Fiber-Bundle models in fracture

A tentative overview

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The Fiber Bundle Model
Modeling Failure in Materials
Complexity in fracture

Elastic response, elastic properties

\[ E_{global} \cong \text{mean}(E_{local}) \]
Complexity in fracture

Failure, strength in materials

\[ \sigma_{f\ global} \not\approx \text{mean}(\sigma_{f\ local}) \]
Complexity in fracture

causes: stress field distortion by defects and microcracks
Complexity in fracture

causes: stress field distortion by defects and microcracks

- Driven by rare & extreme defects, macroscale sensitivity to microscale inhomogeneities,
- Non-trivial statistics & size effects for strength
- Unpredictability at the macroscale

- Not really compatible with continuum engineering mechanics
Complexity in fracture: illustrations (1/2)

Extreme event statistics on strength [Here Duxbury-Leath law on ceramics components]

Van der Born et al. 91

Non-trivial size dependency on strength [Here on concrete]

C.C. Vu et al. 2018
Complexity in fracture: illustrations (2/2)

X-ray tomography imaging of damage in rocks  
*Renard et al. 2018*

Acoustic emission in compressed wood  
*Makinen et al. 2015*

Highly intermittent dynamics  
Damage localization in space

**Sismicity in California, 2012**  
*Bares. 2013*
Complexity in fracture: illustrations (2/2)

X-ray tomography imaging of damage in rocks  \textit{Renard et al. 2018}

Acoustic emission in compressed wood

Highly intermittent dynamics
Damage localization in space
Scale-free avalanches of damage
Power-law distributed size/energy
Fiber Bundle Models (FBM): the approach

Material simplified to:

- N parallel brittle fibers
- Same stiffness $\kappa = 1$
- Random failure threshold
- Prescribed rules for load redistribution

F.T. Pierce, 1926
H.E. Daniels, 1945
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Load transfer decreases with distance from failed fiber

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ELS -- FBM: continuum (average) behavior

- $N \to \infty$ parallel brittle fibers
- Overall stretching force $F$, $\Rightarrow$ stress $\sigma = F/N$
- Same stiffness $\kappa = 1$
- Same fiber extension: $x$
- Random critical loads $x_i$ with distribution $p(x)$

[cumulative $P(x)$]

Fully tractable analytically
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- Same stiffness \( \kappa = 1 \)
- Same fiber extension: \( x \)
- Random critical loads \( x_i \) with distribution \( p(x) \) [cumulative \( P(x) \)]

Fully tractable analytically
Displacement-driven “experiment”, at displacement \( x \):

Number of intact fibers: \( n(x) = N \left(1 - P(x)\right) \)

Density: \( \rho = n/N : \rho(x) = 1 - P(x) \)
ELS -- FBM: continuum (average) behavior

- $N \to \infty$ parallel brittle fibers
- Overall stretching force $F \Rightarrow$ stress $\sigma = F / N$
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Number of intact fibers: $n(x) = N\left(1 - P(x)\right)$

$\Rightarrow$ Density: $\rho = n/N : \rho(x) = 1 - P(x)$

Stretching force $F(x) = xn(x)$

$\Rightarrow$ Applied stress $\sigma(x) = x \left(1 - P(x)\right)$
ELS -- FBM: continuum (average) behavior

- \( N \rightarrow \infty \) parallel brittle fibers
- Overall stretching force \( F \), => stress \( \sigma = F / N \)
- Same stiffness \( \kappa = 1 \)
- Same fiber extension: \( x \)
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Failure at the maximum point:

- Strength \( \sigma_c = \sigma(x_c) \), so that \( \sigma'(x_c) = 0 \)
- Critical damage: \( \rho_c = \rho(x_c) \)

Uniform distribution \( P(x) = x \)
ELS -- FBM: continuum (average) behavior

- \( N \to \infty \) parallel brittle fibers

Displacement-driven "experiment", at displacement \( x \):

Number of intact fibers: \( n(x) = N(1 - P(x)) \)

\( \Rightarrow \) Density: \( \rho = n/N : \rho(x) = 1 - P(x) \)

Stretching force \( F(x) = xn(x) \)

\( \Rightarrow \) Applied stress \( \sigma(x) = x (1 - P(x)) \)

Failure at the maximum point:

\( \Rightarrow \) Strength \( \sigma_c = \sigma(x_c) \), so that \( \sigma'(x_c) = 0 \)

\( \Rightarrow \) Critical damage: \( \rho_c = \rho(x_c) \)

\[ \begin{align*}
\text{Control parameter} & : \sigma \\
\text{Order parameter} & : \rho - \rho_c \\
\rho - \rho_c & \sim (\sigma_c - \sigma)^{1/2} \\
\frac{d\rho}{d\sigma} & \sim (\sigma_c - \sigma)^{-1/2} \\
\text{Universal, independent of } P(x) \\
\end{align*} \]
ELS -- FBM: size effect and fluctuations

- Finite $N$ parallel brittle fibers
- Overall stretching force $F$, $\Rightarrow$ stress $\sigma = F/N$
- Same stiffness $\kappa = 1$
- Same fiber extension: $x$
- Random critical loads $x_i$ with distribution $p(x)$

Uniform distribution $P(x) = x, N = 20$

$pdf(\sigma|x, N)$ known analytically (roughly Gaussian)

$Daniels, Prc. R. Soc. London A, 1945$

- $\langle \sigma_N \rangle(x) = x\left(1 - P(x)\right)$
- $\langle (\sigma_N - \langle \sigma_N \rangle)^2 \rangle(x) = x^2 P(x)\left(1 - P(x)\right)/N$
ELS -- FBM: size effect and fluctuations

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Overall stretching force $F$, $\Rightarrow$ stress $\sigma = F / N$
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$\langle \sigma_N \rangle(x) = x(1 - P(x))$
$\langle (\sigma_N - \langle \sigma_N \rangle)^2 \rangle(x) = x^2 P(x)(1 - P(x))/N$

(Tiny) size effect on strength

$\langle \sigma_{max} | N \rangle = \sigma_c + K/N^{2/3}$
ELS -- FBM: avalanches precursors

- $N$ parallel brittle fibers
- Overall stretching force $F$, $\Rightarrow$ stress $\sigma = F/N$
- Same stiffness $\kappa = 1$
- Same fiber extension: $x$
- Random critical loads $x_i$ with distribution $p(x)$
  [cumulative $P(x)$]

Displacement imposed experiment $\Rightarrow$ monotonic $\sigma(x)$

Uniform distribution $P(x) = x, N = 100$
ELS -- FBM: avalanches precursors

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- Overall stretching force $F$, $\Rightarrow$ stress $\sigma = F/N$
- Same stiffness $\kappa = 1$
- Same fiber extension: $x$
- Random critical loads $x_i$ with distribution $p(x)$
  [cumulative $P(x)$]

Displacement imposed experiment $\Rightarrow$ monotonic $\sigma(x)$

Avalanche size $S = \text{number of fibers involved in microfailure}$
ELS -- FBM: avalanches precursors

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Displacement imposed experiment $\Rightarrow$ monotonic $\sigma(x)$

Avalanche size $S =$ number of fibers involved in microfailure

Size distribution fully computable analytically

Hansen & Hemmer 1992, 1994

at failure: $P(S|\sigma_c) \sim S^{-3/2}$
before failure: $P(S|\sigma) \sim S^{-3/2} \exp(-S(\sigma - \sigma_c))$

integrated $P_{int}(S) \sim S^{-5/2}$

Interpretable via random walk
ELS -- FBM: avalanches precursors

Uniform distribution $P(x) = x$, $N = 100$

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Interpretable via random walk
**ELS -- FBM: avalanches precursors**

**Qualitatively compatible with experiments**

Energy distribution of Earthquakes

Acoustic emission in compressed wood

*Makinen et al, 2015*

Energy (A.U.)

Size distribution fully computable analytically

*Hansen & Hemmer 1992, 1994*

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Interpretable via random walk
ELS -- FBM: more on avalanches precursors

Experimental observable, energy release by avalanche

\[ E = \frac{1}{2} S x^2 \]

Energy distribution fully determined analytically

*Pradhan et al, Rev. Mod. Phys. 2010*

High \( E \): universal

\[ P_{int}(E) \sim E^{-5/2} \]

20000 sample, \( N=10^6 \), from Pradhan et al. 2010
ELS -- FBM: more on avalanches precursors

Experimental observable, energy release by avalanche

$$E = \frac{1}{2} S x^2$$

Energy distribution fully determined analytically

Pradhan et al, Rev. Mod. Phys. 2010

High $E$: universal

$$P_{int}(E) \sim E^{-5/2}$$

Low $E$: non-universal

$$P_{int}(E) \approx \frac{Np(\sqrt{2E})}{\sqrt{2E}}$$

(a) Uniform $P(x)$

(b) Weibull $P(x)$, $k=5$
• $N$ parallel brittle fibers
• Overall constant stress $\sigma_0 = F_0/N < \sigma_c$
• Same stiffness $\kappa = 1$
• Same fiber extension $x$
• Random critical loads $x_i$, pdf Gaussian $pdf$, variance $T_d$
• Thermal fluctuations via a new noise term $\eta(t)$, variance $T$
ELS -- FBM: thermal noise & creep

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Analytical determination of failure rate during creep due the constant stress loading

Omori law

$$\frac{dn}{dt} \approx \frac{A\sqrt{T/T_d}}{t^p}$$

... with temperature/stress dependent Omori exponent

$$p \approx 1 + \frac{T}{T_d} - K(\sigma_c - \sigma_0) \frac{\sqrt{T}}{2T_d}$$
ELS -- FBM: thermal noise & creep

\[ N \text{ parallel brittle fibers} \]
\[ \text{Overall constant stress } \sigma_0 = \frac{F_0}{N} < \sigma_c \]
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\[ \text{Random critical loads } x_i, \text{ pdf Gaussian } pdf, \text{ variance } T_d \]
\[ \text{Thermal fluctuations via a new noise term } \eta(t), \text{ variance } T \]

 פעולה זו מיקнской את התוצאות הממורכבות של התא קרישה ב- e-glass bundle instrumented in acoustic

Omori law
\[ \frac{dn}{dt} \approx \frac{A \sqrt{T/T_d}}{t^p} \]

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Prescribed rules for load redistribution

**Equal load Sharing (ELS)**
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LLS -- FBM: 1D models

- $N$ parallel brittle fibers
- Overall stretching force $F$, $\Rightarrow$ stress $\sigma = F/N$
- Same stiffness $\kappa = 1$
- Random critical loads $x_i^c$, pdf $p(x^c)$, cumulative $P(x^c)$
- Fiber extension: $x_i = \sigma(1 + k_i/2)$
  
  $k_i$ = nb. of failed nearest-neighbor fibers

Full analytical solutions not available anymore
LLS -- FBM: 1D models

- $N$ parallel brittle fibers
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👉 Full analytical solutions not available anymore

Pdf of hole size after $k$ breaking

Uniform: $P(x) = x \ N = 1000, k = 200$

_Hansen et al. (Wiley & son, 2015)_

😊 Localization captured
LLS -- FBM: 1D models

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Localization captured

highly dependent of $P(x)$…
... importance: $x_{min}^c$ and $P(x \approx x_{min}^c)$

- $x_{min}^c = 0, P(x) = x^\beta, \beta \rightarrow \infty$
  infinite disorder, no localization, ELS
- $x_{min}^c = 2/3$, uniform above,
  fully localized, single crack growth
LLS -- FBM: 1D models, size effect on strength

- \( N \) parallel brittle fibers
- Overall stretching force \( F \), \( \Rightarrow \) stress \( \sigma = F/N \)
- Same stiffness \( \kappa = 1 \)
- Fiber extension: \( x_i = \sigma(1 + k_i/2) \)
  - \( k_i \)=nb. of failed nearest-neighbor fibers
- Random critical loads \( x_i^c \) with distribution \( p(x^c) \)
  - [cumulative \( P(x^c) \)]

Size effect can be reproduced
But depend of \( P(x) \), \( x_{\text{min}}^c \) and \( P(x \approx x_{\text{min}}^c) \)

Case of uniform distribution, \( P(x) = x \)

Numerics, simple estimates: \( \sigma_c \approx 1/\log(N) \)

More refined: \( \sigma_c \approx \log(\log(N^2))/\log(N^2) \)

\[ \text{Hansen et al. (Wiley & son, 2015)} \]
LLS -- FBM: 1D models, avalanche distribution

- $N$ parallel brittle fibers
- Overall stretching force $F$, $\Rightarrow$ stress $\sigma = F/N$
- Same stiffness $\kappa = 1$
- Fiber extension: $x_i = \sigma(1 + k_i/2)$
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- Random critical loads $x_i^c$ with distribution $p(x^c)$
  [cumulative $P(x^c)$]

Non-universal size-distributions, Depend on critical load distribution, $P(x)$


\[ P(x) = x^\beta, \ 0 \leq x \leq 1 \]

\[ P(x) = 1 - \exp(-x^k), \ 0 \leq x < \infty \]

Hansen et al. (Wiley & son, 2015)
LLS – FBM in higher dimensions

Localization less important in 2d

⇒ finite $x_{min}^c$ now required to activate localization
**LLS – FBM in higher dimensions**

Localization less important in 2d

➤ finite $x_{\text{min}}^c$ now required to activate localization

Stress-strain curve in LLS for uniform distribution, $P(x^c) = x^c$, over $[0,1]$ 

*From Hansen et al. (Wiley & son, 2015)*

When dimensionality increases, LLS-FBM looks more and more to ELS-FBM!
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An example of FLS -- FBM: Soft-clamp model

*Stormo, Gjerden, Hansen* PRE, 2012

- $N$ parallel brittle fibers, Overall force $F$, $\Rightarrow$ stress $\sigma = F / N$
- Same stiffness $\kappa = 1$
- Random critical loads $x_i^c$, pdf $p(x^c)$, cumulative $P(x^c)$
- Forces between fibers redistributed via the elastic clamp

\[
\sigma_i = \kappa (x_i - x), \quad x_i = \sum_j G_{ij} \sigma_j, \\
G_{ij} = \text{green function for 2D elastic infinite half space}
\]
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$$\sigma_i = \kappa (x_i - x), x_i = \sum_j G_{ij}\sigma_j, \quad G_{ij} = \text{green function for 2D elastic infinite half space}$$

Control parameter for interaction: $e = E / N$

$\Rightarrow$ Increasing size $\equiv$ softer clamp

Transition from LLS (small $e$) to ELS (large $e$)

$e_{\text{stiff}} = 2^6$  $e_{\text{soft}} = 2^{-17}$

From Hansen et al. (Wiley & son, 2015)
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Transition from LLS (small $e$) to ELS (large $e$)

Size effect on strength at small $e$, absence of size effect at large $e$

From Stormo et al. PRE, 2012
FBM: Soft-clamp model & experiments

Gjerden, Stormo, Hansen PRL, 2013

Interfacial cracks

Maloy et al. PRL 2006
FBM: Soft-clamp model & experiments

Gjerden, Stormo, Hansen PRL, 2013

Interfacial cracks
Maloy et al. PRL 2006

Small scale, multi-affine, $\zeta \sim 0.6$

Large scale, self affine, $\zeta \sim 0.35$
FBM: Soft-clamp model & experiments

Gjerden, Stormo, Hansen PRL, 2013

Interfacial cracks
Maloy et al. PRL 2006

Large $N$ (soft $e$)
Self-affine (single) growing front

Small $N$ (stiff $e$)
Fractal percolated clusters

→ In (experimental) fracture: ELS/percolation at small scales, LLS/single crack growth at large scales
FBM: few take-home messages

- FBMs address fracture problems via models of increasing complexity/description ability depending on the redistribution rules

- **Simplest one: Equal-load-Sharing FBM:**
  - Fully tractable analytically, fracture $\leftrightarrow$ critical phenomena
  - Capture qualitatively gradual damaging before failure, scale-free precursor avalanches, response in creep,
  - Ail capturing localization and size effects

- **Local-load-Sharing FBM**
  - Not tractable numerically anymore
  - Capture localization and size effects
  - No universality, very dependent on threshold distribution (and its minimum)
  - As dimension increases, looks more and more to ELS

- **Soft-clamp FBM**
  - LLS at large scales, ELS at small scales
  - Explain 2 scaling regimes in fracture depending on length-scales
The Fiber Bundle Model

Modeling Failure in Materials

To know more

Thank you for your attention