Application of averaging technique for the numerical simulation of highly oscillatory problems

Florian Méhats

IRMAR, Univ. Rennes 1, INRIA MINGUS team

Joint work with P. Chartier, M. Lemou and G. Vilmart

"Space-Time Multiscale Methods", Paris, June 5, 2018

Highly-oscillatory problems with periodic time-dependence

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We consider a highly oscillatory problem in a (functional) Banach space X:

$$(\mathcal{P}_\varepsilon) \qquad \begin{aligned} \frac{d}{dt}u^\varepsilon(t) &= f_{t/\varepsilon}\left(u^\varepsilon(t)\right), \quad u^\varepsilon(t) \in X, \\ u^\varepsilon(0) &= u_0 \in X, \end{aligned}$$

where

- ε is a possibly small parameter (scales as the inverse of a frequency).
- $(\theta, u) \mapsto f_\theta(u)$ is given, smooth and P-periodic with respect to θ.

Numerical difficulties: Standard schemes lead to $\|u^\varepsilon - u^{\varepsilon,\Delta t}\| \leq C\frac{(\Delta t)^p}{\varepsilon^q}$, $q > 0$, forcing $\Delta t \leq \varepsilon$ and thus formidable costs for small values of ε.

Partial remedy: Averaging methods lead to $\|u^\varepsilon - u^{\varepsilon,\Delta t}\| \leq C((\Delta t)^p + \varepsilon^q)$.

Aim: Construct a numerical method which is uniformly accurate with respect to ε, i.e. such that $\|u^\varepsilon - u^{\varepsilon,\Delta t}\| \leq C(\Delta t)^p$ with C independent of ε.

Examples

Our examples are of the form

$$\frac{d}{dt}y^\varepsilon = \frac{1}{\varepsilon}Ay^\varepsilon + f(y^\varepsilon)$$

where $\theta \mapsto e^{A\theta}$ is periodic with respect to θ.

To filter out the highly oscillatory dynamics, we introduce the unknown

$$u^\varepsilon(t) = e^{-At/\varepsilon}y^\varepsilon(t)$$

which satisfies

$$\frac{d}{dt}u^\varepsilon = e^{-At/\varepsilon}f\left(e^{At/\varepsilon}u^\varepsilon(t)\right) = f_{t/\varepsilon}(u^\varepsilon(t))$$

Here we have

$$f_\theta(u) = e^{-A\theta}f\left(e^{A\theta}u^\varepsilon(t)\right)$$

Examples (ii)

- Transport in graphene in semiclassical regime
- Schrödinger equation and Gross-Pitaevskii equation

$$i\partial_t \psi^\varepsilon = -\frac{1}{\varepsilon}\Delta\psi^\varepsilon + \alpha|\psi^\varepsilon|^2\psi^\varepsilon \qquad \text{with periodic boundary conditions}$$

$$i\partial_t \psi^\varepsilon = -\frac{1}{\varepsilon}\Delta\psi^\varepsilon + \frac{\omega}{\varepsilon}|x|^2\psi^\varepsilon + \alpha|\psi^\varepsilon|^2\psi^\varepsilon \qquad \text{on } \mathbb{R}^d$$

- The nonlinear Klein-Gordon equation in the nonrelativistic regime

$$\varepsilon\partial_{tt}u^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon}u^\varepsilon + f(u^\varepsilon) = 0, \qquad x \in \mathbb{R}^d, \quad t > 0,$$

- Vlasov equation in a strong magnetic field

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + E \cdot \nabla_v f^\varepsilon + \frac{v \times B}{\varepsilon} \cdot \nabla_v f^\varepsilon = 0$$

- non relativistic limit of nonlinear Dirac equation

Averaging result (case of the so-called stroboscopic averaging)

Averaging methods assert that for all $k \in \mathbb{N}^*$ there exist

1. a P-periodic change of variables

$$(\tau, u) \in \mathbb{T} \times X \mapsto \Phi_\tau^{[k]}(u) \in X \qquad \text{with } \Phi_0^{[k]}(u) = u$$

2. a smooth, autonomous vector field $F^{[k]}$ and its flow-map $\Psi_t^{[k]}$

   - $u \in X \mapsto F^{[k]}(u) \in X$
   - $(t, u) \mapsto \Psi_t^{[k]}(u) \in X \qquad \dfrac{d}{dt}\Psi_t^{[k]}(u) = F^{[k]}(\Psi_t^{[k]}(u)), \quad \Psi_0^{[k]}(u) = u$

such that

$$\forall t \in [0, T], \quad \left\| u^\varepsilon(t) - \Phi_{\frac{t}{\varepsilon}}^{[k]} \circ \Psi_t^{[k]}(u_0) \right\|_X \leq C\varepsilon^{k+1}.$$

$\Phi_{\frac{t}{\varepsilon}}^{[k]}$: high-oscillations; $\Psi_t^{[k]}$: slow drift

ref.: Bogoliubov-Mitropolsky 1930', Perko 1968, ...
and their adaptation to PDEs: JFOCM 15, Castella-Chartier-M.-Murua

Averaging: an assessment

The standard use of averaging method consist in the simulation of

$$\frac{d}{dt}\Psi_t^{[k]} = F^{[k]}(\Psi_t^{[k]})$$

complemented with the computation of $\Phi_\theta^{[k]}$.

Pros of averaging

- models are non-stiff and do not suffer from severe constraints on the time step when ε is small
- preserve part or all geometric structures

Cons

- methods based on the (numerical or not) computation of the averaged vector field $F^{[k]}$ lead to an incompressible error term $\mathcal{O}(\varepsilon^{k+1})$ owing to the truncation of the series
- for values of ε away from 0 one needs to include many terms in the expansion leading to important costs

New approach: averaging with corrections (i)

Our approach, in the micro/macro spirit, consists in adding an equation to the averaged equation. The unknown is split as follows:

$$u^\varepsilon(t) = \Phi^{[k]}_{t/\varepsilon}\left(\Psi^{[k]}_t\right) + w^\varepsilon(t)$$

where $\Psi^{[k]}_t$ solves the macro (averaged) equation

$$\frac{d}{dt}\Psi^{[k]}_t = F^{[k]}(\Psi^{[k]}_t), \qquad \Psi^{[k]}_0 = u_0.$$

The remainder w^ε solves the micro equation

$$\frac{d}{dt}w^\varepsilon(t) = f_{t/\varepsilon}\left(\Phi^{[k]}_{t/\varepsilon}\left(\Psi^{[k]}_t\right) + w^\varepsilon(t)\right) - \left(\frac{1}{\varepsilon}\partial_\theta \Phi^{[k]}_{t/\varepsilon} - \partial_u \Phi^{[k]}_{t/\varepsilon} F^{[k]}\right)(\Psi^{[k]}_t),$$
$$w^\varepsilon(0) = 0.$$

Theorem

$$\forall \varepsilon \in]0,1], \quad \forall 0 \le s \le k+1, \quad \forall t \in [0,T], \quad \left\| \frac{d^s}{dt^s} w^\varepsilon \right\| \le C\varepsilon^{k+1-s}$$

Advantages of this method

1. It is clear that if Ψ^ε and w^ε satisfy the micro/macro equations, then $u^\varepsilon(t) = \Phi^{[k]}_{t/\varepsilon}\left(\Psi^\varepsilon_t(u_0)\right) + w^\varepsilon(t)$ satisfies the original equation

$$\frac{d}{dt}u^\varepsilon(t) = f_{t/\varepsilon}(u^\varepsilon(t)).$$

   In contrast with usual averaging, our micro/macro method is not an approximation and contains whole the information of the original problem.

2. The fact that w^ε has bounded time-derivatives with respect to ε allows to use standard numerical methods for the micro/macro system with uniform accuracy with respect to ε.

Take-away message:

A p-th order standard scheme is uniformly accurate of order p when applied to the micro-macro system provided $\Phi^{[p]}$ and $F^{[p]}$ are used.

Numerical tests for the Hénon-Heiles model

It is a Hamiltonian system, for the unknown (q_1, q_2, p_1, p_2) with

$$H(p,q) = \frac{p_1^2}{2\varepsilon} + \frac{p_2^2}{2} + \frac{q_1^2}{2\varepsilon} + \frac{q_2^2}{2} + q_1^2 q_2 - \frac{1}{3} q_2^3.$$

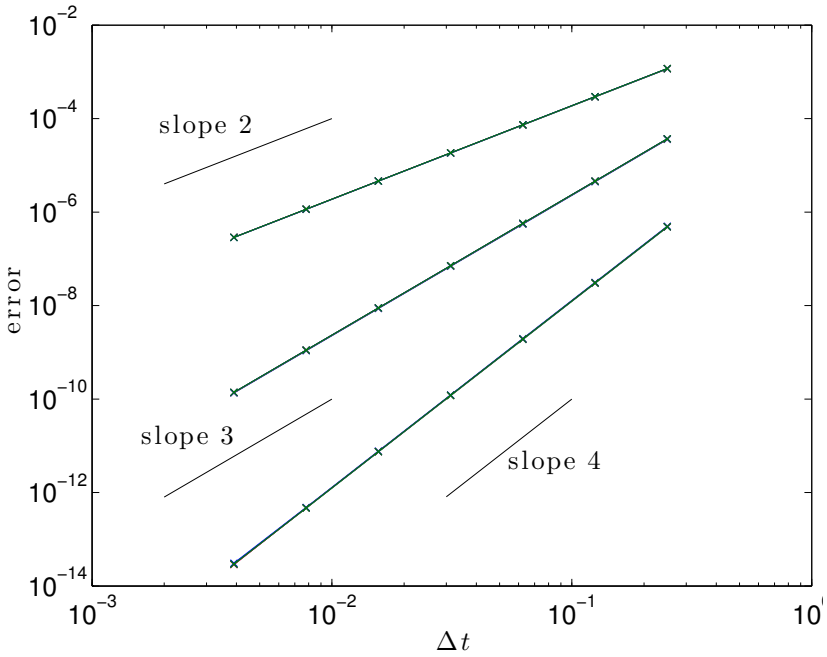
When ε is small, the variables $(q_1^\varepsilon, p_1^\varepsilon)$ is highly oscillatory:

$$\begin{cases} \dot{q}_1^\varepsilon = \dfrac{p_1^\varepsilon}{\varepsilon} \\ \dot{q}_2^\varepsilon = p_2^\varepsilon \\ \dot{p}_1^\varepsilon = -\dfrac{q_1^\varepsilon}{\varepsilon} - 2 q_1^\varepsilon q_2^\varepsilon \\ \dot{p}_2^\varepsilon = -q_2^\varepsilon - \left(q_1^\varepsilon\right)^2 + \left(q_2^\varepsilon\right)^2 \end{cases}$$

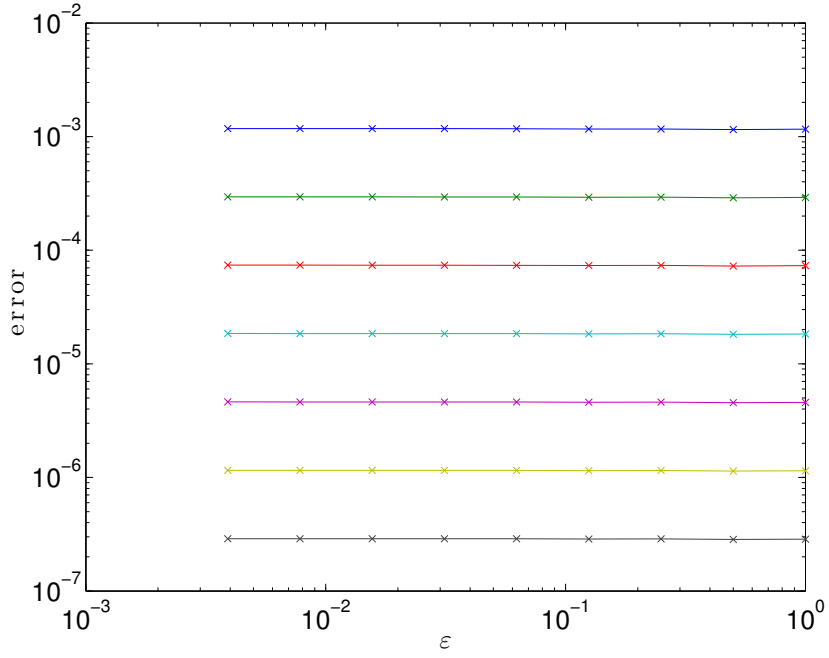
We test the following numerical methods:

- Our second order method: UA of order 2
- A third order method constructed with $\Phi^{[2]}$ and $F^{[2]}$ by extrapolation: UA of order 3
- A fourth order method constructed with $\Phi^{[3]}$ and $F^{[3]}$ by extrapolation: UA of order 4

Our UA schemes (orders 2, 3 and 4)



(a) Error with respect to Δt



(b) Error with respect to ε, scheme of order 2

Thank you for your attention