Effective dynamics for elastic waves in an array of non-linear interfaces

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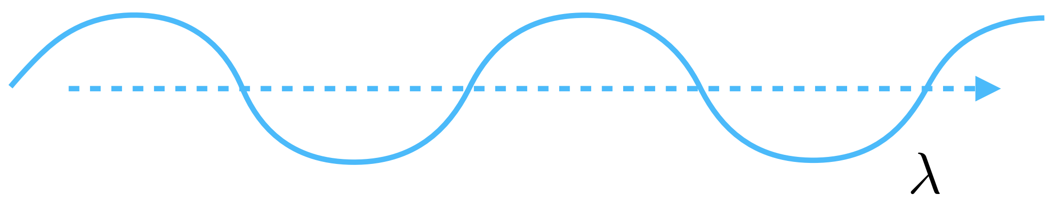
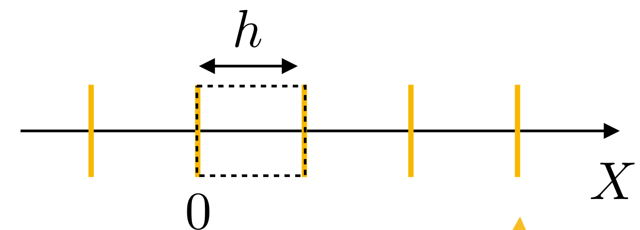
Microstructured problem

- 1D array of non-linear imperfect interfaces





- wave equation:

$$\rho_h(X)\frac{\partial^2 U_h}{\partial t^2}(X,t) = \frac{\partial \Sigma_h}{\partial X}(X,t) + F(X,t)$$

- linear stress-strain relation:

$$\Sigma_h(X,t) = E_h(X)\frac{\partial U_h}{\partial X}(X,t)$$

- interface conditions:

$$\begin{cases} M\left\langle\!\!\left\langle \dfrac{\partial^2 U_h}{\partial t^2}(\cdot,t) \right\rangle\!\!\right\rangle_{X_n} = [\![\Sigma_h(\cdot,t)]\!]_{X_n} \\ \langle\!\langle \Sigma_h(\cdot,t)\rangle\!\rangle_{X_n} = K\,\mathcal{R}\left([\![U_h(\cdot,t)]\!]_{X_n}\right) \end{cases}$$

M, K : interface mass and stiffness

from phenomenological models or homogenization of interphases

- jump and mean operators: $[\![g]\!]_{X_n} = g(X_n^+) - g(X_n^-)$ and $\langle\!\langle g\rangle\!\rangle_{X_n} = \frac{1}{2}\left(g(X_n^+) + g(X_n^-)\right)$

- **Specific setting:**
  - long-wavelength regime $h \ll \lambda$
  - non-linear interface behavior $\mathcal{R}$

- **Objective:** effective dynamical model at the 1st-order, i.e. approximation $U_h(X,t) = U^{(1)}(X,t) + o(h)$

Constitutive interface law

- **Constitutive assumptions:**

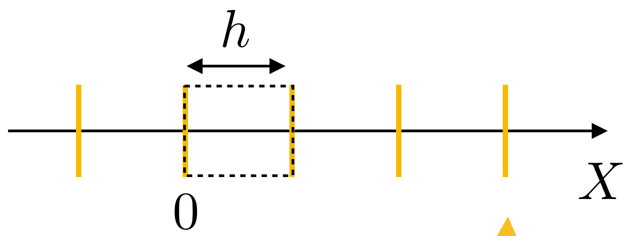
1) $M \geq 0$ and $K > 0$

2) $\mathcal{R} : (-d, +\infty) \longrightarrow \mathbb{R}$ smooth

maximum *compressibility length*: $d \in \mathbb{R}_+ \cup \{+\infty\}$

technical assumptions: $\mathcal{R}(0) = 0, \quad \mathcal{R}' > 0$

$$\mathcal{R}'' < 0 \text{ or } \mathcal{R}'' = 0$$



- interface conditions:

$$\begin{cases} M \left\langle\!\left\langle \dfrac{\partial^2 U_h}{\partial t^2}(\cdot, t) \right\rangle\!\right\rangle_{X_n} = [\![\Sigma_h(\cdot, t)]\!]_{X_n} \\ \langle\!\langle \Sigma_h(\cdot, t) \rangle\!\rangle_{X_n} = K\, \mathcal{R}\left([\![U_h(\cdot, t)]\!]_{X_n}\right) \end{cases}$$

M, K : interface mass and stiffness

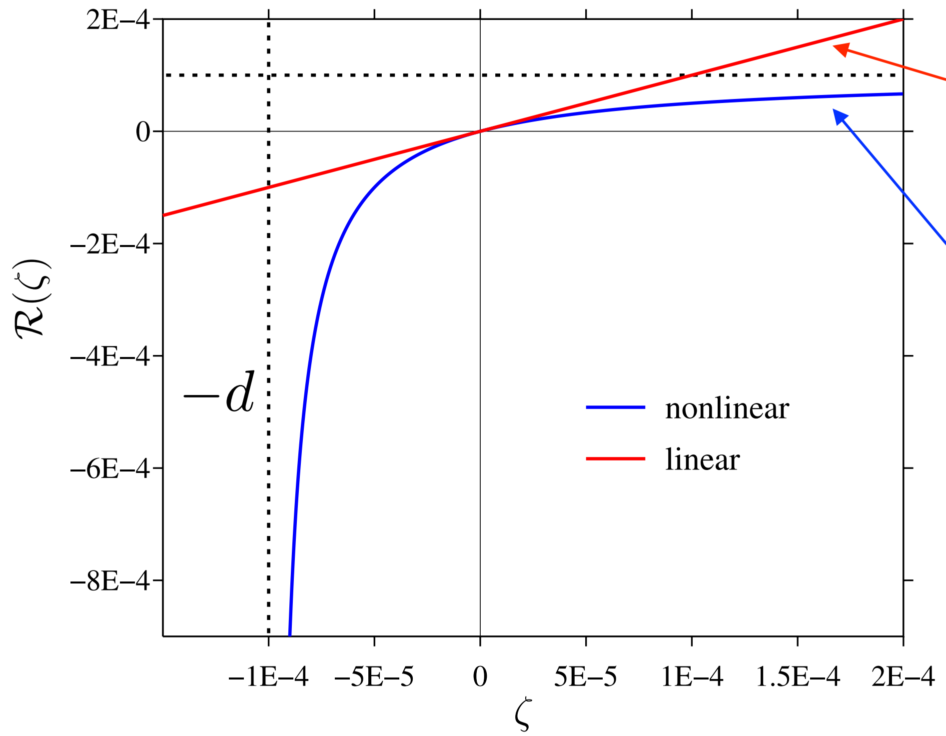
- Phenomenological models

[Achenbach, Norris, Bandis, Sevostianov, Broda, ...]

- **Examples:**

model degenerates to

- perfect transmission cond.
- linear law
- unilateral contact
- ...



$\mathcal{R}(\zeta) = \zeta$ linear (spring-like)

$\mathcal{R}(\zeta) = \dfrac{\zeta}{1 + \zeta/d}$ hyperbolic

Microstructure behavior

- For a **single** interface:
    - existence and uniqueness of a solution to elastodyn. equations [Junca, Lombard, 2009]
    - generation of harmonics
    - amplitude of harmonics increases with source amplitude and non-monotonic behavior w.r.t. freq.

- **Energy analysis in array:** $\mathscr{E}_h = \mathscr{E}_h^m + \mathscr{E}_h^i$ such that $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_h = 0$ without source

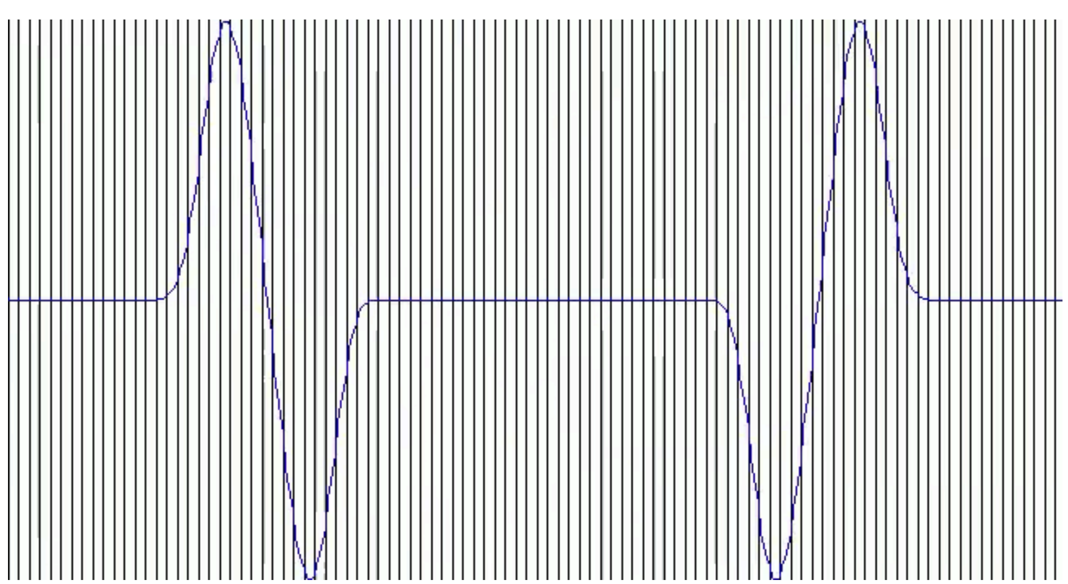
medium mech. energy

interface energy

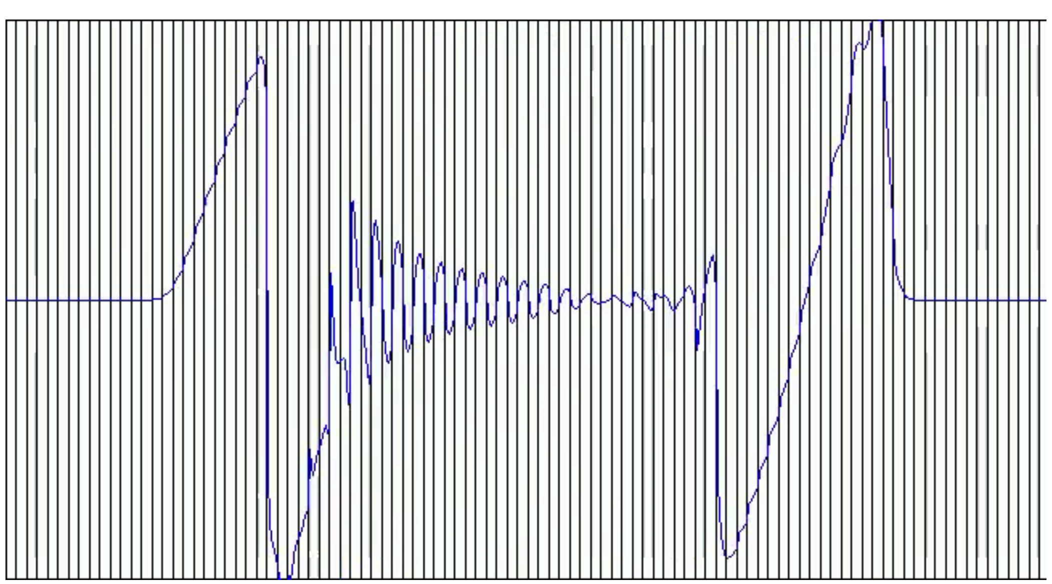
$$\mathscr{E}_h^m(t) = \frac{1}{2}\int_I \left\{ \rho_h(X) V_h(X,t)^2 + \frac{1}{E_h(X)} \Sigma_h(X,t)^2 \right\} \mathrm{d}X \qquad \mathscr{E}_h^i(t) = \sum_{X_n^I} \left\{ \frac{1}{2} M \langle\!\langle V_h(\cdot,t) \rangle\!\rangle_{X_n^I}^2 + K \int_0^{\mathcal{R}^{-1}\left(\langle\!\langle \Sigma_h(\cdot,t) \rangle\!\rangle_{X_n^I}/K\right)} \mathcal{R}(\zeta)\,\mathrm{d}\zeta \right\}$$

- **Numerical simulation:** source with *fixed* central frequency and amplitude A

$A = 0.1$



$A = 120$



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medium mech. energy ($\mathscr{E}_h^m$) — interface energy ($\mathscr{E}_h^i$)

$$\mathscr{E}_h^m(t) = \frac{1}{2}\int_I \left\{ \rho_h(X) V_h(X,t)^2 + \frac{1}{E_h(X)} \Sigma_h(X,t)^2 \right\} \mathrm{d}X \qquad \mathscr{E}_h^i(t) = \sum_{X_n^I} \left\{ \frac{1}{2} M \langle\!\langle V_h(\cdot,t) \rangle\!\rangle_{X_n^I}^2 + K \int_0^{\mathcal{R}^{-1}\left(\langle\!\langle \Sigma_h(\cdot,t)\rangle\!\rangle_{X_n^I}/K\right)} \mathcal{R}(\zeta)\,\mathrm{d}\zeta \right\}$$

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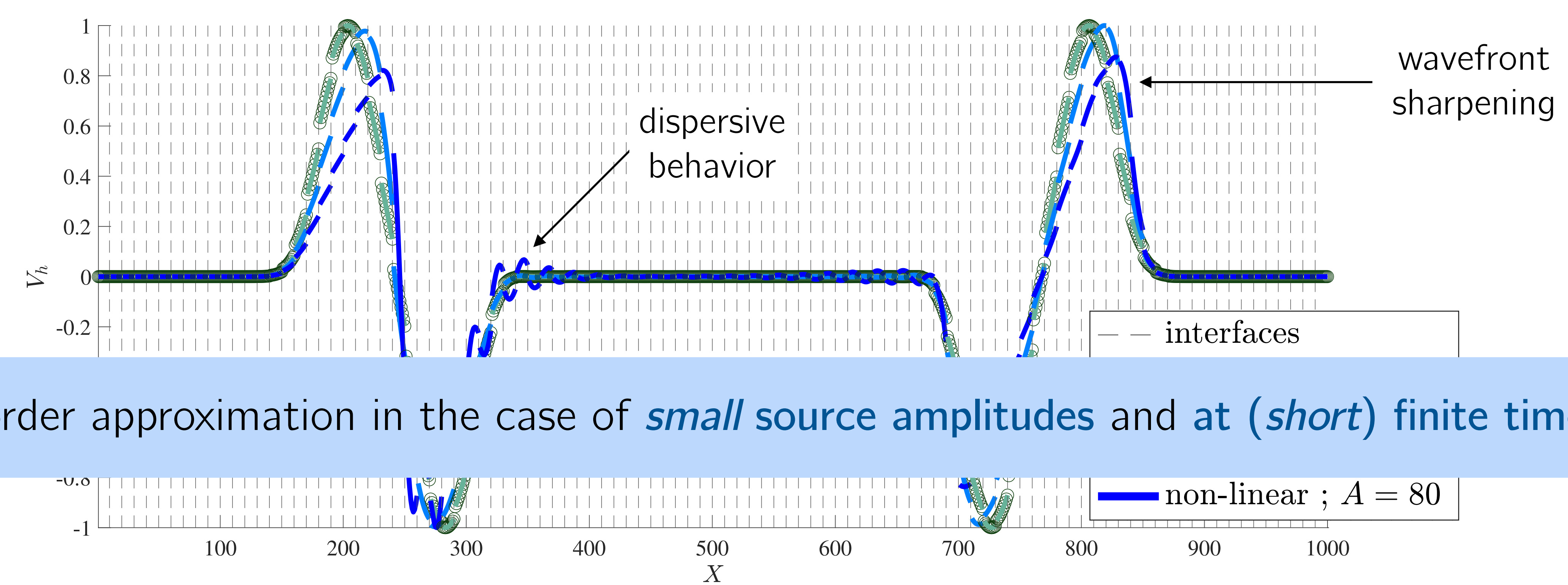
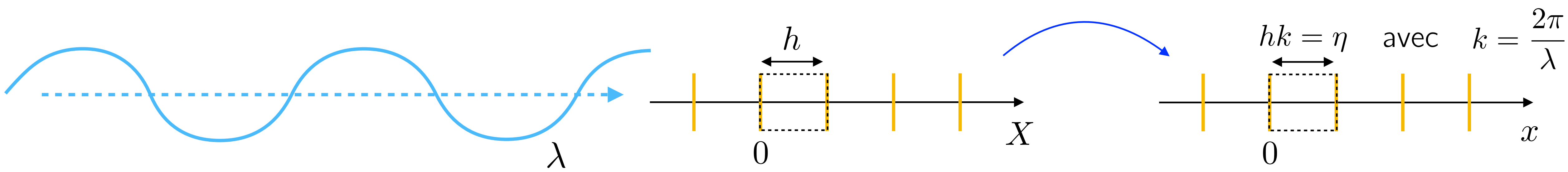


➡ 1st-order approximation in the case of *small* source amplitudes and at (*short*) finite times

Two-scale analysis

- **Non-dimensionalization** of governing equations: $(X,t) \longrightarrow (x,\tau)$



- wave equation: $\alpha\left(\dfrac{x}{\eta}\right)\dfrac{\partial^2 u_\eta}{\partial \tau^2}(x,\tau) = \dfrac{\partial}{\partial x}\left(\beta\left(\dfrac{x}{\eta}\right)\dfrac{\partial u_\eta}{\partial x}(x,\tau)\right) + f(x,\tau)$

- interface conditions: $\begin{cases} m\,\eta\left\langle\!\!\left\langle \dfrac{\partial^2 u_\eta}{\partial \tau^2}(\cdot,\tau)\right\rangle\!\!\right\rangle_{x_n} = \left[\!\!\left[\beta\dfrac{\partial u_\eta}{\partial x}(\cdot,\tau)\right]\!\!\right]_{x_n} \\ \left\langle\!\!\left\langle \beta\dfrac{\partial u_\eta}{\partial x}(\cdot,\tau)\right\rangle\!\!\right\rangle_{x_n} = \dfrac{\not{k}}{h}\,\mathcal{R}\left(\dfrac{h}{\eta}\,[\![u_\eta(\cdot,\tau)]\!]_{x_n}\right) \end{cases}$

→ contributions in η

- **Formal asymptotic expansion:** $u_\eta(x,\tau) = u_0(x,\tau) + \sum_{j\geq 1}\eta^j u_j(x, x/\eta, \tau)$ with *fast variable* $y = x/\eta$

$\eta \ll 1$

Specificities of the setting considered:

- **smooth interface law**: Taylor exp. $\mathcal{R}\left(\dfrac{h}{\eta}[\![u_\eta]\!]_{y_n}\right) = \sum_{\ell\geq 0}\dfrac{(h\eta)^\ell}{\ell!}\left(\sum_{j\geq 2}\eta^{j-2}[\![u_j(x,\cdot,\tau)]\!]_{y_n}\right)^\ell \mathcal{R}^{(\ell)}\left(h\,[\![u_1(x,\cdot,\tau)]\!]_{y_n}\right)$

- **1D magic**: direct integration possible and $\left\langle\dfrac{\mathrm{d}g}{\mathrm{d}y}\right\rangle = \displaystyle\int_0^1 \dfrac{\mathrm{d}g}{\mathrm{d}y}(y)\,\mathrm{d}y = -[\![g]\!]$

Microstructured field approximation

1st-order approximation: $U_h(X,t) = U_0(X,t) + h\,U_1(X,t) + o(h)$

- **Zeroth-order field U_0 :**

  - continuous in X

  - solution of: $\rho_{\text{eff}} \dfrac{\partial^2 U_0}{\partial t^2}(X,t) = \dfrac{\partial \Sigma_0}{\partial X}(X,t) + F(X,t)$

effective mass density

$$\rho_{\text{eff}} = \left(\langle \rho \rangle + \frac{M}{h} \right)$$

local and non-linear stress-strain relation $\Sigma_0 = \mathcal{G}_{\text{eff}}(\mathcal{E}_0)$

$$\left\langle \frac{1}{E} \right\rangle \Sigma_0(X,t) + \frac{1}{h} \mathcal{R}^{-1} \left(\frac{1}{K} \Sigma_0(X,t) \right) = \mathcal{E}_0(X,t)$$

where $\mathcal{E}_0 = \partial U_0 / \partial X$

- **Remarks:**

  - in the case of linear interfaces, i.e. $\mathcal{R}(\zeta) = \zeta$

$$\mathcal{G}_{\text{eff}}(\mathcal{E}_0) = \mathcal{C}^{\ell}_{\text{eff}}\, \mathcal{E}_0 \quad \text{with} \quad \mathcal{C}^{\ell}_{\text{eff}} = \left(\left\langle \frac{1}{E} \right\rangle + \frac{1}{Kh} \right)^{-1}$$

  - in the case of perfect interfaces

$$\rho_{\text{eff}} \sim \langle \rho \rangle \quad \text{and} \quad \mathcal{C}^{\ell}_{\text{eff}} \sim \langle 1/E \rangle^{-1}$$

Microstructured field approximation

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where $\mathcal{E}_0 = \partial U_0 / \partial X$

- Rewriting as a **non-linear first-order system**:

$$\frac{\partial}{\partial t} \Psi_0(X,t) + \frac{\partial}{\partial X} \Big(\mathbb{G}_{\text{eff}}\big(\Psi_0(X,t)\big) \Big) = \mathbb{F}(X,t) \quad \text{with} \quad \Psi_0 = \big(\mathcal{E}_0, V_0\big)^{\top}$$

  - strictly hyperbolic system (owing to technical assumptions on $\mathcal{R}$)

  - characteristic speeds (possibly strain dependent) $\nu_{\pm}(\mathcal{E}_0) = \pm \sqrt{\dfrac{1}{\rho_{\text{eff}}} \dfrac{\partial \Sigma_0}{\partial \mathcal{E}_0}}$

Microstructured field approximation

➡ 1st-order approximation: $U_h(X,t) = U_0(X,t) + h\,U_1(X,t) + o(h)$

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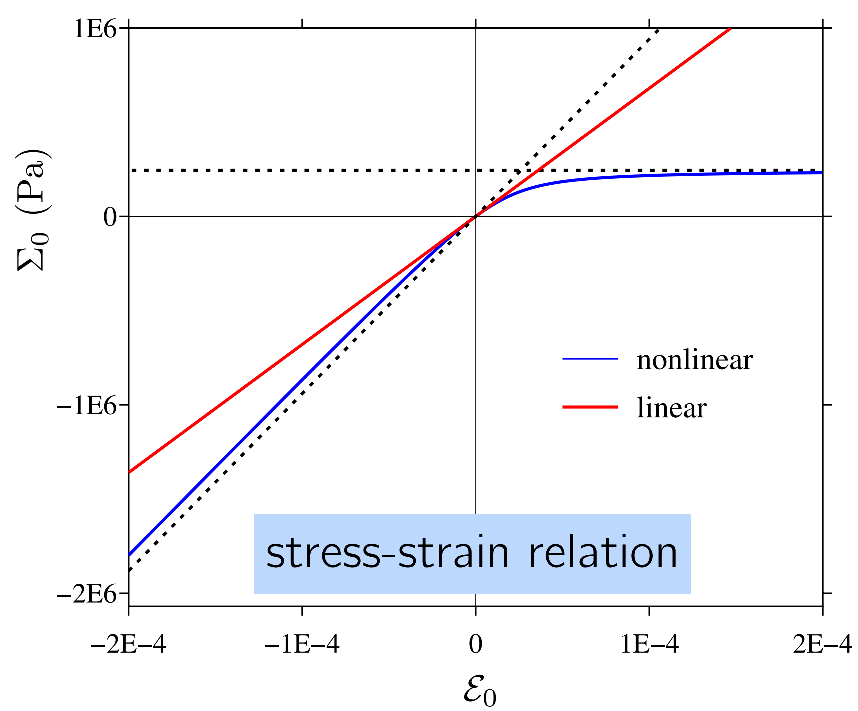
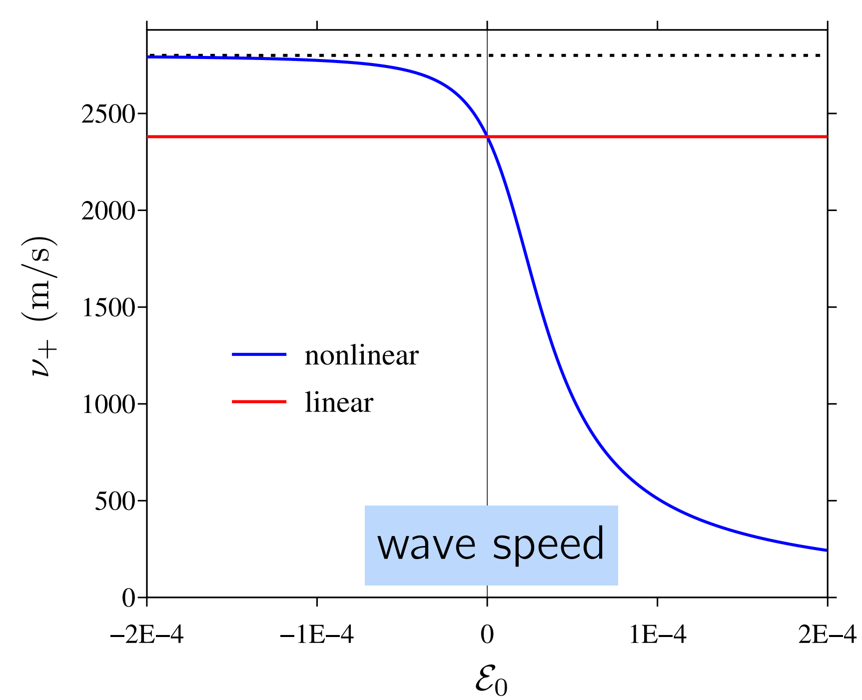
- **Examples:**

$\mathcal{R}(\zeta) = \zeta$

$$\mathcal{R}(\zeta) = \frac{\zeta}{1 + \zeta/d}$$





Microstructured field approximation

1st-order approximation: $U_h(X,t) = U_0(X,t) + h\,U_1(X,t) + o(h)$

- **Zeroth-order field U_0 :**

$$\frac{\partial}{\partial t}\Psi_0(X,t) + \frac{\partial}{\partial X}\Big(\mathbb{G}_{\text{eff}}\big(\Psi_0(X,t)\big)\Big) = \mathbb{F}(X,t) \quad \text{with} \quad \Psi_0 = \big(\mathcal{E}_0, V_0\big)^\top$$

- **Energy analysis:**

  - recall $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_h = 0$ in microstructured medium
  - here: $\mathscr{E}_0(t) = \mathscr{E}_0^m + \mathscr{E}_0^i$ such that for *sufficiently smooth fields*, it holds $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_0 = 0$ without source

    $\mathscr{E}_0(t) = \displaystyle\int_I \left\{\frac{1}{2}\rho_{\text{eff}} V_0^2 + \mathcal{J}_{\text{eff}}\right\} \mathrm{d}X$ with $\mathcal{J}_{\text{eff}}(\mathcal{E}_0) = \displaystyle\int_0^{\mathcal{E}_0} \mathcal{G}_{\text{eff}}(\tilde{\mathcal{E}})\,\mathrm{d}\tilde{\mathcal{E}}$

    bulk energy $\mathscr{E}_0^m(t) = \dfrac{1}{2}\displaystyle\int_I \left\{\langle\rho\rangle V_0^2 + \left\langle\frac{1}{E}\right\rangle \Sigma_0^2\right\} \mathrm{d}X$

    interface energy $\mathscr{E}_0^i(t) = \dfrac{1}{h}\displaystyle\int_I \left\{\frac{1}{2} M V_0^2 + K\int_0^{\mathcal{R}^{-1}(\Sigma_0/K)} \mathcal{R}(\zeta)\,\mathrm{d}\zeta\right\} \mathrm{d}X$

formally analogous to so-called *p-system* in gas dynamics [Lax, Dafermos, ...] ⟶ formation of shocks in finite time $t^\star$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_0 \le 0 \quad \text{for} \quad t \ge t^\star$$

for linearized stress-strain relation $\Sigma_0 \underset{\mathcal{E}_0\to 0}{\sim} \mathcal{C}^\ell_{\text{eff}}\,\mathcal{E}_0\,(1 - \gamma\,\mathcal{E}_0) + o(\mathcal{E}_0^2)$ $\Rightarrow \quad t^\star \propto \dfrac{1}{\mathcal{E}_{\max}\gamma\,\omega_c}$

time-harmonic forcing at ω_c and amplitude $\mathcal{E}_{\max}$

➡ 1st-order approximation: $U_h(X,t) = U_0(X,t) + h\, U_1(X,t) + o(h)$

- **First-order field** U_1 (corrector):

- decomposed as $U_1(X,t) = \overline{U}_1(X,t) + \mathcal{P}\big(y, \mathcal{E}_0(X,t)\big)\, \mathcal{E}_0(X,t)$

mean field — cell-function

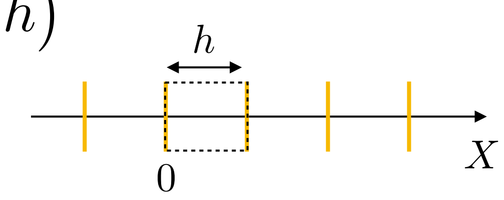
- **Mean field:** solution of **linear** and **heterogeneous problem**

$$\ast \quad \rho_{\text{eff}} \frac{\partial^2 \overline{U}_1}{\partial t^2}(X,t) = \frac{\partial \overline{\Sigma}_1}{\partial X}(X,t) + \mathcal{S}\big(U_0(X,t)\big) \quad \text{with} \quad \overline{\Sigma}_1(X,t) = \mathcal{G}'_{\text{eff}}\big(\mathcal{E}_0(X,t)\big) \frac{\partial \overline{U}_1}{\partial X}(X,t)$$

source term and stiffness (may) depend non-linearly on U_0

- **Cell-function:** $\mathcal{P}\big(y, \mathcal{E}_0(X,t)\big) \quad \text{where} \quad y = (X - nh)/h \quad \text{for} \quad X \in \big(nh, (n+1)h\big)$

(possibly) non-linear fct of U_0



- **Remarks:**
  - problem $\ast$ is well-posed (owing to technical assumptions on $\mathcal{R}$)
  - in the case of linear interfaces: $\mathcal{G}'_{\text{eff}}\big(\mathcal{E}_0(X,t)\big) = \mathcal{C}^{\ell}_{\text{eff}}$; $\mathcal{S}\big(U_0(X,t)\big) = 0$; $\mathcal{P} \equiv \mathcal{P}(y)$
  - rewriting of $\ast$ as a linear and strictly hyperbolic system with charac. speeds $\nu_{\pm}(\mathcal{E}_0)$

Numerical results

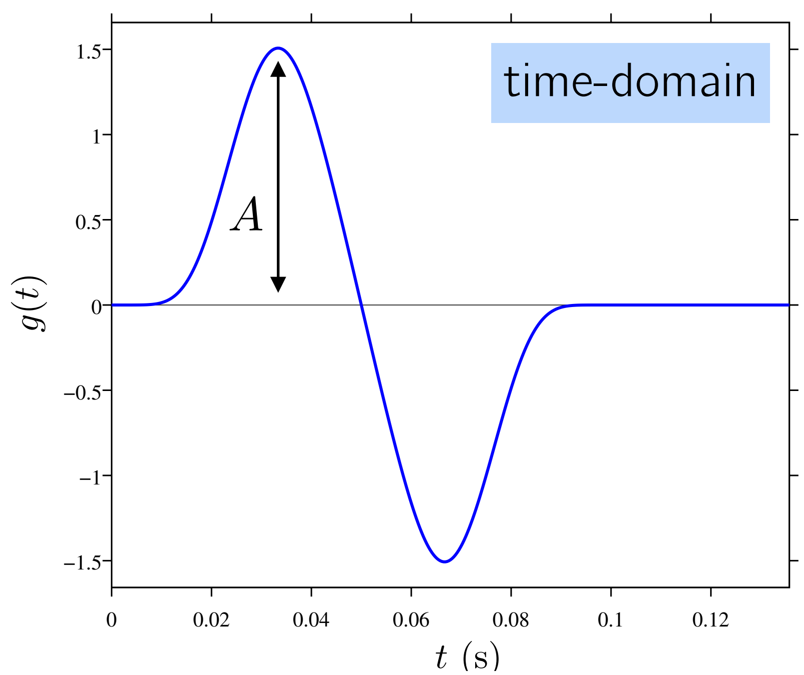
Objectives: comparisons of full-field simulations (velocity-based) V_h with homogenized solutions, i.e. V_0 and $V^{(1)}(X,t) = V_0(X,t) + hV_1(X,t)$

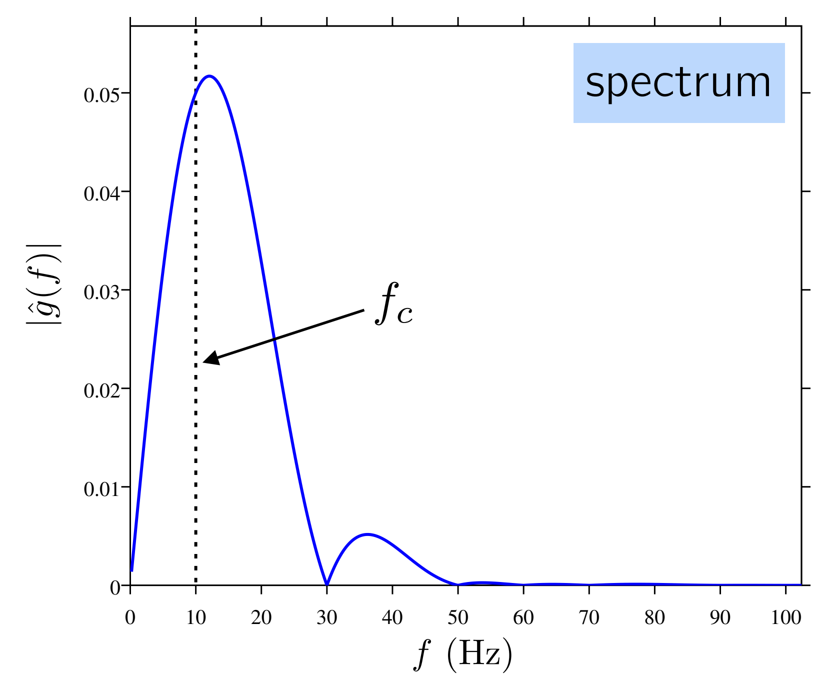
- **Numerical scheme:** - finite-volume scheme [LeVeque, 2002] to capture sharp wavefronts and shocks
  - hyperbolic systems solved sequentially
  - adjusted CFL condition $\max_{X_j} \nu_+\big(\mathcal{E}_0(X_j,t_n)\big)\dfrac{\Delta t_n}{\Delta X} \leq 1$

- **Excitation:**

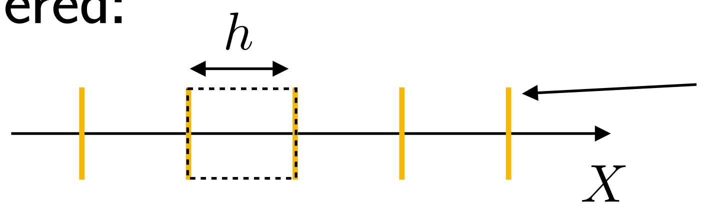
$$F(X,t) = \delta(X - X_s)\, g(t)$$

with time-domain signal $g(t)$

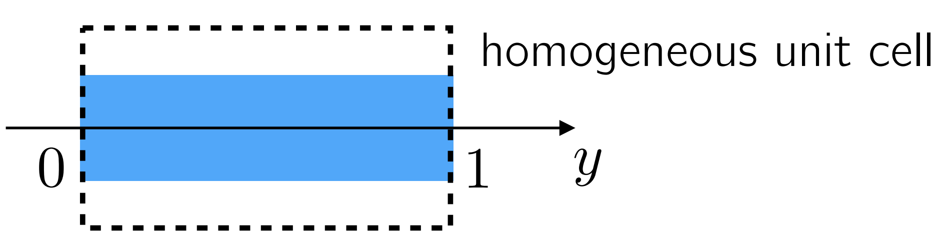




- **Medium considered:**



$$\begin{cases} M = 0 \\ \mathcal{R}(\zeta) = \dfrac{\zeta}{1 + \zeta/d} \end{cases}$$



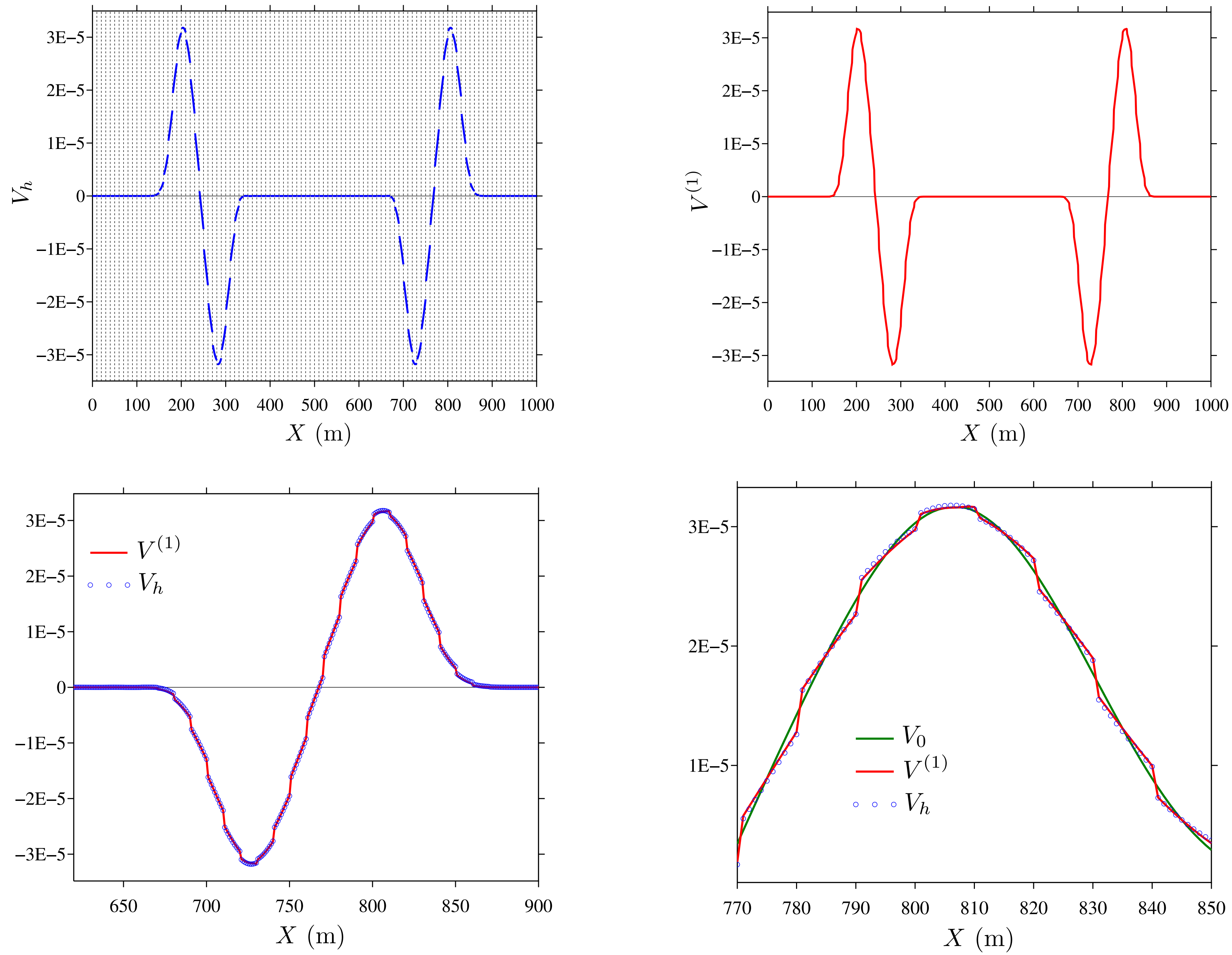
Numerical results: *agreements*

Objectives: comparisons of full-field simulations (velocity-based) V_h with homogenized solutions, i.e. V_0 and $V^{(1)}(X,t) = V_0(X,t) + hV_1(X,t)$

- **Excitation:**

$$\begin{cases} f_c = 10\,\mathrm{Hz} \\ A = 0.1 \end{cases}$$

$\eta = 0.26$



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- **Excitation:**

$$\begin{cases} f_c = 10\,\mathrm{Hz} \\ A = 40 \end{cases}$$

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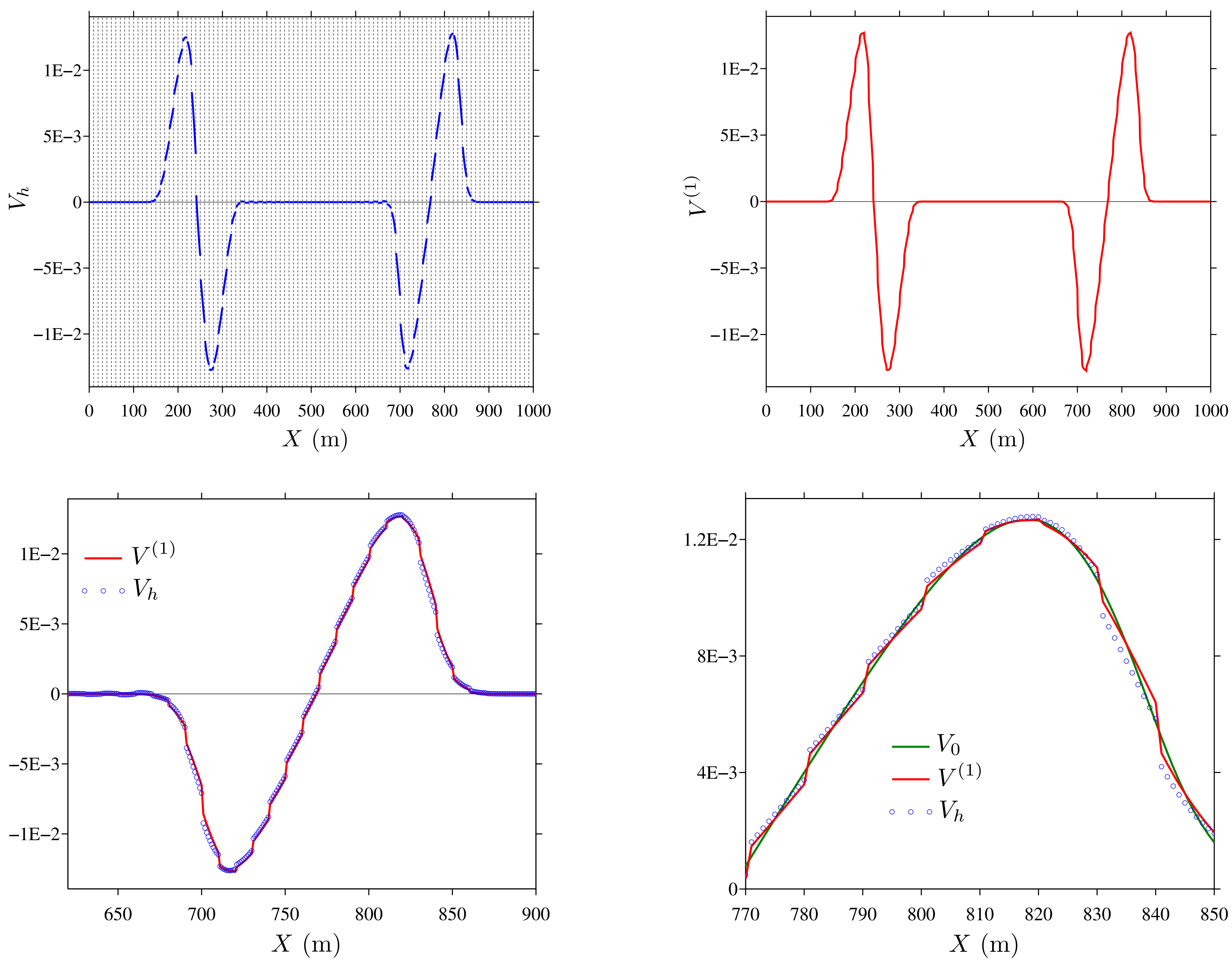

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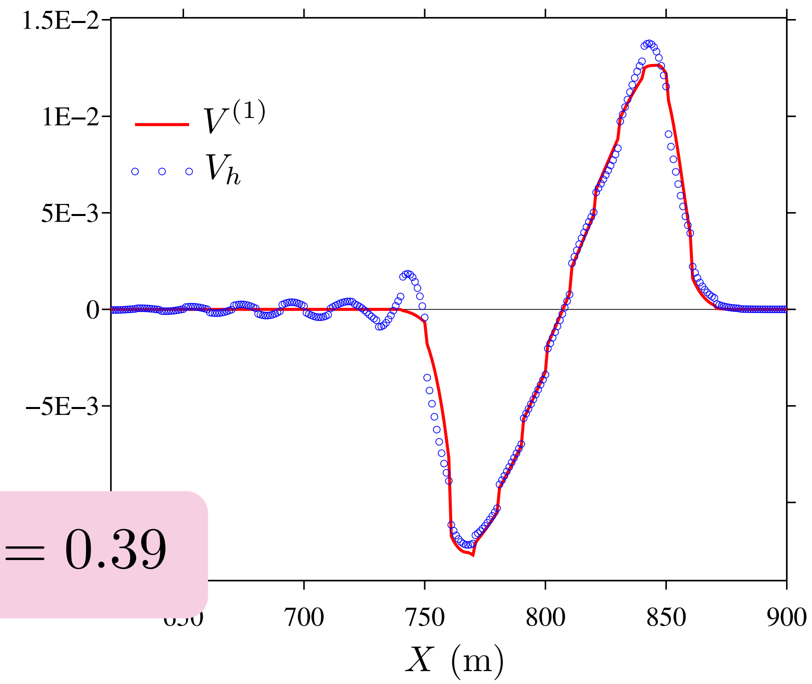
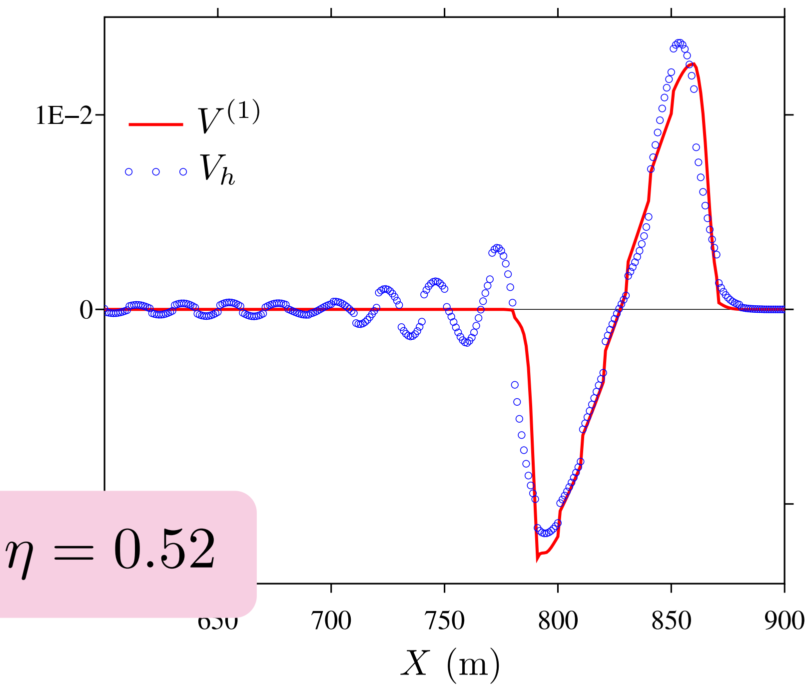
- Excitation with **fixed amplitude**: $A = 40$

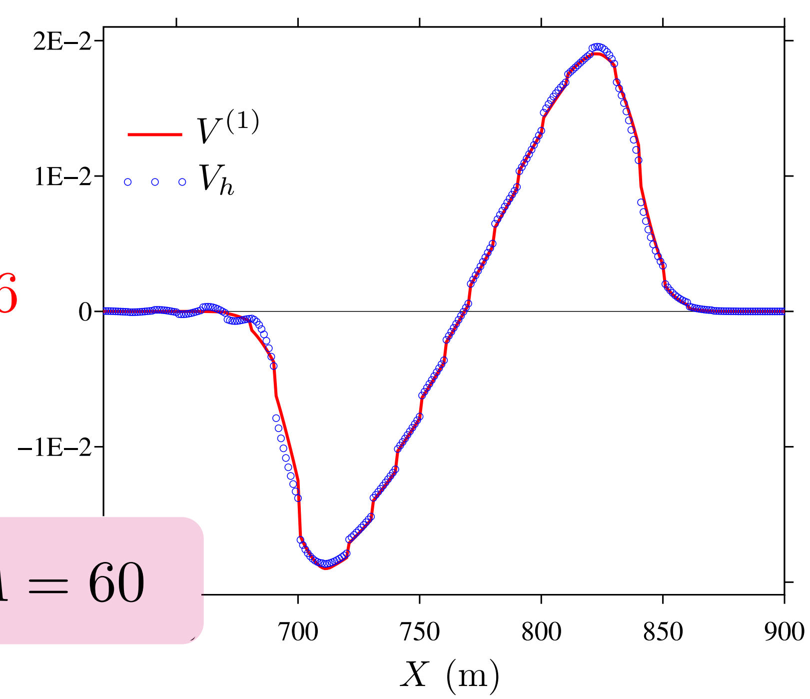


$\eta = 0.39$

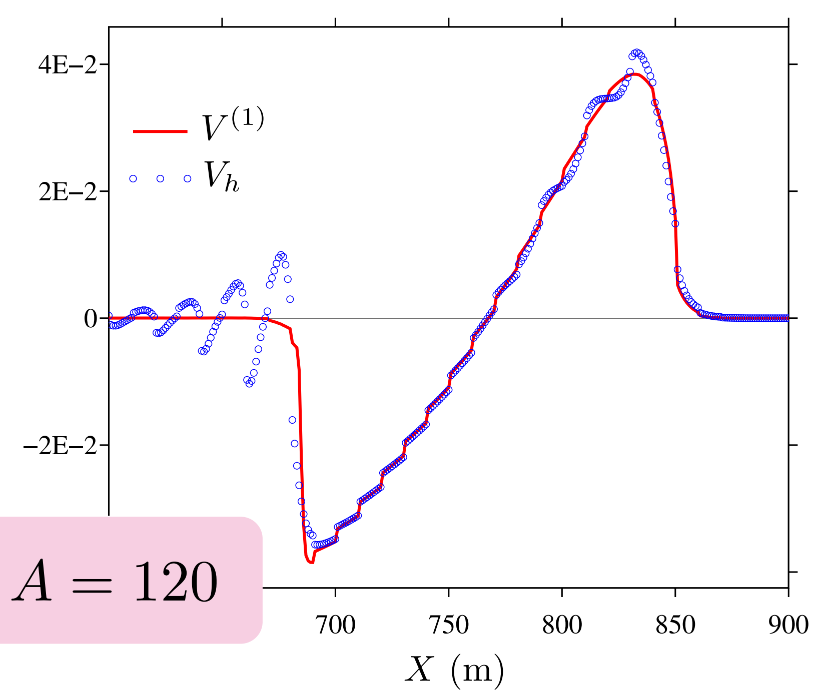


$\eta = 0.52$

- Excitation with **fixed frequency**: $\eta = 0.26$



$A = 60$



$A = 120$

Numerical results: formation of shocks

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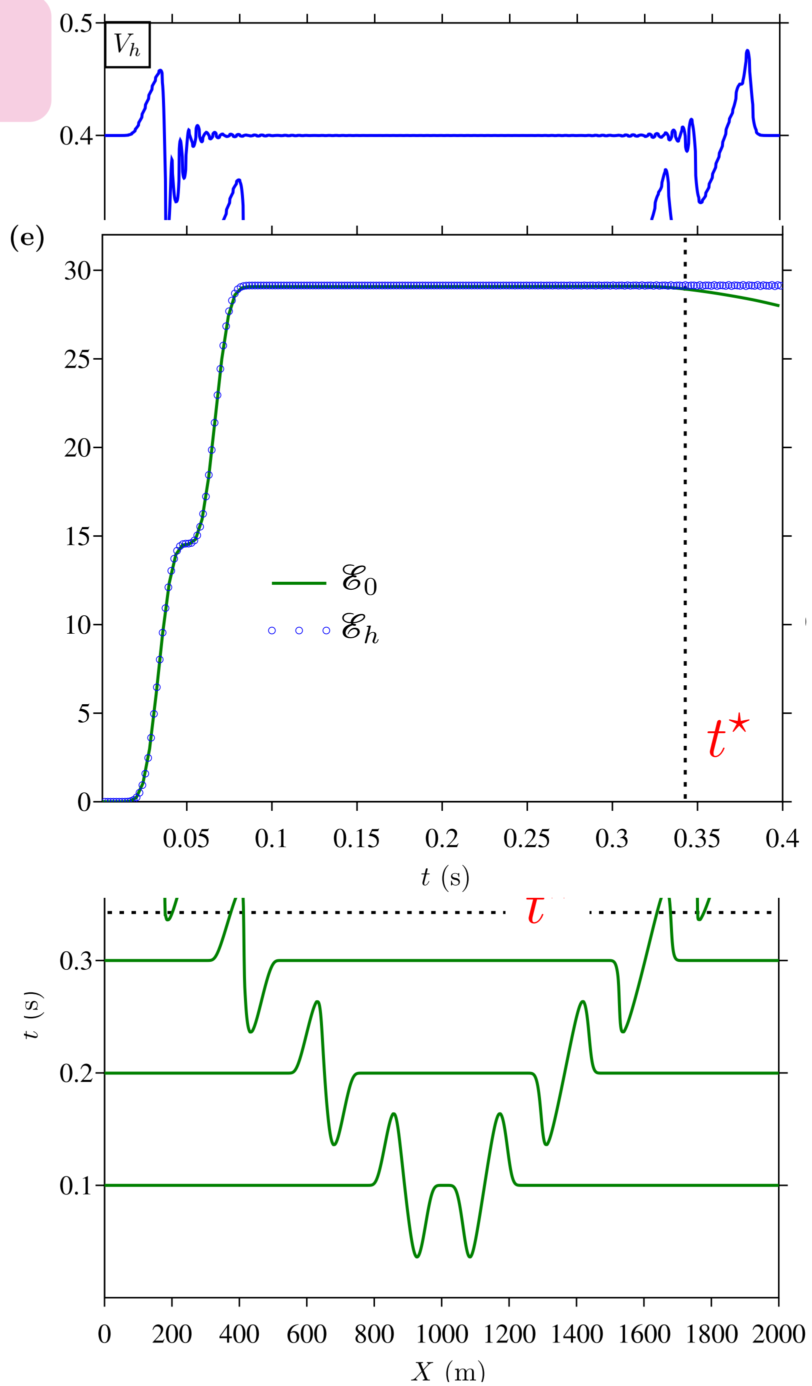

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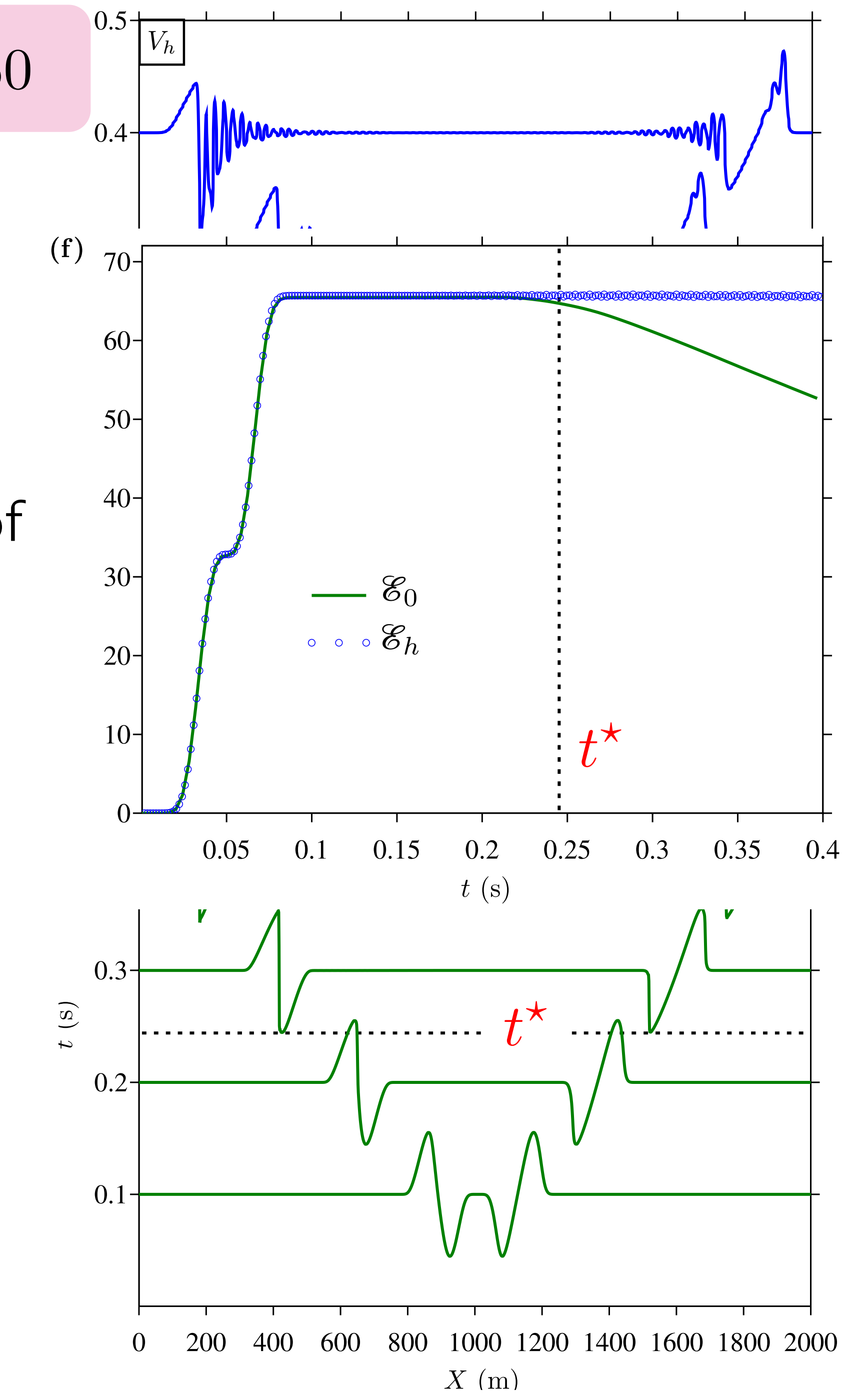


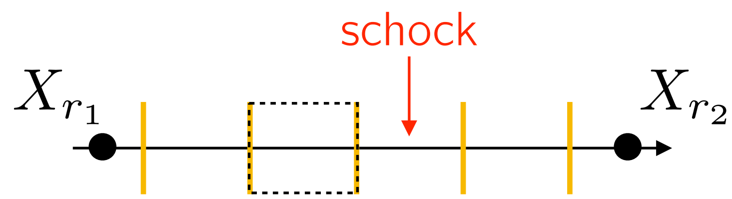
- What happens if shocks occur?

in terms of energies

(as expected with effective model with estimated time $t^\star$)

Numerical results: formation of shocks

- What about the **accuracy of spectra**?

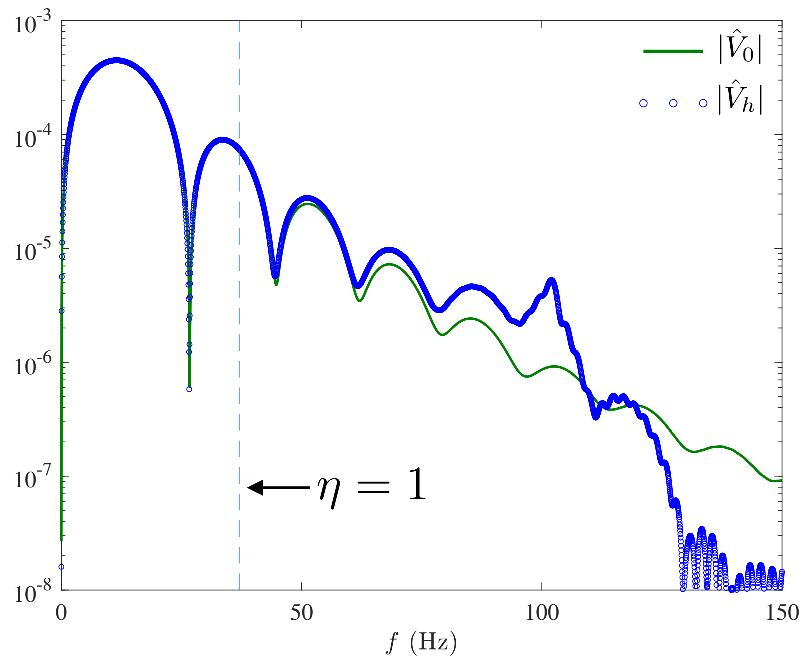


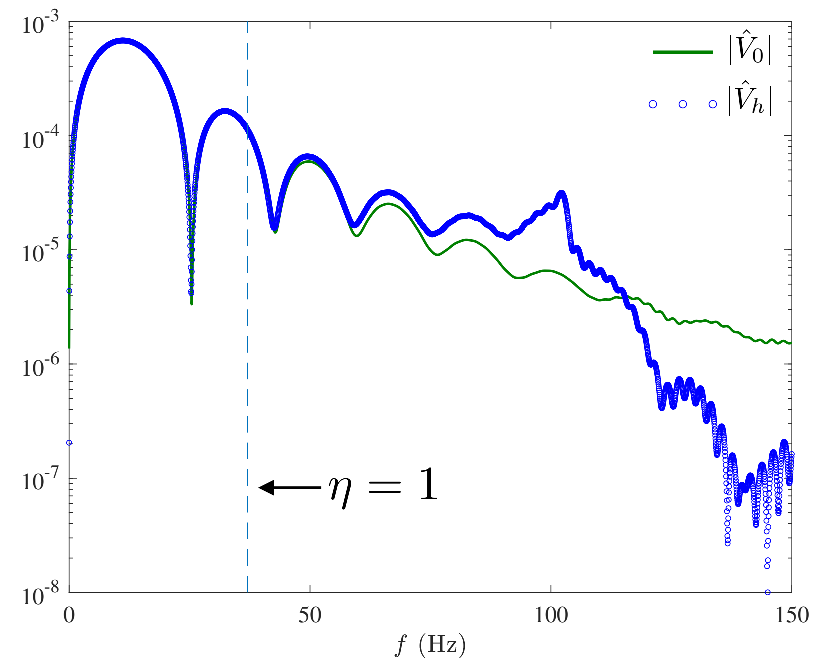
$A = 40$

$A = 60$

spectra
before the shock

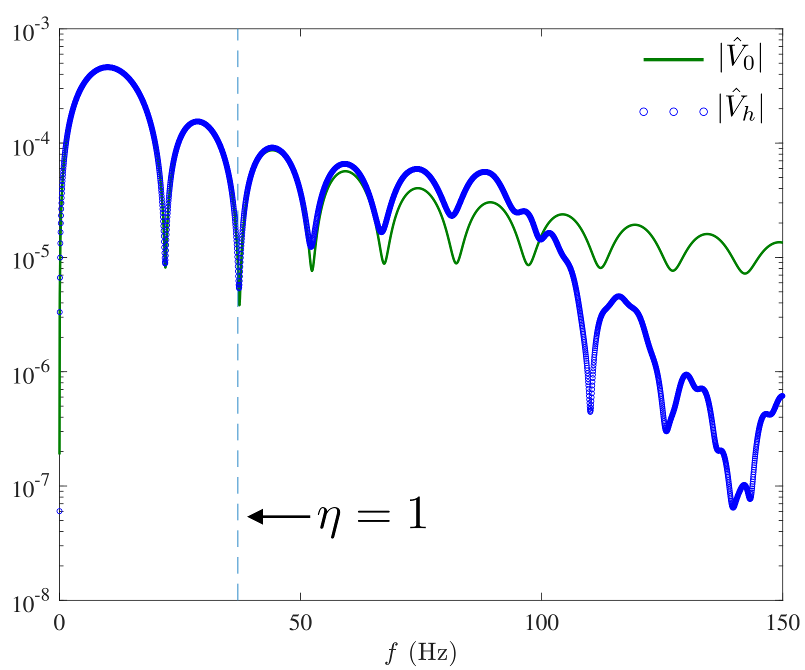
X_{r_1}

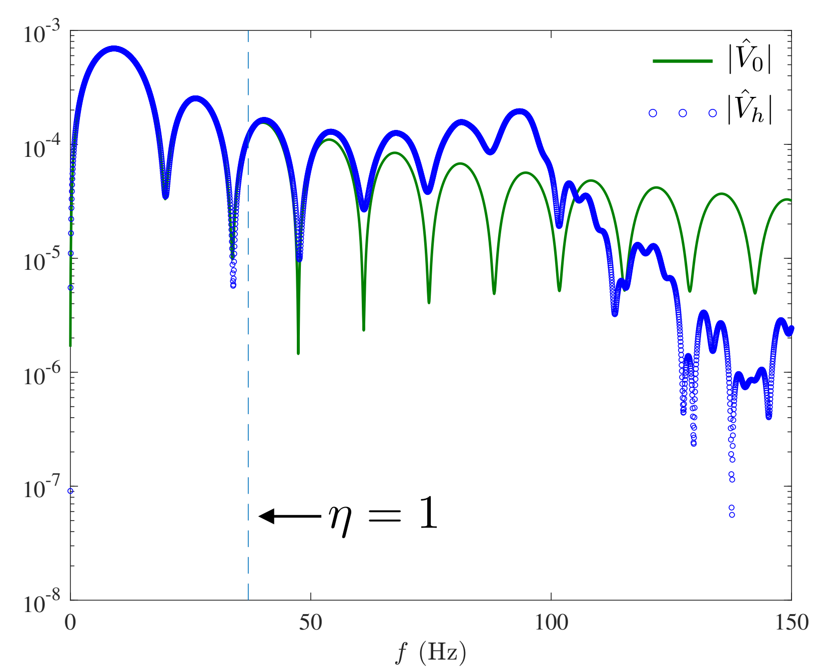




spectra
after the shock

X_{r_2}





Conclusions and perspectives

- **1st-order approximate model** accurate for short times, with an estimated upper bound
- **Zeroth-order effective model is non-linear** hence does not prevent **shocks** to occur
- **Dispersive effects** develop with time and source amplitude **in the microstructure** but they are **not captured in approximate model**

[CB, B. Lombard, M. Touboul, R. Assier, JMPS 2021]

- Derivation of a homogenized model accounting for **dispersive effects** (second-order model ?)
- Perform asymptotics for **larger source amplitudes**
- Handle **non-periodic** (random) arrays of interfaces
- Derivation of an effective model in **2D or 3D**?

Effective dynamics for elastic waves in an array of non-linear interfaces

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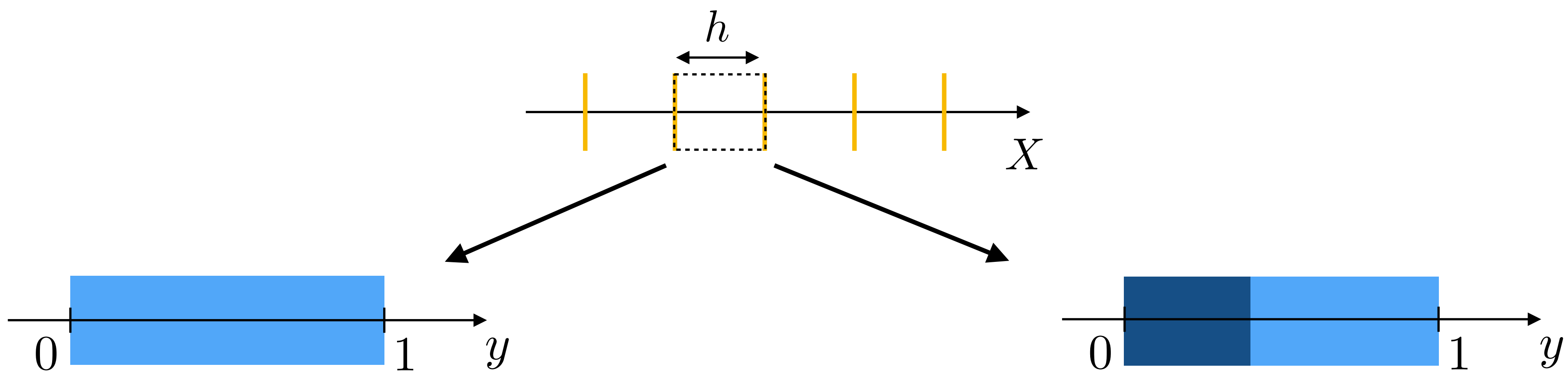
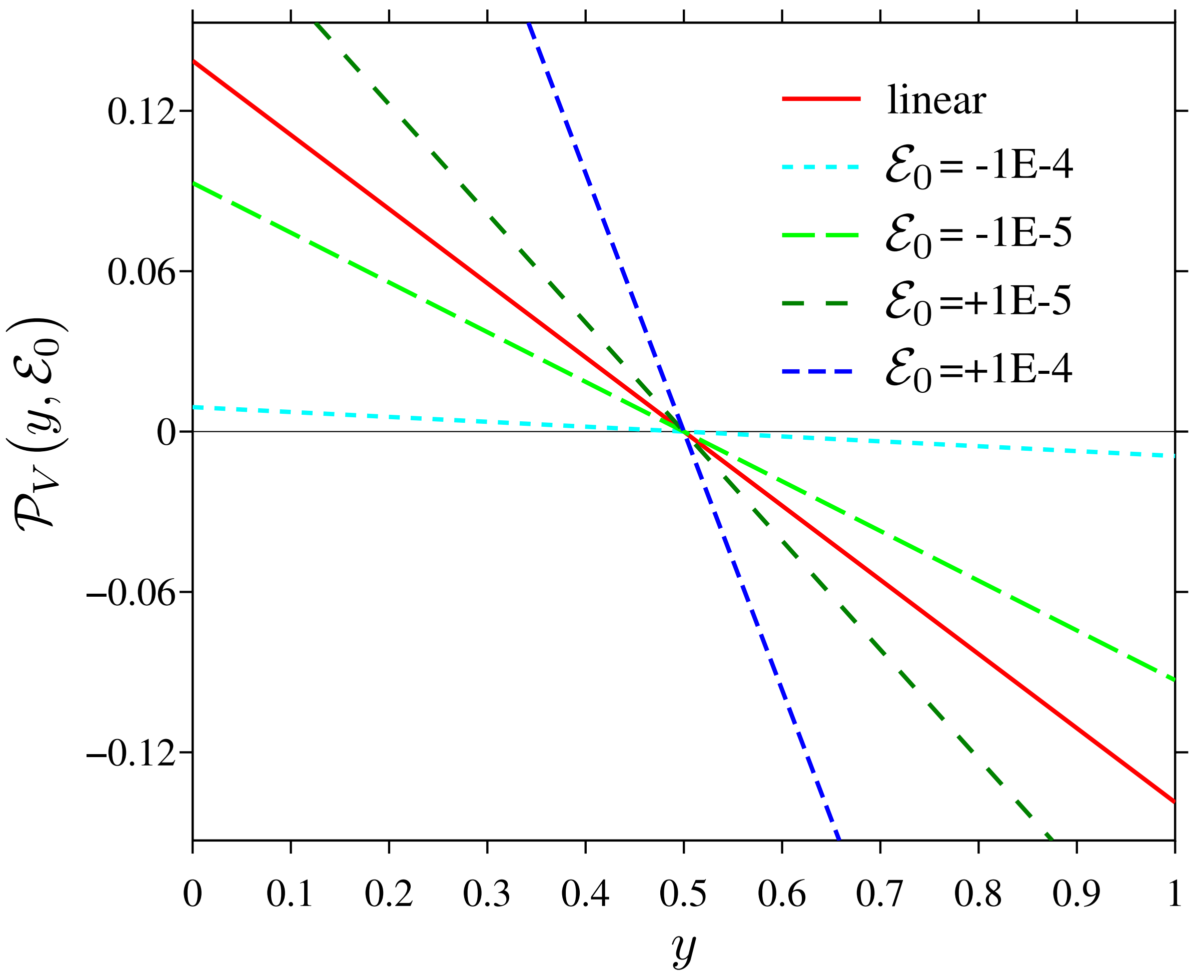
Thank you for your attention

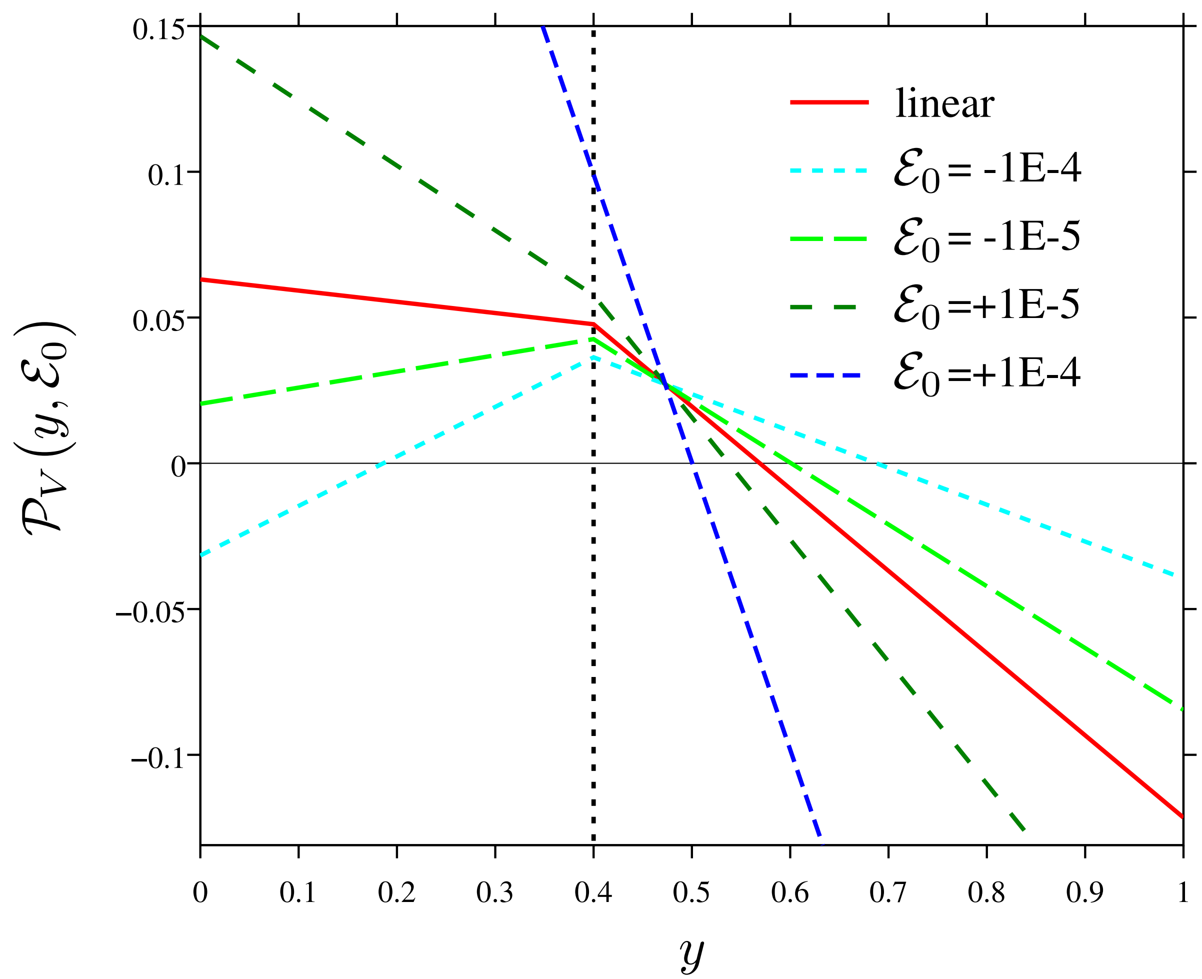
Cell functions

$$V^{(1)}(X,t) = \frac{\partial U^{(1)}}{\partial t}(X,t) = V_0(X,t) + h\overline{V}_1(X,t) + h\,\mathcal{P}_V\big(y, \mathcal{E}_0(X,t)\big)\,\frac{\partial V_0}{\partial X}(X,t)$$

$V_0 = \partial U_0/\partial t$ $\qquad$ $\overline{V}_1 = \partial \overline{U}_1/\partial t$







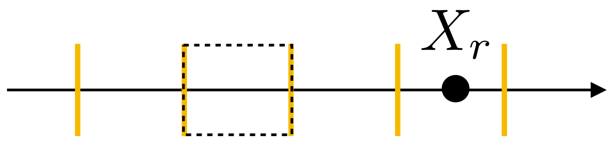
Numerical results: harmonics

Objectives: comparisons of full-field simulations (velocity-based) V_h

with homogenized solutions, i.e. V_0 and $V^{(1)}(X,t) = V_0(X,t) + hV_1(X,t)$

- What about the **accuracy of the harmonics**?

→ monochromatic source $(f_c = 10\,\mathrm{Hz})$ and spectrum of recorded signal



Amplitude of the normalized Fourier coefficients