Homogenisation for transient waves in periodic 1D media: dispersion, interfaces and source points

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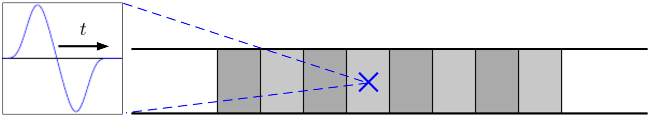
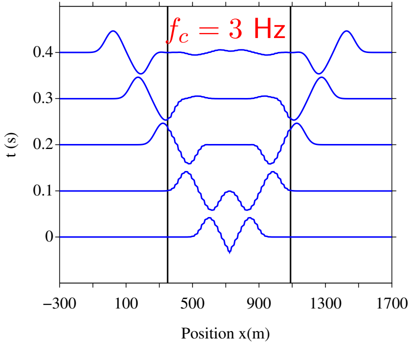
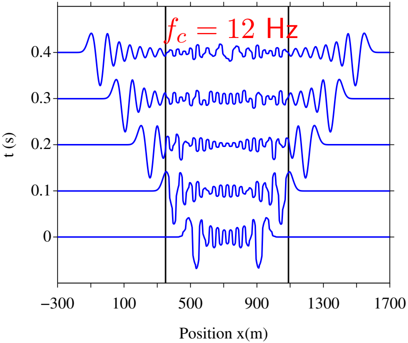
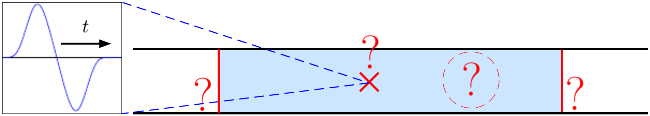
Dispersion, interfaces and source points





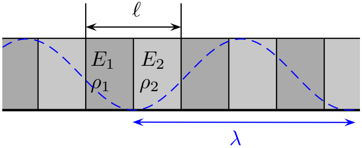




Contents

Two-scale asymptotic homogenization in a 1D periodic medium

[Sanchez-Palencia, 1974, Bensoussan et al., 1978, Cioranescu and Donato, 1999] ...



- Wave equation in a ℓ-periodic medium

$$\rho\left(\frac{x}{\ell}\right)\frac{\partial^2 u_\ell}{\partial t^2} - \frac{\partial}{\partial x}\left[E\left(\frac{x}{\ell}\right)\frac{\partial u_\ell}{\partial x}\right] = 0$$

- *Reference wavelength* $\lambda > \ell$

- **Separated variable solution** featuring a mean field $U(x,t)$ and cell functions $P_j(x/\ell)$:

$$u_\ell(x,t) = U(x,t) + \ell U_{,x}(x,t) P_1\left(\frac{x}{\ell}\right) + \ell^2 U_{,xx}(x,t) P_2\left(\frac{x}{\ell}\right) + o(\ell^2),$$

- **Effective wave equations** for the mean field, from [Wautier and Guzina, 2015]

$$\frac{1}{c_0^2}U_{,tt} - U_{,xx} - \ell^2\left(\beta_x U_{,xxxx} - \frac{\beta_m}{c_0^2}U_{,xxtt} - \frac{\beta_t}{c_0^4}U_{,tttt}\right) = 0, \qquad \beta_x - \beta_m - \beta_t = \beta$$

- ... featuring **effective coefficients** computed from the cell functions:

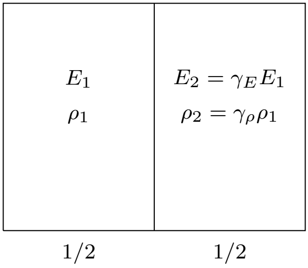
$$c_0^2 = \frac{\mathcal{E}_0}{\varrho_0} \quad \text{and} \quad \beta = \left[\frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\varrho_2}{\varrho_0}\right] \quad \text{with} \quad \begin{cases} \rho_j = \langle \rho P_j \rangle, \\ \mathcal{E}_j = \langle E(P_j + P_{j+1,y}) \rangle \end{cases}$$

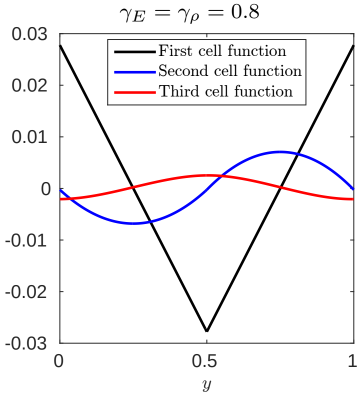
- ... and **two degrees of freedom** to choose $(\beta_x, \beta_m, \beta_t)$.

Cell functions

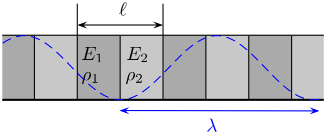
(P_1, P_2, P_3): solutions of **static equilibrium problems** posed in the unit cell $Y = [0,1]$

$$\partial_y \left[E \partial_y P_j \right] = \mathcal{F}(E, \rho, P_{j-1}, P_{j-2}), \quad P_j \text{ is 1-periodic}, \quad \langle P_j \rangle = 0$$





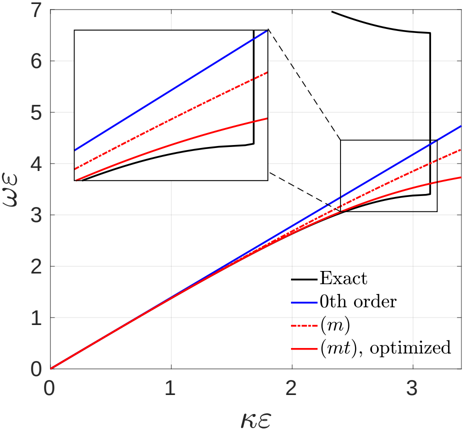
Choosing a (mt) model by fitting dispersion curves



Material contrasts: $\frac{E_2}{E_1} = 6, \quad \frac{\rho_2}{\rho_1} = 1.5$

Phase ratio: $\frac{\ell_2}{\ell_1} = 3$

($\approx$ maximal dispersion [Santosa and Symes, 1991])



- Bloch wave: $u(x, y; t) = \phi(y)e^{\mathrm{i}(\kappa x - \omega t)}$
  $\Longrightarrow$ *Dispersion relation* $\kappa = f(\omega)$
- 0th-order: $U_{,tt} - c_0^2 U_{,xx} = 0$
  $\Longrightarrow$ non-dispersive
- 2nd-order (m):
  $\Longrightarrow O((\kappa\ell)^2)$ approximation
- Optimized (mt):
  $\Longrightarrow O((\kappa\ell)^4)$ approximation
  [Pichugin et al., 2008, Cornaggia and Guzina, 2020]

Equivalent hyperbolic system for the (mt) model

Second-order homogenization, (mt) model:

$$\frac{1}{c_0^2}\left[1+\ell^2\beta_m\partial_{xx}+\ell^2\frac{\beta_t}{c_0^2}\partial_{tt}\right]U_{,tt}=U_{,xx},$$ (β_m,β_t) obtained by fitting dispersion curve.

"Stress gradient" system by [Forest and Sab, 2017]:

$$\begin{cases}\sigma = E\partial_x(u+\phi)\\ r = D\phi\end{cases} \qquad \begin{cases}\rho\partial_{tt}u=\partial_x\sigma\\ \rho J\partial_{tt}\phi=\partial_x\sigma-r\end{cases}$$

So that: $\frac{\rho}{E}\left[1-\frac{E(1+J)}{D}\partial_{xx}+\frac{\rho J}{D}\partial_{tt}\right]u_{,tt}=u_{,xx}$

$$\Rightarrow\begin{cases}\partial_t V-\frac{a}{\varrho_0}\partial_x S &= -\frac{a-1}{\varrho_0}r\\ \partial_t S-\mathcal{E}_0\partial_x V &= 0\\ \partial_t\varphi-\frac{a-1}{\varrho_0}\partial_x S &= -\frac{a-1}{\varrho_0}r\\ \partial_t r &= \frac{\mathcal{E}_0}{\ell^2\beta}\varphi\end{cases}$$

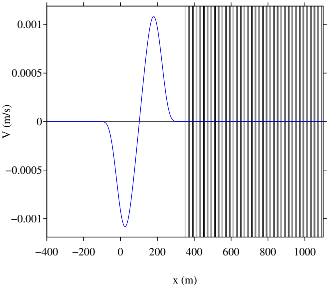
- Macroscopic fields $V=\partial_t(v+\phi)$ and $S=\sigma$
- **Auxiliary fields** $\varphi=\partial_t\phi$ and r
- Parameter $a=-\beta_m/\beta_t$ to select a (mt) model.

Hyperbolicity and **stability** for:

$$\beta_m<0 \quad \text{and} \quad \beta_t>0$$

Outline

First-order transmission conditions - regularization



- Fields of interest:

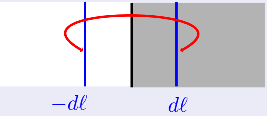
$$\begin{cases} (V,S) = (\partial_t U, E_- \partial_x U) & x < 0 \\ (V,S,\varphi,r) & x > 0 \end{cases}$$

- Transmission conditions from **total fields continuity**:

$$\begin{cases} V(0^-) = V(0^+) + \ell P_1(0) V_{,x}(0^+) \\ S(0^-) = S(0^+) + \ell \Sigma_1(0) S_{,x}(0^+) \end{cases}$$

$\Longrightarrow$ Assymetric conditions, sometimes **unstable**

Spring-mass conditions across a thin interface $I_d = [-d\ell, d\ell]$ (additional $o(\ell)$ error):



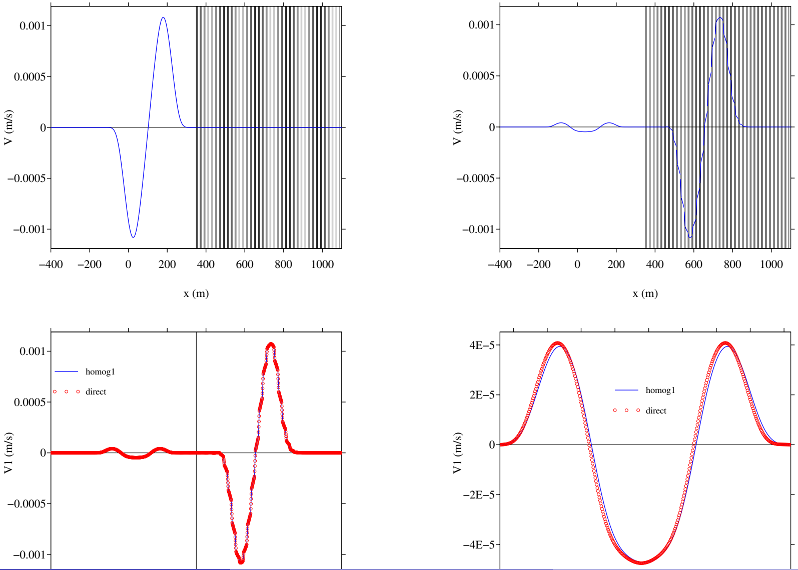
$$\begin{cases} [\![V]\!]_{I_d} = \ell A_1 \partial_t \langle S \rangle_{I_d} & A_1 = -P_1(0)\mathcal{E}_0^{-1} + d(E_-^{-1} + \mathcal{E}_0^{-1}) \\ [\![S]\!]_{I_d} = \ell B_1 \partial_t \langle V \rangle_{I_d} & B_1 = -\Sigma_1(0)\varrho_0 \quad + d(\rho_- + \varrho_0) \end{cases}$$

Stability of {Hyperbolic system + transmission conditions} for $A_1 \geq 0$ and $B_1 \geq 0$

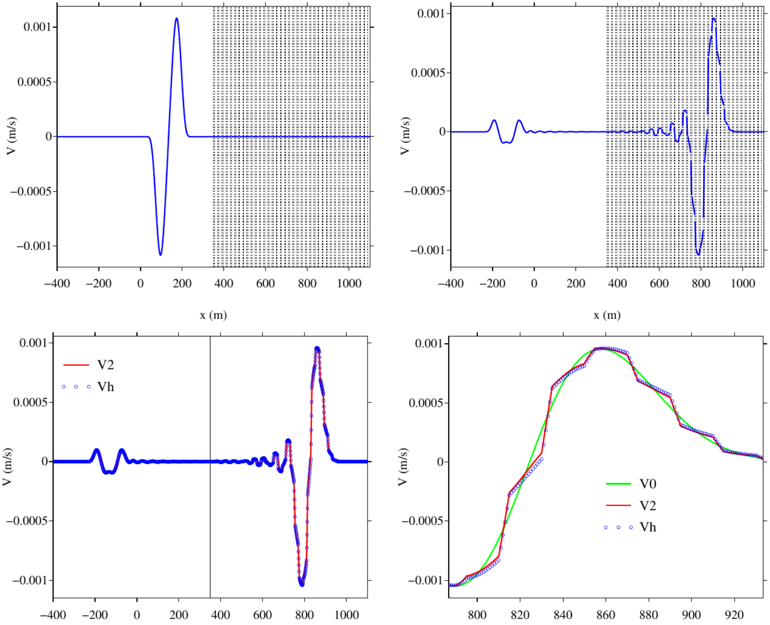
$$\Rightarrow \quad \text{Choice of } d \geq d_{\min} := \max\left(\frac{E_- P_1(0)}{E_- + \mathcal{E}_0}, \ \frac{\varrho_0 \Sigma_1(0)}{\rho_- + \varrho_0}, \ 0 \right)$$

Transmission conditions for $\varepsilon = \ell/\lambda_c \approx 0.04$ (Numerics by B. Lombard)

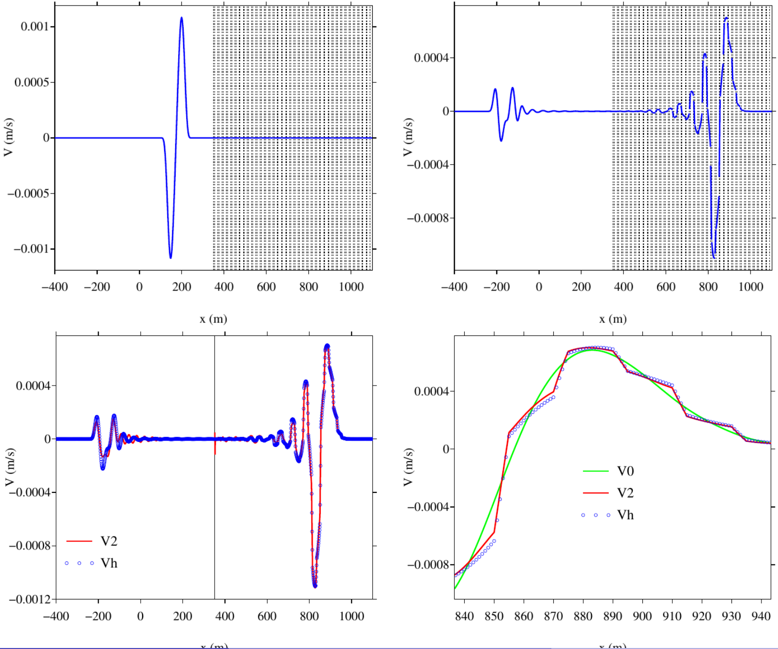
$E_-\rho_- = \mathcal{E}_0\varrho_0 \implies$ reflected wave is a 1st-order effect



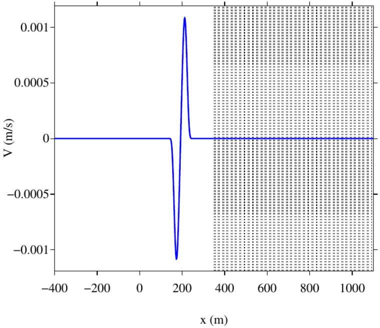
Transmission conditions, increasing central frequency, $\varepsilon \approx 0.09$

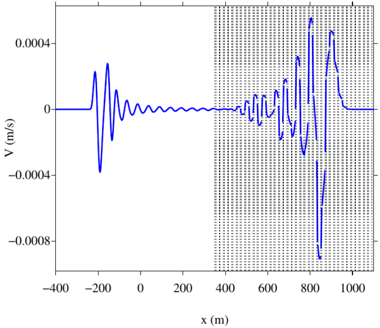


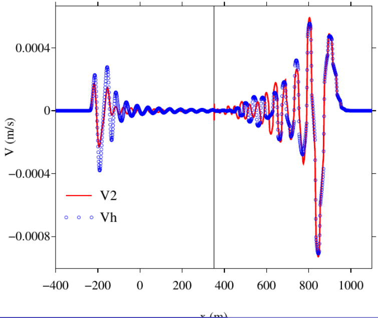
Transmission conditions, increasing central frequency, $\varepsilon \approx 0.13$

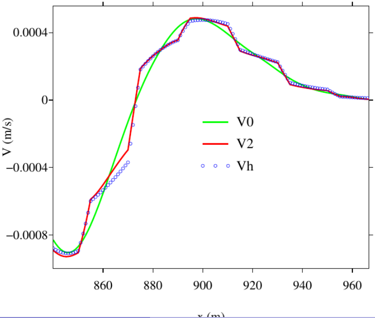


Transmission conditions, increasing central frequency, $\varepsilon \approx 0.17$





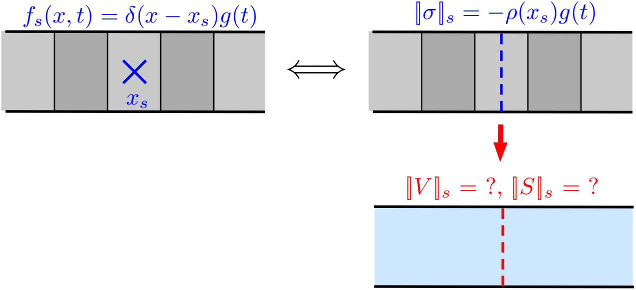




Outline

In 1D, source points ⇔ interfaces with stress jumps

$$\begin{cases} \partial_t v_\ell - \dfrac{1}{\rho_\ell}\partial_x \sigma_\ell = \delta_s g(t) & x \in \mathbb{R} \\ \partial_t \sigma_\ell - E_\ell\, \partial_x v_\ell = 0 & x \in \mathbb{R} \end{cases} \iff \begin{cases} \partial_t v_\ell - \dfrac{1}{\rho_\ell}\partial_x \sigma_\ell = 0 & x < x_{\mathrm{s}} \text{ and } x > x_{\mathrm{s}}, \\ \partial_t \sigma_\ell - E_\ell\, \partial_x v_\ell = 0 & x < x_{\mathrm{s}} \text{ and } x > x_{\mathrm{s}}, \\ [\![v_\ell]\!]_{\mathrm{s}} = 0 & x = x_{\mathrm{s}}, \\ [\![\sigma_\ell]\!]_{\mathrm{s}} = -\rho_{\mathrm{s}}\, g & x = x_{\mathrm{s}}, \end{cases}$$



First-order correction

- **Same tools** (and no need for an enlarged interface):

$$\begin{cases} [\![V]\!]_{\mathrm{s}} = \ell P_1(x_{\mathrm{s}}/\ell)\,\dfrac{\rho_{\mathrm{s}}}{\mathcal{E}_0}\,\partial_t g, \\ [\![S]\!]_{\mathrm{s}} = -\rho_{\mathrm{s}}\, g, \end{cases}$$

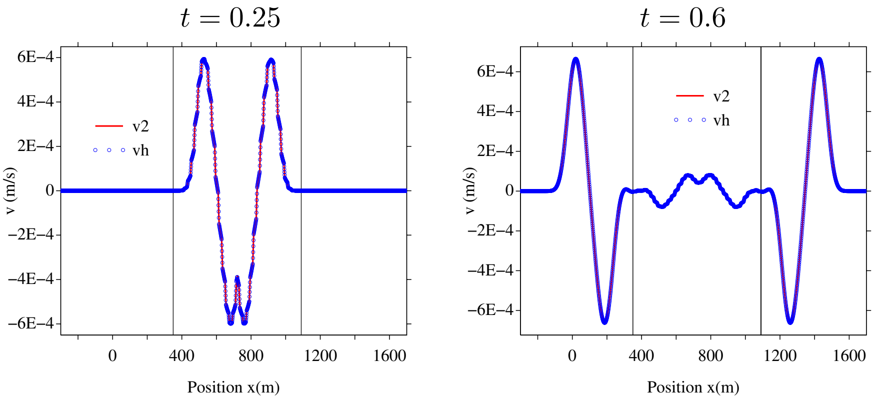
  - First-order corrector on V involving $\partial_t g$ and the local values ρ_{s} and $P_1(x_{\mathrm{s}}/\ell)$.
  - Already given by [Capdeville et al., 2010] with another argument.

- Equivalent system with Dirac notation:

$$\begin{cases} \partial_t V - \dfrac{a}{\varrho_0}\partial_x S & = -\dfrac{a-1}{\varrho_0} r + \dfrac{a}{\varrho_0}\rho_{\mathrm{s}}\, g\, \delta_{\mathrm{s}}, \\ \partial_t S - \mathcal{E}_0 \partial_x V & = -\ell\, \rho_{\mathrm{s}} P_1(x_{\mathrm{s}}/\ell)\, \partial_t g\, \delta_{\mathrm{s}}, \\ \partial_t \varphi - \dfrac{a-1}{\varrho_0}\partial_x S & = -\dfrac{a-1}{\varrho_0} r + \dfrac{a-1}{\varrho_0}\rho_{\mathrm{s}}\, g\, \delta_{\mathrm{s}}, \\ \partial_t r & = \dfrac{\mathcal{E}_0}{\ell^2 \beta}\varphi. \end{cases}$$

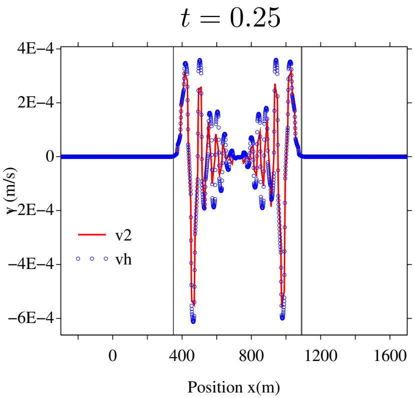
Velocity fields in a slab with a source point

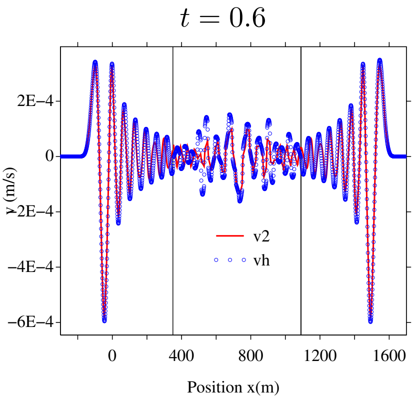
$f_c = 3$ Hz, $\varepsilon = 0.04$



Velocity fields in a slab with a source point

$f_c = 12$ Hz, $\varepsilon = 0.29$





Key ideas

- The second-order asymptotic homogenization accounts for dispersive effects.
- Boundary and transmission conditions can be designed to complement the inner expansion. (including in 2-3D [Vinoles, 2016, Maurel and Marigo, 2018, Cakoni et al., 2019, Beneteau, 2021] ...)
- For transient waves, **stability** can be adressed using (i) hyperbolic formalism and (ii) enlarged interfaces (in 1D for now).
- Source points can be addressed using the same tools (specific to 1D).

Perspectives

- Pursue the transient case up to *full second-order model* (as done in the stationary case [Cornaggia and Guzina, 2020])
- Apply the built framework to other 1D configurations:
  - High-frequency homogenization [Craster et al., 2010, Guzina et al., 2019]
  - Solids with inner imperfect interfaces
  - Non-linearities

  $\Longrightarrow$ *See Cédric Bellis' talk before lunch.*
- Hyperbolic formalism for higher dimensions and other models ? (2D acoustics, elasticity ...)

Thanks for your attention !

Second-order homogenization of boundary and transmission conditions for one-dimensional waves in periodic media
Rémi Cornaggia, Bojan B. Guzina
International Journal of Solids and Structures, 2020

An homogenized model accounting for dispersion, interfaces and source points for transient waves in 1D periodic media
Rémi Cornaggia, Bruno Lombard
submitted, preprint available on HAL: https://hal.archives-ouvertes.fr/hal-03652455

References

Allaire, G., Briane, M., and Vanninathan, M. (2016).
A comparison between two-scale asymptotic expansions and Bloch wave expansions for the homogenization of periodic structures.
SeMA Journal, 73(3):237–259.

Beneteau, C. (2021).
Modèles homogénéisés enrichis en présence de bords : Analyse et traitement numérique.
Ph. D. thesis, Institut Polytechnique de Paris.

Bensoussan, A., Lions, J. L., and Papanicolau, G. (1978).
Asymptotic Analysis for Periodic Structures.
North-Holland, Amsterdam.

Cakoni, F., Guzina, B. B., Moskow, S., and Pangburn, T. (2019).
Scattering by a bounded highly oscillating periodic medium and the effect of boundary correctors.
SIAM Journal on Applied Mathematics, 79(4):1448–1474.

Capdeville, Y., Guillot, L., and Marigo, J.-J. (2010).
1-D non-periodic homogenization for the seismic wave equation.
Geophysical Journal International.

Cioranescu, D. and Donato, P. (1999).
An introduction to homogenization, volume 17 of *Oxford Lecture Series in Mathematics and its Applications*.
Oxford University Press.

Cornaggia, R. and Guzina, B. B. (2020).
Second-order homogenization of boundary and transmission conditions for one-dimensional waves in periodic media.
International Journal of Solids and Structures, 188-189:88 – 102.

Craster, R. V., Kaplunov, J., and Pichugin, A. V. (2010).
High-frequency homogenization for periodic media.
Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 466(2120):2341–2362.

Dumontet, H. (1986).
Study of a boundary layer problem in elastic composite materials.
ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique, 20(2):265–286.

Forest, S. and Sab, K. (2017).
Finite-deformation second-order micromorphic theory and its relations to strain and stress gradient models.
Mathematics and Mechanics of Solids, page 108128651772084.

Guzina, B. B., Meng, S., and Oudghiri-Idrissi, O. (2019).
A rational framework for dynamic homogenization at finite wavelengths and frequencies.
Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 475(2223):20180547.

Maurel, A. and Marigo, J.-J. (2018).
Sensitivity of a dielectric layered structure on a scale below the periodicity: A fully local homogenized model.
Physical Review B, 98(2).

Pichugin, A., Askes, H., and Tyas, A. (2008).
Asymptotic equivalence of homogenisation procedures and fine-tuning of continuum theories.
Journal of Sound and Vibration, 313(3-5):858 – 874.

Sanchez-Palencia, E. (1974).
Comportements local et macroscopique d'un type de milieux physiques heterogenes.
International Journal of Engineering Science, 12(4):331 – 351.

Santosa, F. and Symes, W. W. (1991).
A dispersive effective medium for wave propagation in periodic composites.
SIAM Journal on Applied Mathematics, 51(4):984–1005.

Vinoles, V. (2016).
Interface problems with metamaterials : modelling, analysis and simulations.
Ph. D. thesis, ENSTA, Université Paris-Saclay.

Wautier, A. and Guzina, B. B. (2015).
On the second-order homogenization of wave motion in periodic media and the sound of a chessboard.
Journal of the Mechanics and Physics of Solids, 78:382 – 414.

Near boundaries ? From statics ... [Dumontet, 1986]

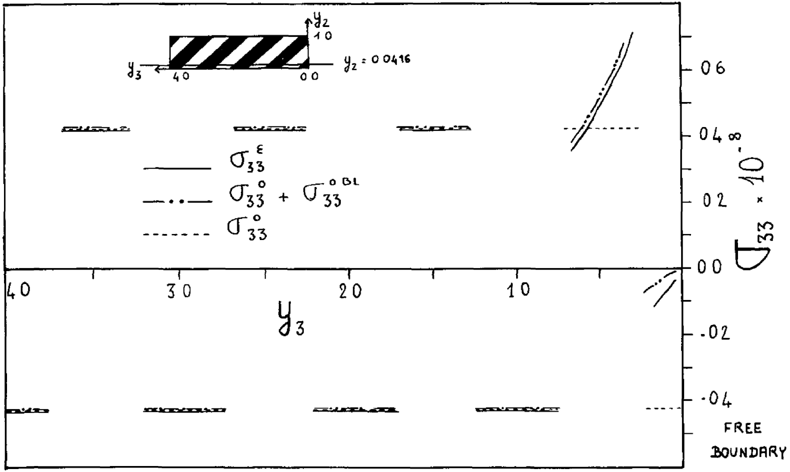


Figure 4. — The (3, 3) components of the real stresses σ^{ε}, of the micro-stresses σ^{0} and of the boundary layer stresses σ^{OBL} added to σ^{0}, ploted against y_3, at $y_2 = 0.0416$ fixed.

... to time-harmonic dynamics [Beneteau, 2021] with S. Fliss, X. Claeys

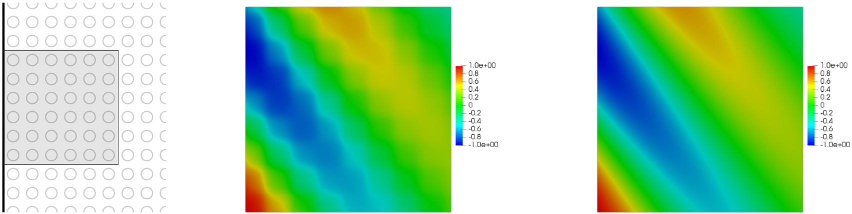


FIGURE 2.4 – u_ε pour $\varepsilon = 1/8$ (à gauche) et u_0 (à droite) dans $\Omega = (0,1)^2$

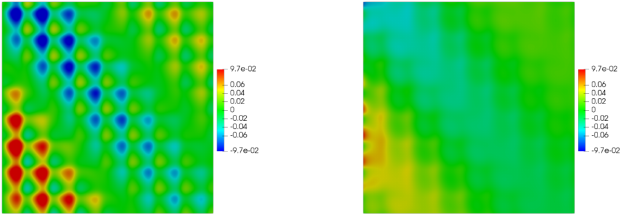
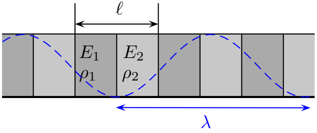


FIGURE 2.7 – $u_\varepsilon - u_0$ (à gauche) et $u_\varepsilon - (u_0 + \varepsilon u_1^{osc})$ (à droite) dans $\Omega = (0,1)^2$ pour $\varepsilon = 1/8$

Choosing a (mt) model from dispersion relations in layered media



- *Bloch wave* $u_\ell(X,t) = \tilde{u}(x)e^{\mathrm{i}(\kappa x - \omega t)} \implies$ *dispersion relation* $\omega = f(\kappa)$.
  About $(\omega, \kappa) = (0, 0)$ (on the *acoustic branch*):

$$\frac{\omega}{c_0} = \kappa \left[1 - \frac{\beta}{2}(\kappa\ell)^2 + \frac{\beta\left(2 - 27\beta - 8\overline{\beta}\right)}{40}(\kappa\ell)^4 + O(\ell^6) \right],$$

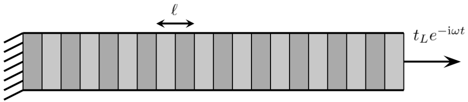
  ($\overline{\beta}$ known for layered media)

- *Plane wave* $U(x,t) = \tilde{U}e^{\mathrm{i}(\kappa x - \omega t)}$ in the second-order (mt) homogenized model:

$$\frac{\omega}{c_0} = \kappa \left[1 + \frac{\beta_m + \beta_t}{2}(\kappa\ell)^2 + \frac{(\beta_m + \beta_t)(3\beta_m + 7\beta_t)}{8}(\kappa\ell)^4 + O(\ell^6) \right].$$

- *Second-order* approximation of ω/c_0 obtained for any (β_m, β_t) satisfying $-\beta_m - \beta_t = \beta$.
  Also true in 2-3D [Allaire et al., 2016].
- *Fourth-order* approximation for $\beta_m = \dfrac{-1 - 4\beta + 4\overline{\beta}}{10}$ and $\beta_t = \dfrac{1 - 6\beta - 4\overline{\beta}}{10}$
  Similar approximation for spring-mass lattice in [Pichugin et al., 2008]

Model problem and leading-order approximation



- Time-harmonic model problem:

$$\begin{cases} \left[E(x/\ell)u_{,x}\right]_{,x} + \rho(x/\ell)\,\omega^2 u = 0 & x \in Y_L :=]0, L[\\ u = 0 & x = 0 \\ \sigma = E(x)u_{,x} = \sigma_L & x = L \end{cases}$$

- Leading-order homogenization: $(u, \sigma) \to (U, \mathcal{E}_0 U_{,x})$

$$\begin{cases} U_{,xx} + k_0^2\, U = 0 & x \in Y_L, \qquad k_0 := \omega / c_0 \\ U = 0 & x = 0 \\ U_{,x} = \sigma_L / \mathcal{E}_0 & x = L \end{cases}$$

Using full-field approximations [Cornaggia and Guzina, 2020]

First-order approximations of displacement and stress (u, σ):

$$\begin{cases} \tilde{u}^{(1)}(x) = U(x) + \ell P_1\left(\frac{x}{\ell}\right)U_{,x}(x) \\ \tilde{\sigma}^{(1)}(x) = \mathcal{E}_0\left[U_{,x}(x) + \ell\Sigma_1\left(\frac{x}{\ell}\right)U_{,xx}(x)\right] \end{cases}$$

Using **cell stress functions:** $\Sigma_j := (E/\mathcal{E}_0)\left[P_j + P_{j+1,y}\right]$

Boundary-value problem for the mean field U:

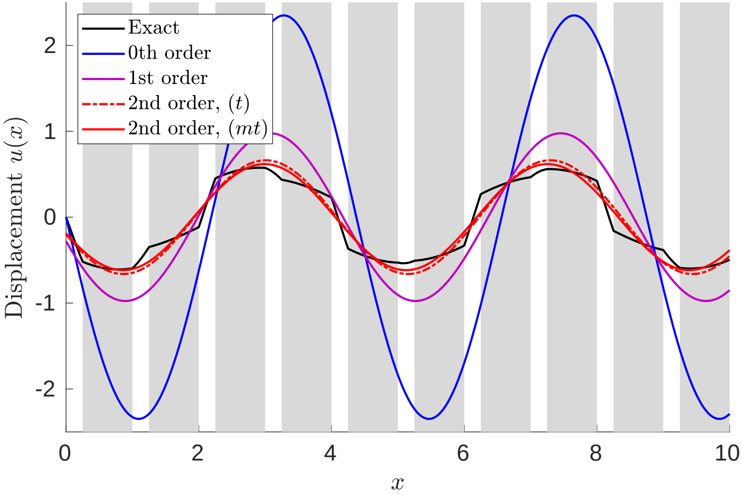
$$\left|\begin{array}{ll} U_{,xx} + k_0^2\, U = 0 & x \in Y_L \\ \tilde{u}^{(1)} = 0 & x = 0 \\ \tilde{\sigma}^{(1)} = \sigma_L & x = L \end{array}\right.$$

$$\left|\begin{array}{ll} U_{,xx} + k_0^2\, U = 0 & x \in Y_L \\ U + \ell P_1(0) U_{,x} = 0 & x = 0 \\ U_{,x} + \ell\Sigma_1(0) U_{,xx} = \sigma_L/\mathcal{E}_0 & x = L \end{array}\right.$$

$$\left|\begin{array}{ll} U_{,xx} + k_0^2\, U = 0 & x \in Y_L \\ U + \ell P_1(0) U_{,x} = 0 & x = 0 \\ U_{,x} - \ell\Sigma_1(0) k_0^2\, U = \sigma_L/\mathcal{E}_0 & x = L \end{array}\right.$$

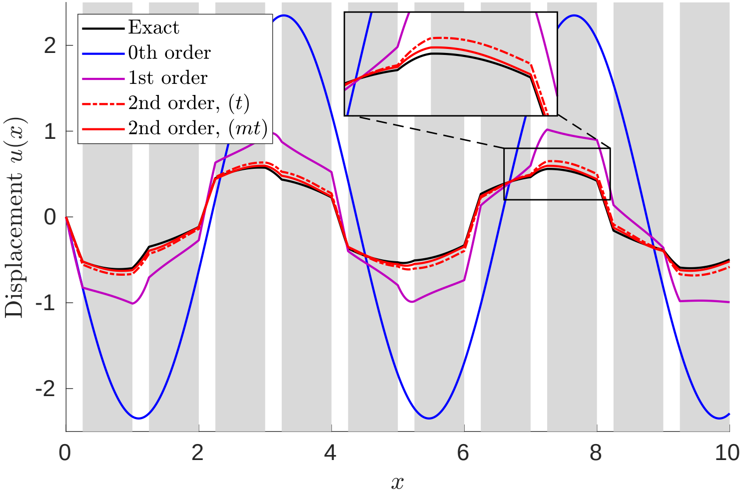
Example for a layered material - mean fields U

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$



Example for a layered material - total displacement u

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$



Example for a layered material - axial stress σ

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$