Topology optimization for designing structures with optimal load-bearing capacity

Jérémy Bleyer
coll. Leyla Mourad, Romain Mesnil, Karam Sab

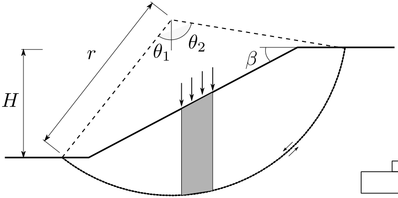
Workshop GDR MePhy : From Computational Fabrication to Material Design

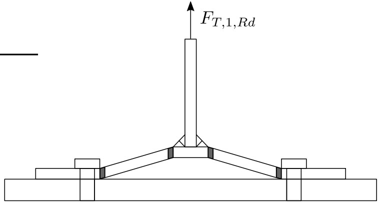
June, 22$^{\text{nd}}$ 2021

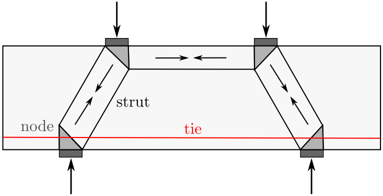
Outline

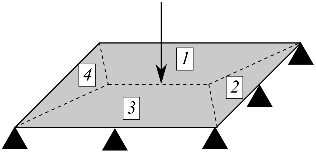
Limit analysis-based design methods

Ultimate Limit State design methods (Eurocodes)









slip/yield lines, strut and ties: **hand-based** limit analysis methods for ULS

Automated limit analysis in civil engineering

finite element computations in CE: global models are **elastic**, **very few nonlinear computations** (cost, lack of robustness, time and expertise)

Avantages of limit analysis

- situations outside the scope of design norms
- improve verification: critical yield mechanisms, bracketing the limit load
- simple input parameters: geometry, loading, strength properties
- robustness of numerical computations

Fondation Louis Vuitton

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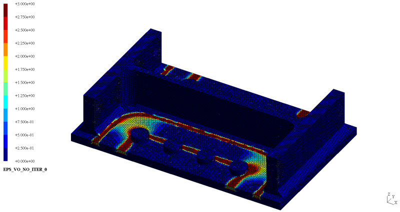

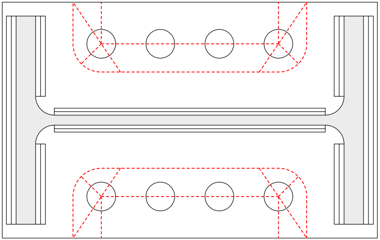

Th. Chadi EL BOUSTANI

Outline

Formulation : two bracketing approaches

Lower bound static approach

$$\begin{aligned}\lambda^+ \geq \lambda_s = \max_{\lambda, \boldsymbol{\sigma} \in \mathcal{W}_h} \quad & \lambda \\ \text{s.t.} \quad & \boldsymbol{\sigma} \text{ in equilibrium with } \lambda \boldsymbol{F} \\ & \boldsymbol{\sigma}(\boldsymbol{x}) \in G(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega\end{aligned}$$

G: **strength criterion** (convex set)

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Dual approach with virtual work principle: for $\boldsymbol{\sigma} \in G$ and in equilibrium

$$W_{ext}(\boldsymbol{u}) = \int_\Omega \boldsymbol{\sigma} \cdot \varepsilon[\boldsymbol{u}] \, \mathrm{dx} \leq \int_\Omega \sup_{\boldsymbol{\sigma} \in G} \{\boldsymbol{\sigma} \cdot \varepsilon[\boldsymbol{u}]\} \, \mathrm{dx} = W_{mr}(\boldsymbol{u})$$

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Upper bound kinematic approach

$$\begin{aligned} \lambda^+ \leq \lambda_k = \min_{\boldsymbol{u} \in \mathcal{V}_h} \quad & W_{mr}(\boldsymbol{u}) = \int_\Omega \pi(\boldsymbol{\varepsilon}[\boldsymbol{u}]) \,\mathrm{dx} \qquad \text{(maximum resisting work)} \\ \text{s.t.} \quad & W_{ext}(\boldsymbol{u}) = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{u} \,\mathrm{dx} = 1 \end{aligned}$$

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convex optimization problems

Numerical aspects of limit analysis problems

Variational approach: minimizing a convex, non-smooth, large-scale functional
Challenge : **numerical optimization** solver efficiency

`fenics_optim` package: Bleyer, TOMS, 2020; Bleyer and Hassen, Comp & Struct, 2021

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Numerical tools

- **conic programming** solvers — mosek http://www.mosek.com
- **non traditional** FE (stress-based, discontinuous interpolations)
- **automated formulation** using FEniCS http://fenicsproject.org

$$\begin{array}{rl} \max & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathcal{K}^1 \times \ldots \times \mathcal{K}^p \end{array}$$

where $\mathcal{K}^j$ are **cones** e.g. :

- positive orthant : $x_i \geq 0 \Rightarrow$ LP
- Lorentz second-order ("ice-cream") cone : $\|\bar{\boldsymbol{x}}\| \leq x_0 \Rightarrow$ SOCP
- cone of positive semi-definite matrix $\boldsymbol{X} \succeq 0 \Rightarrow$ SDP

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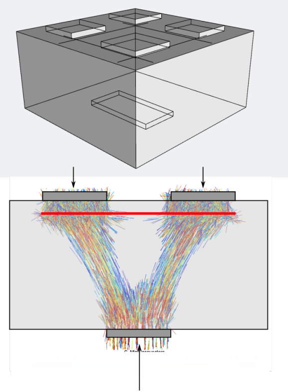
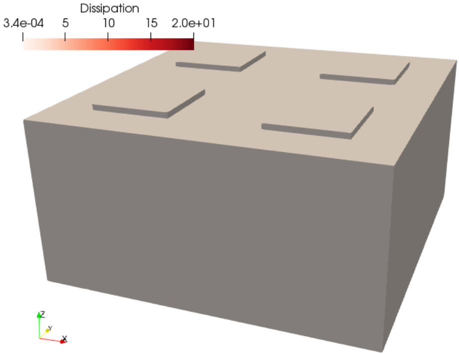
3D strength criterion for concrete and SDP

concrete = **Rankine** 3D criterion :

$$-f_c \leq \sigma_{I,II,III} \leq f_t \Leftrightarrow \begin{cases} \boldsymbol{\Sigma} \preceq f_t \mathbf{I} \\ -f_c \mathbf{I} \preceq \boldsymbol{\Sigma} \end{cases}$$

semi-definite positive constraints





Vincent, Arquier, Bleyer, de Buhan, Int. J. Num. Anal. Meth. Geomech., 2018

Outline

Topology optimization in limit analysis

Load-bearing capacity maximization

$$
\begin{aligned}
\lambda^+(\eta) = \max_{\lambda,\boldsymbol{\sigma},\rho} \quad & \lambda \\
\text{s.t.} \quad & \boldsymbol{\sigma} \text{ in eq. with } \lambda \boldsymbol{F} \\
& \boldsymbol{\sigma} \in \rho G \\
& \int_{\mathcal{D}} \rho \, \mathrm{dx} \leq \eta |\mathcal{D}| \\
& 0 \leq \rho \leq 1
\end{aligned}
$$

Mourad et al., Topology optimization of load-bearing capacity, *Struct Multidisc Optim*, 2021

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Volume minimization

$$\eta^-(\lambda) = \min_{\boldsymbol{\sigma},\rho} \quad \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} \rho \,\mathrm{dx} \quad \text{s.t.} \quad \begin{array}{l} \boldsymbol{\sigma} \text{ in eq. with } \lambda \boldsymbol{F} \\ \boldsymbol{\sigma} \in \rho G \\ 0 \leq \rho \leq 1 \end{array}$$

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Results

- both problems are **convex** and **equivalent**
- link with **optimal truss theory** [Michell, 1904]

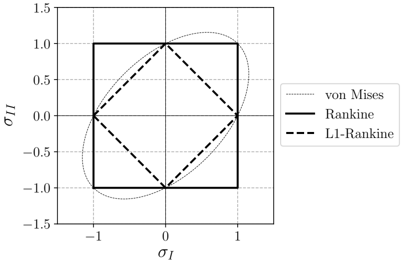
Numerical resolution

- **same tools** (conic programming)
- can adapt some strategies e.g. SIMP, filtering

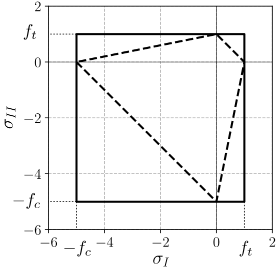
Mourad et al., Topology optimization of load-bearing capacity, *Struct Multidisc Optim*, 2021

Strength criterion choice

steel: von Mises, **concrete**: Rankine/Mohr-Coulomb



(a) symmetric strengths $f_t = f_c = 1$



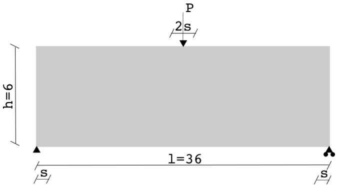
(b) asymmetric strengths $f_c = 5$, $f_t = 1$

In fact...

want to favor unixial stress states ("trusses") ⇒ L1-Rankine

$$\sum_{J=I,II} \max\left\{-\frac{\sigma_J}{f_c}; \frac{\sigma_J}{f_t}\right\} \leq 1$$

Simply supported beam

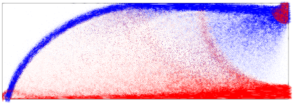


(a) L_1-Rankine

(b) Rankine

(c) von Mises

non penalized, $\eta = 20\%$ imposed



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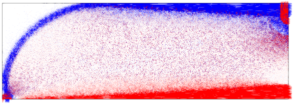
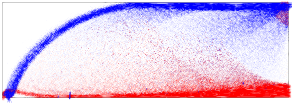


(b) Rankine

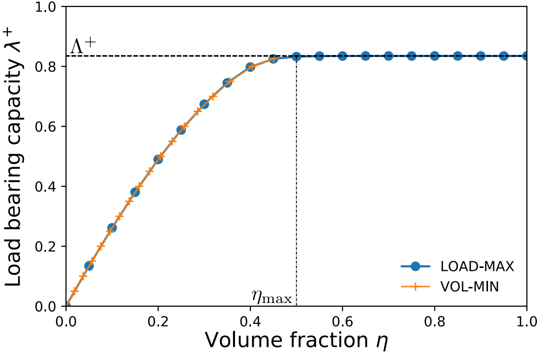


(c) von Mises

principal stresses (**tension**/**compression**)

Simply supported beam

Λ^+ = classical load-bearing capacity



- VOL-MIN = LOAD-MAX
- can save up to 50% of the volume

Solutions after penalization

(a) L_1-Rankine

(b) Rankine

(c) von Mises

penalized LOAD-MAX

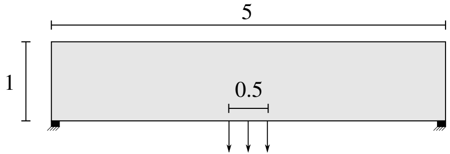
(a) L_1-Rankine

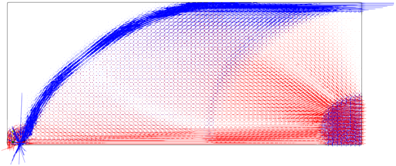
(b) Rankine

(c) von Mises

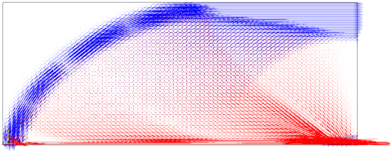
penalized VOL-MIN

Bridge : asymmetric strengths





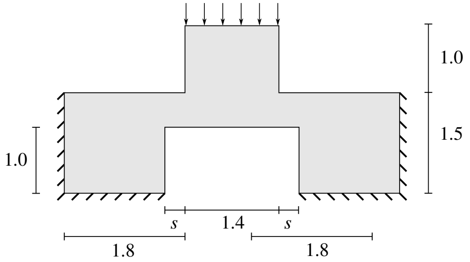
(a) $f_c/f_t = 10$

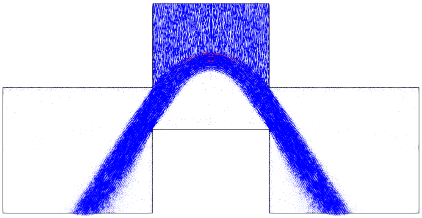


(b) $f_c/f_t = 0.1$

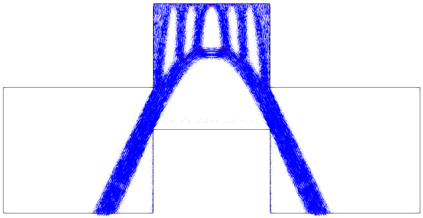
different topologies for different strength ratios

Material without tensile strength





(a) Initial unpenalized solution



(b) Final penalized solution

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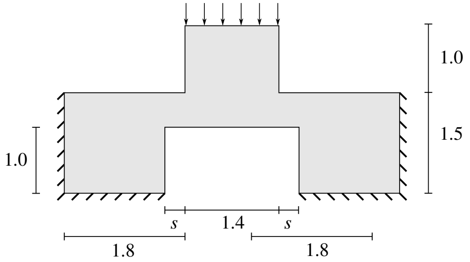

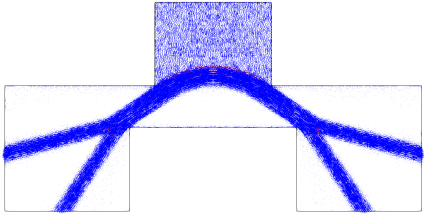

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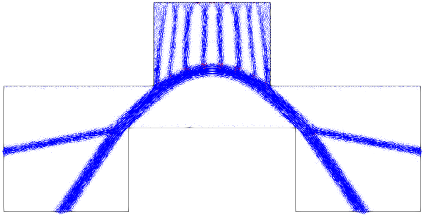

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Extension to a bi-material (e.g. reinforced concrete)

$$\min_{\rho_c,\rho_s,\boldsymbol{\sigma}_c,\boldsymbol{\sigma}_s} \int_\Omega (\rho_c + \rho_s)\,\mathrm{dx}$$
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_c + \boldsymbol{\sigma}_s \text{ in equilibrium}$$
$$-\rho_c f_c \preceq \boldsymbol{\sigma}_c \preceq 0$$
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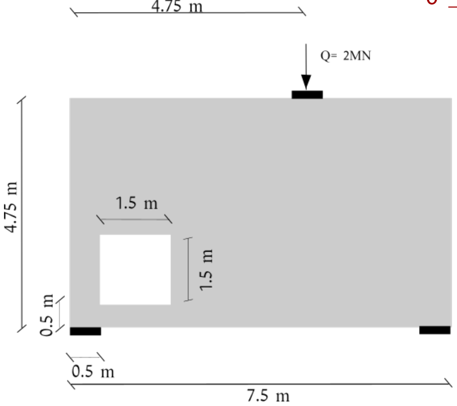

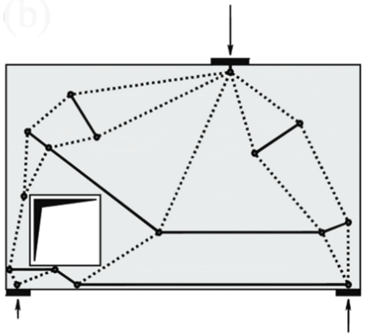

[Muttoni et al., Aci Structural Journal, 2015]

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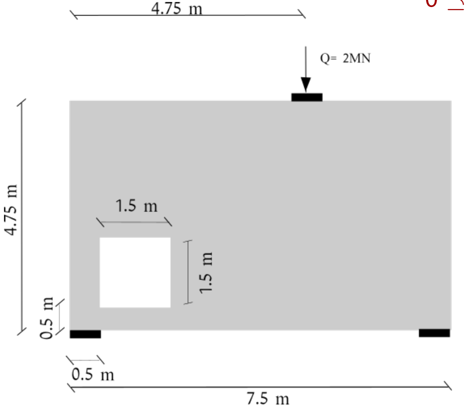

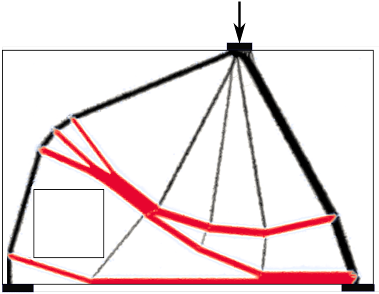

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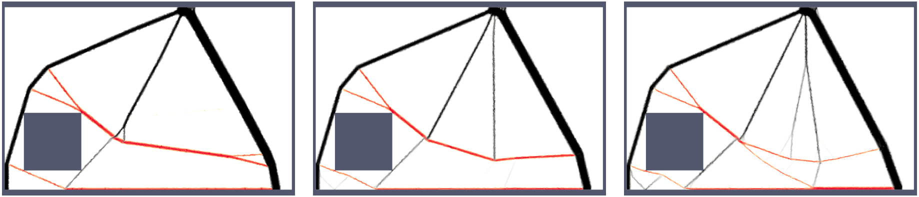
Extension to a bi-material (e.g. reinforced concrete)

Generalization to a **weighted** cost function

$$\begin{aligned} \min_{\rho_c,\rho_s,\boldsymbol{\sigma}_c,\boldsymbol{\sigma}_s} \quad & \int_\Omega (1-w_s)\rho_c + w_s\rho_s)\,\mathrm{dx} \\ & \boldsymbol{\sigma} = \boldsymbol{\sigma}_c + \boldsymbol{\sigma}_s \text{ in equilibrium} \\ & -\rho_c f_c \preceq \boldsymbol{\sigma}_c \preceq 0 \\ & 0 \preceq \boldsymbol{\sigma}_s \preceq \rho_s f_y \end{aligned}$$

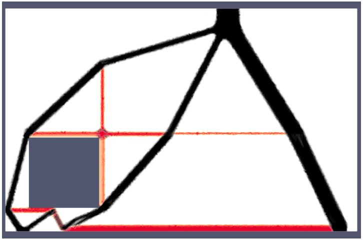


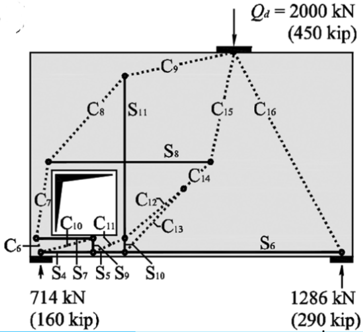
$w_s = 0.25$ $\qquad$ $w_s = 0.5$ $\qquad$ $w_s = 0.75$

Orthogonal steel layout

Further imposing zero shear for steel:

$$\begin{aligned} \min_{\rho_c,\rho_s,\boldsymbol{\sigma}_c,\boldsymbol{\sigma}_s} \quad & \int_\Omega \left((1-w_s)\rho_c + w_s\rho_s\right) \mathrm{d}x \\ & \boldsymbol{\sigma} = \boldsymbol{\sigma}_c + \boldsymbol{\sigma}_s \text{ in equilibrium} \\ & \sigma_{s,xy} = 0 \\ & -\rho_c f_c \preceq \boldsymbol{\sigma}_c \preceq 0 \\ & 0 \preceq \boldsymbol{\sigma}_s \preceq \rho_s f_y \end{aligned}$$





Conclusions & Perspectives

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- efficient numerical procedure
- liberty in strength criterion choice
- extension to bi-materials, automated strut-and-tie models

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- specimen realization and testing
- 3D computations
- robust optimization wrt loading uncertainty

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