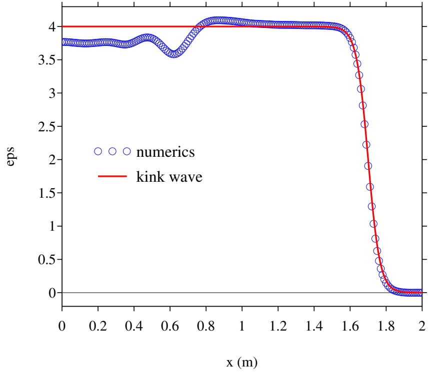


CL-Alpha1m4.m4v **CL-Alpha1m3.m4v**

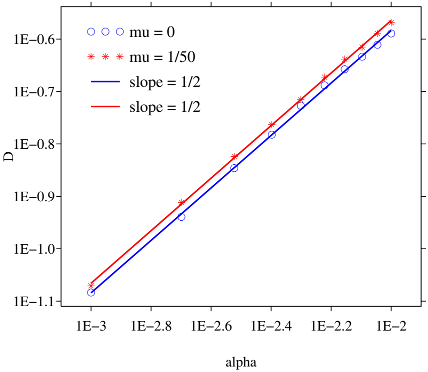
Properties

- ✓ simulation (extended Lagrangian) vs kink wave (master Lagrangian)
- ✓ velocity (measured and exact): $V = 0.5$ m.s^{-1} depends only on the energy
- ✓ dimensional analysis and kink wave: thickness $D \propto \sqrt{\alpha}$

snapshot de ε



D vs α



Part V

Conclusion

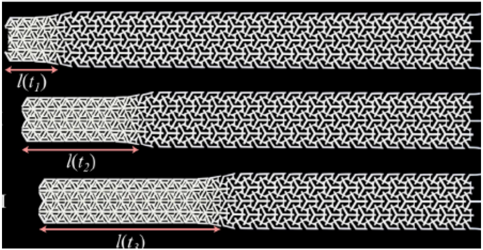
Future directions

✓ mathematical analysis (📚 Duchêne, Nonlinearity 2019)

✓ investigation of the bistable tape spring with 2 dof

📚 Martin-Bourgeois-Cochelin-Guinot, IJSS (2020)

✓ design of innovative deployable structures



📚 Khajehtourian-Kochmann, EML (2020)

Thanks for your attention!

Part VI

Appendix

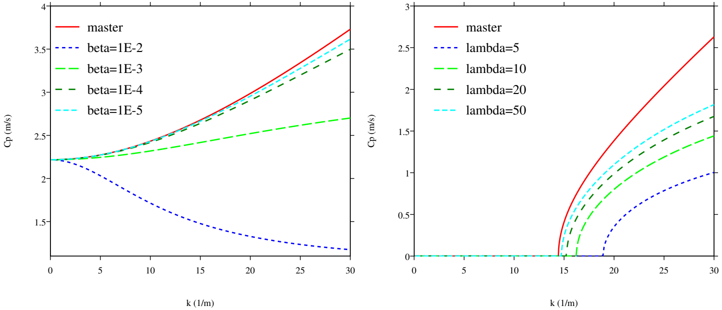
Choice of parameters $\beta \to 0$ and $\lambda \to +\infty$

- ✓ hyperbolicity: $\lambda \geq \lambda_{\min} \equiv 25/12$
- ✓ $W'' \geq 0$: slow wave c_p^- increases like master wave c_p if $\frac{\alpha}{\beta} > \frac{W''(\varepsilon)}{\rho}$

$$\varepsilon \in [0, \varepsilon_2^0] \Rightarrow \Gamma(\beta) = \frac{\rho}{2\, a_2} \frac{\alpha}{\beta} \geq 1$$

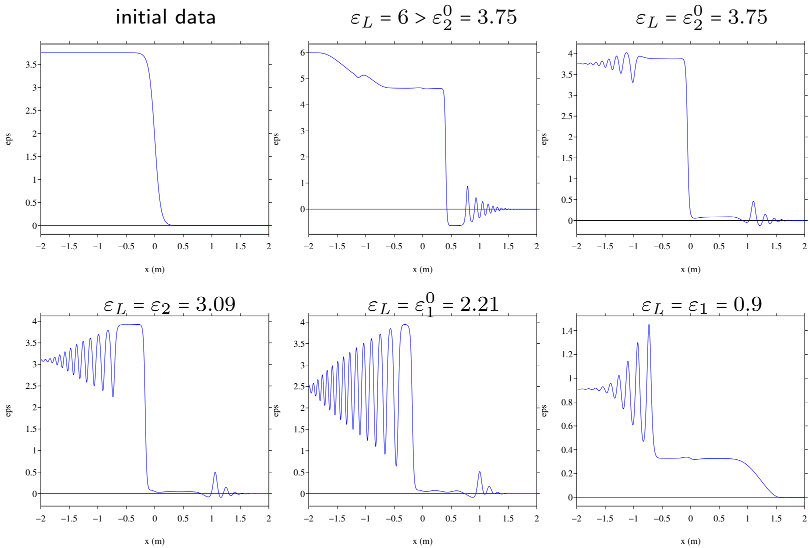
- ✓ $W'' < 0$: critical wavenumber $\left| \frac{k_c^e}{k_c} - 1 \right| \leq \frac{1}{2\lambda} \left| W''_{\min} \right| \equiv \frac{25}{24} \frac{1}{\lambda}$
- ✓ minimization of the numerical dissipation: $\Lambda_e^{(2)}(\varepsilon) = \Lambda_e^{(4)} \Rightarrow \lambda + 2\, a_2 = \rho \frac{\alpha}{\beta}$

$W'' > 0,\ \lambda = 100$ $\qquad$ $W'' < 0,\ \beta = 10^{-3}$



Riemann problem

✓ discontinuous initial data $(\varepsilon_L, 0)$, with smooth transition



Influence of dissipation

- master Lagrangian: Kelvin-Voigt, μ dynamic viscosity

$$\rho\,\partial_t v - \partial_x \sigma = \mu\,\partial^2_{xx} v$$

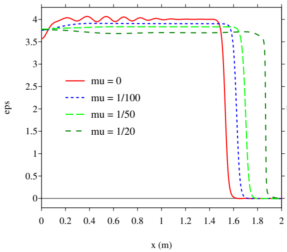
- extented Lagrangian: relaxation term

$$\beta\,\partial_t w - \partial_x(\alpha\, p) = \lambda(\varepsilon - \eta) - \mu\, w$$

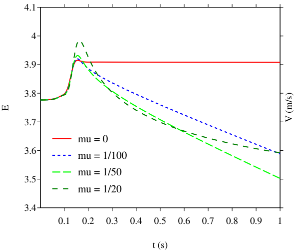
- dimensional analysis

$$V = \Phi_V\left(\sqrt{\frac{a_i}{\rho}},\ \frac{\sqrt{a_i\,\alpha}}{\mu}\right)$$

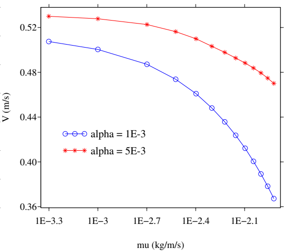
snapshot of ε



energy dissipation



V vs μ



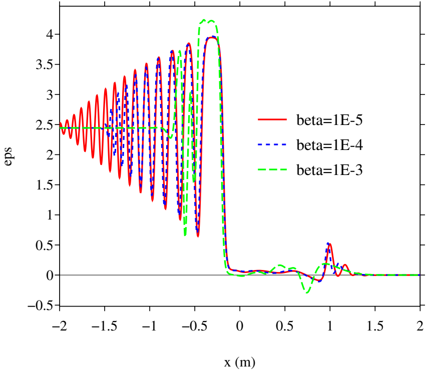
Choice of parameters β and λ

- reference solution: no influence of meshing or (β, λ)
- 2 criteria: $\Gamma(\beta) > 1$, $\lambda > \lambda_c$

✗ criteria not satisfied ($\beta = 10^{-3}$ left, $\lambda = 5$ right) $\curvearrowright$ non-convergence

✓ criteria satisfied (other curves) $\curvearrowright$ convergence towards the reference solution

β variable et $\lambda = 100$



λ variable et $\beta = 10^{-5}$