A landscape method to unveil high-frequency localized modes of the classical wave equation in heterogeneous media

David Colas, Régis Cottereau, Cédric Bellis, Bruno Lombard

Aix Marseille Univ, CNRS, Centrale Marseille, LMA UMR7031, France

Outline

Plan

1. Introduction
   - Eigenvectors in heterogeneous media
   - Rayleigh-Ritz algorithm to estimate eigenvectors
2. The localization landscape method (with lower localized modes)
3. The localization landscape (with lower delocalized modes)
4. Conclusions

Eigenvectors in heterogeneous media

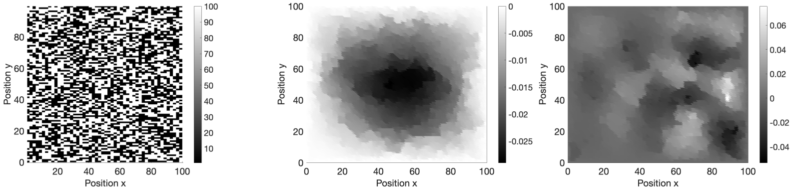
Continuous formulation

We consider the following eigenvalue problem : find $(E_n, \psi_n(\mathbf{x}))$ such that

$$-\nabla \cdot (\kappa(\mathbf{x})\nabla\psi_n(\mathbf{x})) = \rho(\mathbf{x})E_n\psi_n(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega,$$

with appropriate boundary conditions (in most of this talk : $\psi_n(\mathbf{x}) = 0, \forall \mathbf{x} \in \partial\Omega$), and normalization condition :

$$\int_\Omega \rho(\mathbf{x})\psi_n(\mathbf{x})\psi_n(\mathbf{x})d\mathbf{x} = 1$$



(a) Map of $\kappa(\mathbf{x})$ (or $\rho(\mathbf{x})$) (b) 1st mode $\psi_1(\mathbf{x})$ (c) 20th mode $\psi_{20}(\mathbf{x})$

Eigenvectors in heterogeneous media

Discrete formulation

The corresponding discretized formulation (FEM for instance) is

$$\mathbf{K}\psi_n = E_n \mathbf{M}\psi_n,$$

with normalization condition, $\forall n \geq 0$

$$\psi_n^T \mathbf{M}\psi_n = 1.$$

The Rayleigh-Ritz algorithm to estimate eigenvectors

Description of the algorithm

Starting from initial estimates (E_n^0, ψ_n^0), the sequence of updates

$$\psi_n^{k+1} = \left(\mathbf{K} - E_n^k \mathbf{M}\right)^{-1} \mathbf{M}\psi_n^k,$$

$$E_n^{k+1} = \frac{(\psi_n^{k+1})^T \mathbf{K} \psi_n^{k+1}}{(\psi_n^{k+1})^T \mathbf{M} \psi_n^{k+1}}.$$

converges to (E_0, ϕ_0), where E_0 is the smallest eigenvalue for which $\psi_0^T \mathbf{M} \psi_n^0 \neq 0$.

Remarks :

- Typical initialization : $(0, \mathbf{r})$, where $\mathbf{r}$ is a random vector with independent entries distributed uniformly in $[0, 1]$.
- For other eigenvalues : deflation or solving in a subspace orthogonal to previous eigenvectors.

Plan

The localization landscape

Generic eigenvalue problems

We consider the following eigenvalue problem : find $(E_n, \psi_n(\mathbf{x}))$ such that

$$\mathcal{L}\left(\psi_n(\mathbf{x})\right) = E_n \psi_n(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega,$$

with appropriate boundary conditions, and normalization condition.

Examples :

- Schrödinger equation : $\mathcal{L}\left(\psi_n(\mathbf{x})\right) = -\Delta\psi_n(\mathbf{x}) + V(\mathbf{x})\psi_n(\mathbf{x})$
- Classical (acoustic) wave equation : $\mathcal{L}\left(\psi_n(\mathbf{x})\right) = -\rho^{-1}(\mathbf{x})\nabla\cdot(\kappa(\mathbf{x})\nabla\psi_n(\mathbf{x}))$

The localization landscape [1]

The localization landscape is defined as the solution $u(\mathbf{x})$ of

$$\mathcal{L}\left(u(\mathbf{x})\right) = 1, \quad \forall \mathbf{x} \in \Omega,$$

with the same boundary conditions. The landscape is independent of n.

1. M. Filoche et S. Mayboroda. “Universal mechanism for Anderson and weak localization”. In : *Proc. Nat. Acad. Sci. USA* 109.37 (2012), p. 14761-14766. DOI : 10.1073/pnas.1120432109

Properties of the localization landscape

The localization landscape

- The localization landscape $u(\mathbf{x})$ is "cheap" to compute (compared to eigenvalue problem or Rayleigh-Ritz)
- One function contains information on "all" modes

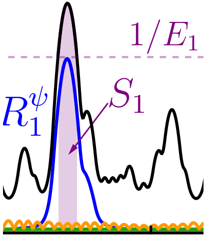
Property 1 [1]

$$R_n^{\psi} = \frac{\psi_n(\mathbf{x})}{E_n \|\psi_n\|_\infty} \leq u(\mathbf{x})$$

Property 2 [2]

A mode $\psi_n(\mathbf{x})$ is "mostly" supported on

$$S_n = \{x \in \Omega,\ 1/E_n \leq u(\mathbf{x})\}$$



1. M. Filoche et S. Mayboroda. “Universal mechanism for Anderson and weak localization”. In : *Proc. Nat. Acad. Sci. USA* 109.37 (2012), p. 14761-14766. DOI : 10.1073/pnas.1120432109

2. D. N. Arnold et al. “Computing spectra without solving eigenvalue problems”. In : *SIAM J. Sci. Comp.* 41.1 (2019), B69-B92. DOI : 10.1137/17M1156721

Sketch of how property 1 works

Introducing the Green's function $G_{\mathbf{y}}(\mathbf{x})$ of the adjoint operator

$$\mathcal{L}\left(G_{\mathbf{y}}(\mathbf{x})\right) = \delta_{\mathbf{y}}(\mathbf{x}), \quad \forall \mathbf{x}, \mathbf{y} \in \Omega,$$

Observe that (with $(f,g) = \int_{\Omega} f(\mathbf{x})g(\mathbf{x})d\mathbf{x}$) and assuming $\mathcal{L}$ is self-adjoint :

$$u(\mathbf{y}) = (u, \delta_{\mathbf{y}}) = (u, \mathcal{L}\left(G_{\mathbf{y}}\right)) = \left(\mathcal{L}\left(u\right), G_{\mathbf{y}}\right) = (1, G_{\mathbf{y}})$$

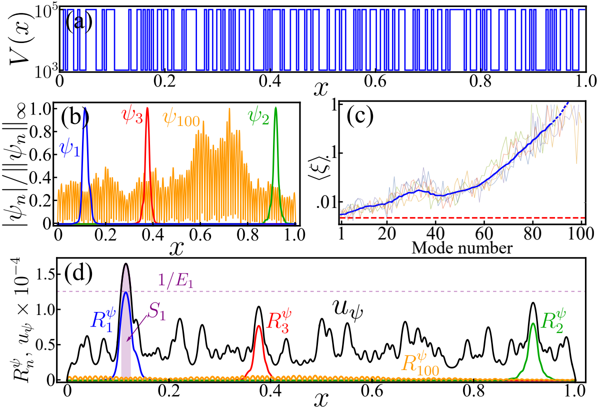
and

$$\psi_n(\mathbf{y}) = (\psi_n, \delta_{\mathbf{y}}) = (\psi_n, \mathcal{L}\left(G_{\mathbf{y}}\right)) = \left(\mathcal{L}\left(\psi_n\right), G_{\mathbf{y}}\right) = E_n\left(\psi_n, G_{\mathbf{y}}\right).$$

Since $G_{\mathbf{y}}(\mathbf{x}) \geq 0$, the modes verify

$$|\psi_n(\mathbf{y})| \leq E_n \|\psi_n\|_{\infty} \left(1, G_{\mathbf{y}}\right) = E_n \|\psi_n\|_{\infty} u(\mathbf{y})$$

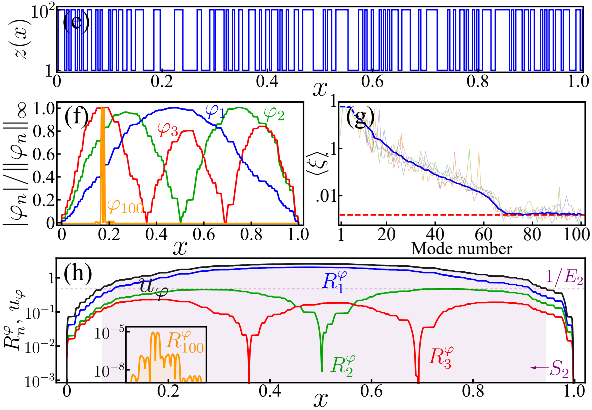
Example (1D) for Schrödinger equation [1]



- Localization landscape $u(\mathbf{x})$ identifies the (low-frequency) localized modes
- The bound is correct but uninformative at higher frequencies

1. D. Colas et al. "Crossover between quantum and classical waves in high frequency localization landscapes". In : *Phys. Rev. B* (2022). Submitted for publication

Example (1D) for the classical wave equation ($\rho(x) = \kappa(x)$)



- Localization landscape $u(\mathbf{x})$ is uninformative at all frequencies
- The lowest (delocalized) eigenmode hides all other modes

Plan

The localization landscape for higher-order modes

Symmetrization for classical waves

The original eigenvalue problem : find $(E_n, \psi_n(\mathbf{x}))$ such that

$$-\frac{1}{\rho(\mathbf{x})}\nabla\cdot(\kappa(\mathbf{x})\nabla\psi_n(\mathbf{x})) = E_n\psi_n(\mathbf{x}), \quad \forall\mathbf{x}\in\Omega,$$

is transformed, through $\phi_n(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}\psi_n(\mathbf{x})$, into : find $(E_n, \psi_n(\mathbf{x}))$ such that

$$-\frac{1}{\sqrt{\rho(\mathbf{x})}}\nabla\cdot\left(\kappa(\mathbf{x})\nabla\frac{\phi_n(\mathbf{x})}{\sqrt{\rho(\mathbf{x})}}\right) = E_n\phi_n(\mathbf{x}), \quad \forall\mathbf{x}\in\Omega,$$

which corresponds to a symmetric operator.

A mode $\phi_n(\mathbf{x})$ (or $\psi_n(\mathbf{x})$) is "mostly" supported on

$$S_n = \left\{x\in\Omega,\ \frac{1}{E_n}\leq\sqrt{\rho(\mathbf{x})}u(\mathbf{x})\right\}$$

where $u(\mathbf{x})$ is the solution of

$$-\nabla\cdot(\kappa(\mathbf{x})\nabla u(\mathbf{x})) = \sqrt{\rho(\mathbf{x})}$$

The localization landscape for higher-order modes

Shifting of the eigenvalue problem

For any $\mathcal{E}_s$, the solutions of : find $(E_n, \chi_n(\mathbf{x}))$ such that

$$-\frac{1}{\sqrt{\rho(\mathbf{x})}}\nabla\cdot\left(\kappa(\mathbf{x})\nabla\frac{\chi_n(\mathbf{x})}{\sqrt{\rho(\mathbf{x})}}\right)+\mathcal{E}_s\chi_n(\mathbf{x})=E_n\chi_n(\mathbf{x}),\quad \forall\mathbf{x}\in\Omega,$$

are $(E_n+\mathcal{E}_s, \psi_n(\mathbf{x}))$.

A mode $\psi_n(\mathbf{x})$ is "mostly" supported (for any $\mathcal{E}_s$) on

$$S_n^s=\left\{x\in\Omega,\ \frac{1}{E_n+\mathcal{E}_s}\leq\sqrt{\rho(\mathbf{x})}u^s(\mathbf{x})\right\}=\left\{x\in\Omega,\ \frac{1}{E_n}\leq\left(\frac{1}{\sqrt{\rho(\mathbf{x})}u^s(\mathbf{x})}-\mathcal{E}_s\right)^{-1}\right\}$$

The localization landscape for higher-order modes

Shifting of the eigenvalue problem

For any $\mathcal{E}_s$, the solutions of : find $(E_n, \chi_n(\mathbf{x}))$ such that

$$-\frac{1}{\sqrt{\rho(\mathbf{x})}}\nabla\cdot\left(\kappa(\mathbf{x})\nabla\frac{\chi_n(\mathbf{x})}{\sqrt{\rho(\mathbf{x})}}\right)+\mathcal{E}_s\chi_n(\mathbf{x})=E_n\chi_n(\mathbf{x}),\quad\forall\mathbf{x}\in\Omega,$$

are $(E_n+\mathcal{E}_s, \psi_n(\mathbf{x}))$.

A mode $\psi_n(\mathbf{x})$ is "mostly" supported (for any $\mathcal{E}_s$) on

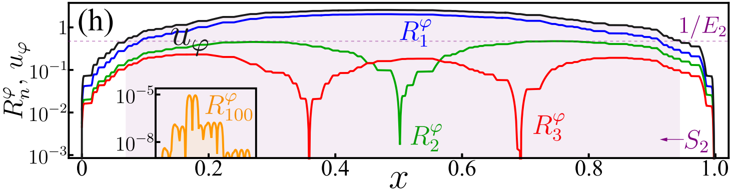
$$S_n^s=\left\{x\in\Omega,\ \frac{1}{E_n+\mathcal{E}_s}\leq\sqrt{\rho(\mathbf{x})}u^s(\mathbf{x})\right\}=\left\{x\in\Omega,\ \frac{1}{E_n}\leq\left(\frac{1}{\sqrt{\rho(\mathbf{x})}u^s(\mathbf{x})}-\mathcal{E}_s\right)^{-1}\right\}$$

The localization landscape for higher-order modes

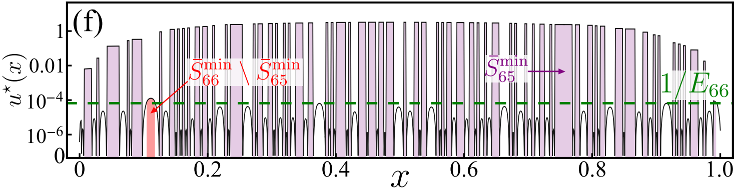
Defining (for several $\mathcal{E}_s$) the landscape $u^*(\mathbf{x})=\min_s(1/(\sqrt{\rho(\mathbf{x})}u^s(\mathbf{x}))-\mathcal{E}_s)^{-1}$, a mode $\psi_n(\mathbf{x})$ is mostly supported on

$$S_n=\left\{x\in\Omega,\ \frac{1}{E_n}\leq u^*(\mathbf{x})\right\}.$$

Example (1D) for classical waves



(d) The "low-frequency" (classical) landscape



(e) The "higher-frequency" landscape

Plan

1. Introduction
   - Eigenvectors in heterogeneous media
   - Rayleigh-Ritz algorithm to estimate eigenvectors
2. The localization landscape method (with lower localized modes)
3. The localization landscape (with lower delocalized modes)
4. Conclusions

Conclusions

The localization landscape method

- The localization landscape method can be adapted when lower modes are delocalized [1]
- The (classical) Localization landscape is just one Rayleigh-Ritz iteration initialized with $(0, 1)$. Potential gains from using different loading functions or initial shift ?
- An interesting visualization tool to identify localized modes

1. D. Colas et al. “Crossover between quantum and classical waves in high frequency localization landscapes”. In : *Phys. Rev. B* (2022). Submitted for publication

A landscape method to unveil high-frequency localized modes of the classical wave equation in heterogeneous media

David Colas, Régis Cottereau, Cédric Bellis, Bruno Lombard

Aix Marseille Univ, CNRS, Centrale Marseille, LMA UMR7031, France