Coarse Graining Method for scaling

Application to Quasi Periodic beam lattices

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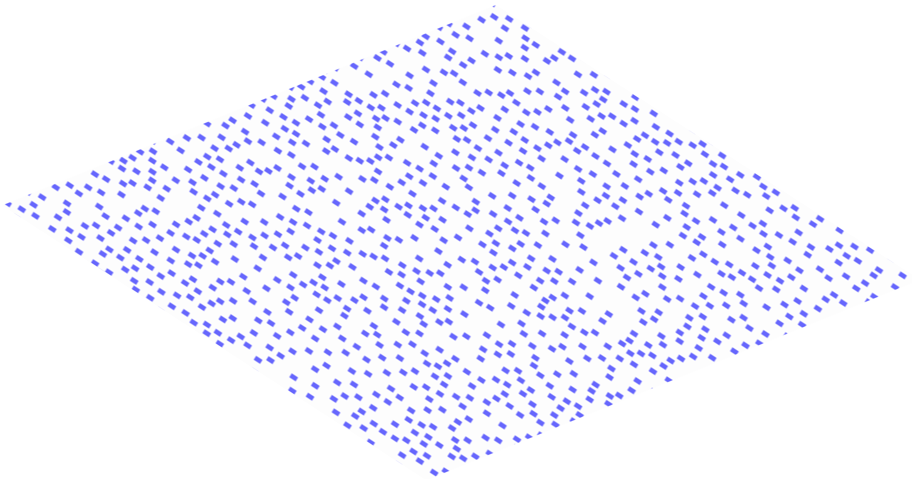
Coarse Graining Method :

Goldhirsch, I. (2010). Stress, stress asymmetry and couple stress: from discrete particles to continuous fields. *Granular Matter*,

Density definition :

$$\rho^{mic}(\mathbf{r}) = \sum_i m^i \delta(\mathbf{r} - \mathbf{r}_i)$$



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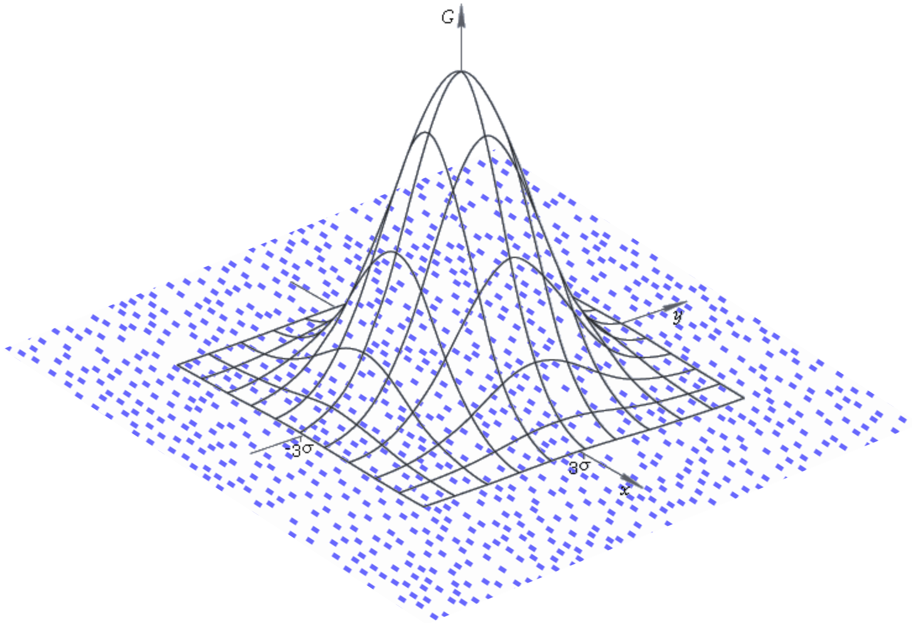

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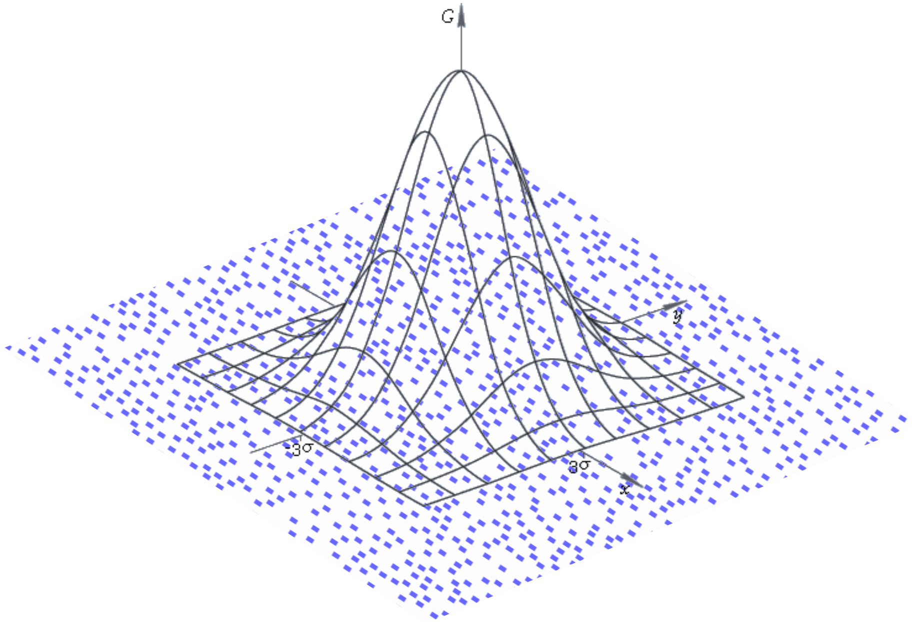
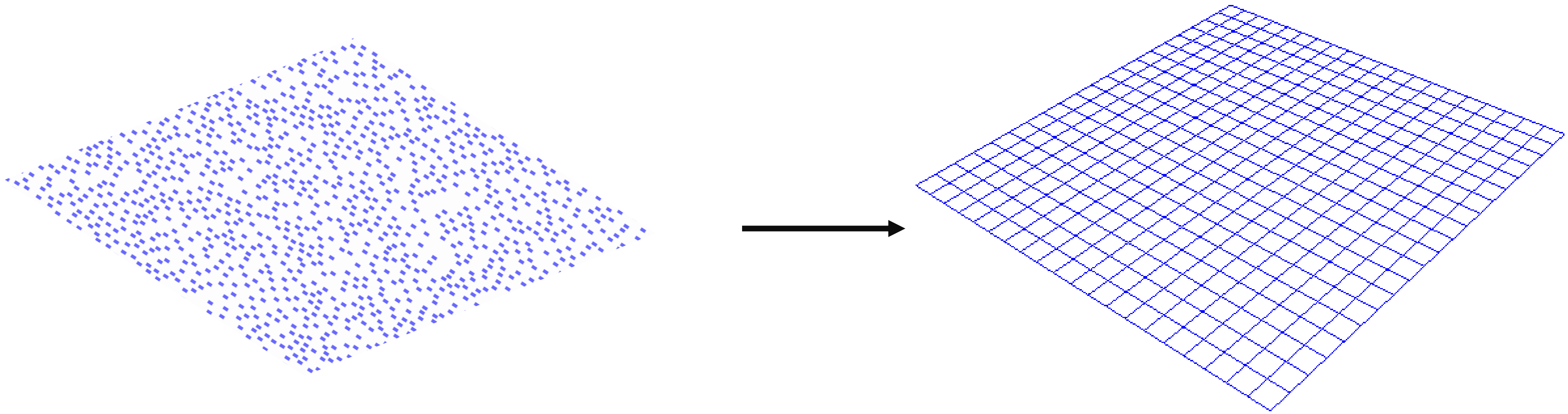
Mass Conservation :

$$div(\mathbf{p^{cg}}) = -\dot{\rho}_{cg}$$

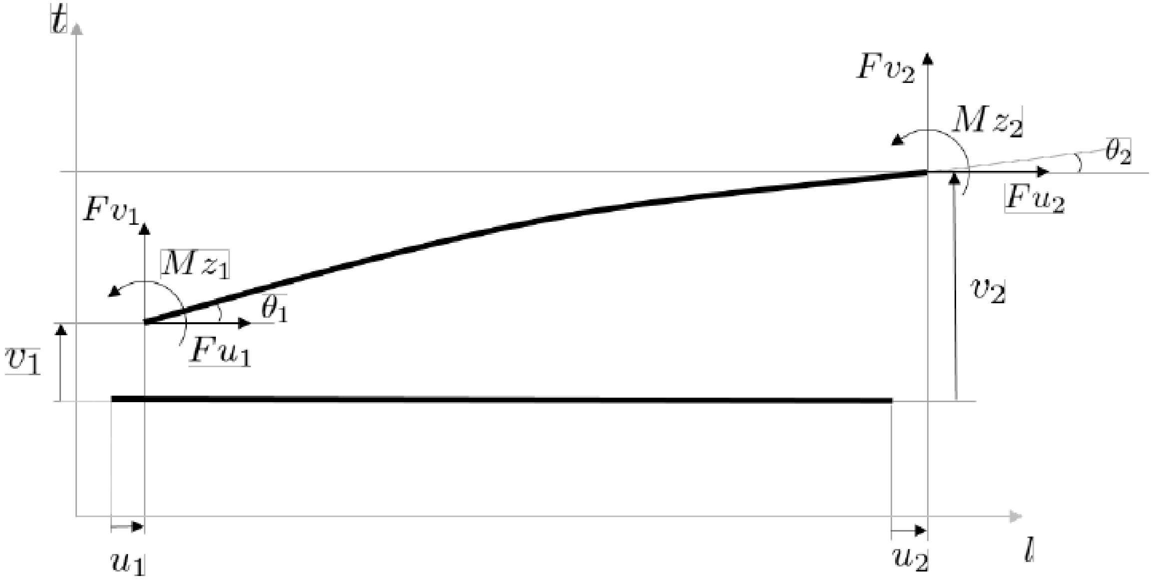


Momentum Definition:

$$p_\alpha^{cg}(\mathbf{r}) = \sum_i^N p_\alpha^i \Phi^i(\mathbf{r})$$



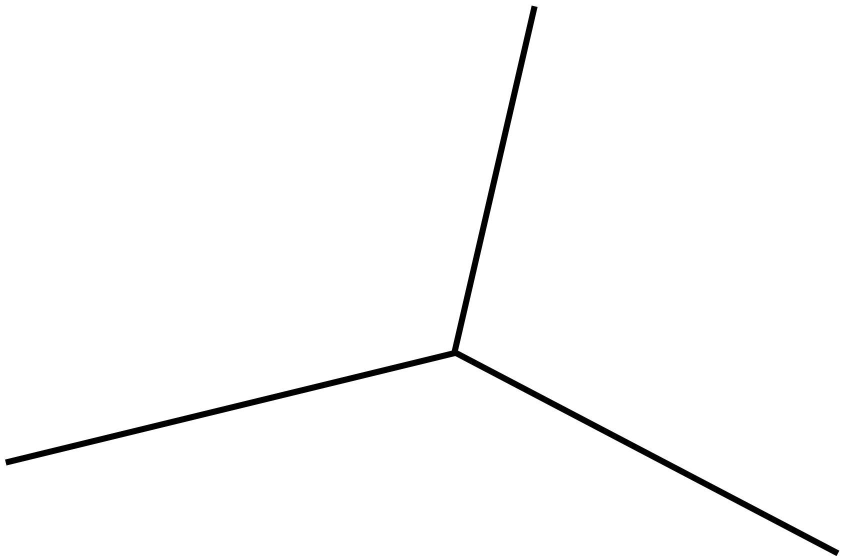
Beam Model :



$$\mathbf{q}_e = \{\ u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad \theta_2\ \}^t$$

$$\mathbf{F}_e = \{\ Fu_1 \quad Fv_1 \quad Mz_1 \quad Fu_2 \quad Fv_2 \quad Mz_2\ \}^t$$

$$\mathbf{K}_e\mathbf{q}_e + \mathbf{M}_e\ddot{\mathbf{q}}_e = \mathbf{F}_e$$



$$\mathbf{K}\mathbf{q} + \mathbf{M}\ddot{\mathbf{q}} = \mathbf{F}$$

$$f^i_\alpha = \sum_{j=1}^{N}\sum_{\gamma=1}^{n+m} K^{ij}_{\alpha\gamma} q^j_\gamma = -\sum_{j=1}^{N}\sum_{\gamma=1}^{n+m} M^{ij}_{\alpha\gamma}\ddot{q}^j_\gamma$$

$$p^i_\alpha = \sum_{j=1}^{N}\sum_{\gamma=1}^{n+m} M^{ij}_{\alpha\gamma}\dot{q}^j_\gamma = \dot{w}^i_\alpha$$

Coarse Graining Method :

Density definition :

$$\rho_{cg}(\mathbf{r}) = \sum_i m^i \Phi^i(\mathbf{r})$$

$$m^i = \frac{1}{n} \sum_j^N \sum_{\alpha=1}^n \sum_{\beta=1}^n M_{\alpha\beta}^{ij}$$

$$\Phi^i(\mathbf{r}) = \Phi(\mathbf{r} - \widehat{\mathbf{r}}^i) = \frac{1}{\sqrt{(2\pi)^n l_{cg}^n}} e^{-\frac{\|\mathbf{r}-\widehat{\mathbf{r}}^i\|^2}{2l_{cg}^2}}$$

$$\widehat{r}_\alpha^i = \sum_j^N \sum_{\gamma=1}^{n+m} \frac{M_{\alpha\gamma}^{ij}}{m^i}(r_\gamma^j + q_\gamma^j)$$

Mass Conservation :

$$div(\mathbf{p^{cg}}) = -\dot{\rho}_{cg}$$

→

Momentum Definition:

$$p_\alpha^{cg}(\mathbf{r}) = \sum_i^N p_\alpha^i \Phi^i(\mathbf{r})$$

$$p_\alpha^i = \sum_{j=1}^N \sum_{\gamma=1}^{n+m} M_{\alpha\gamma}^{ij} \dot{q}_\gamma^j = w_\alpha^i$$

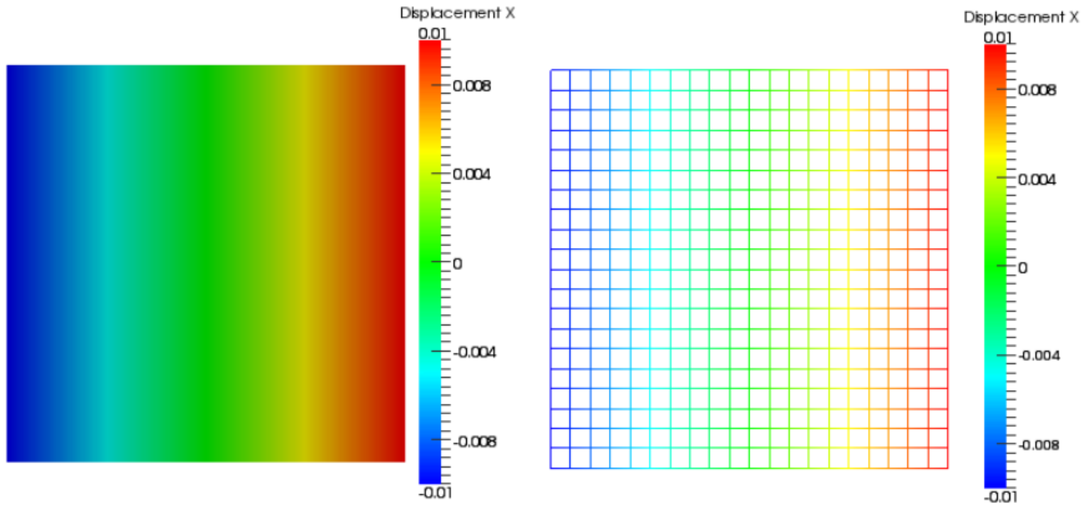
CG Displacements and strains:

$$v_\alpha^{cg} = \frac{p_\alpha^{cg}}{\rho_{cg}}$$

$$u_\alpha^{cg}(\mathbf{r},t) = \int_0^t v^{cg}(\mathbf{r},t')dt'$$

$$u_\alpha^{cg} = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\gamma=1}^{n+m} M_{\alpha\gamma}^{ij} q_\gamma^j \frac{\Phi^i}{\rho_{cg}}$$

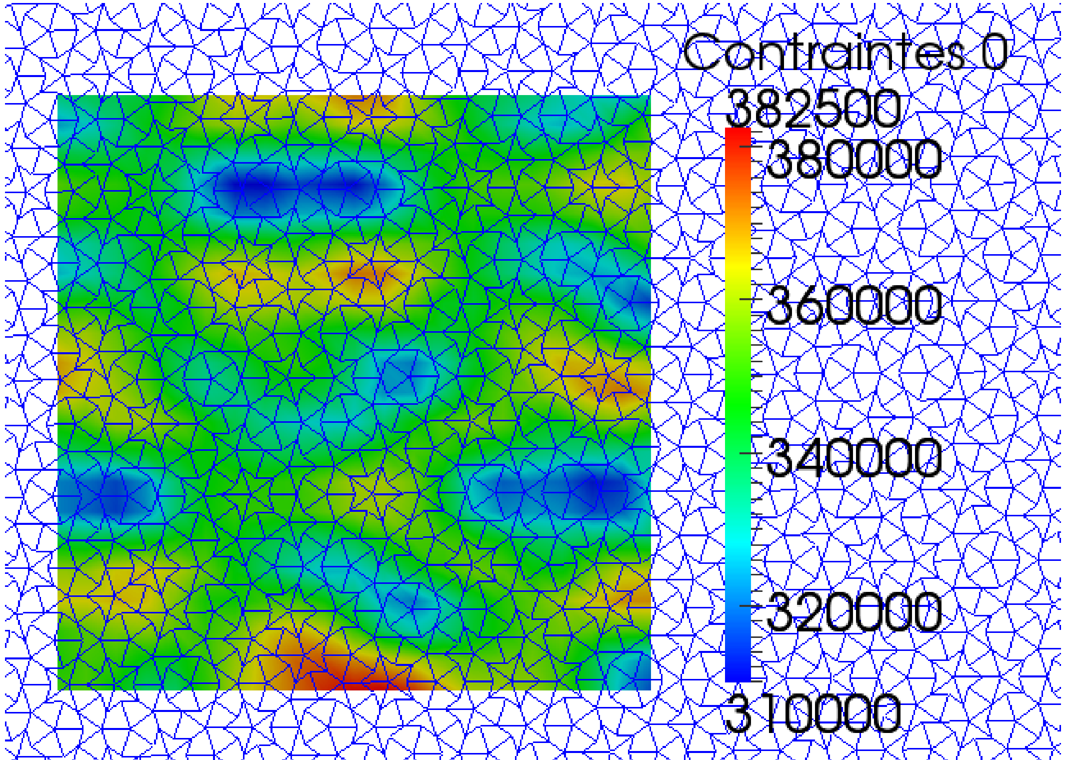
$$\frac{\partial u_\alpha^{cg}}{\partial r_\beta} = \frac{1}{\rho_{cg}^2}\sum_{ij}^{N}\left(w_\alpha^i m^j - w_\alpha^j m^i\right)\frac{\partial \Phi^i}{\partial r_\beta}\Phi^j$$



CG Stresses :

$$\frac{\partial p_\alpha^{cg}}{\partial t} = \sum_\beta \frac{\partial}{\partial r_\beta}\left(\sigma_{\alpha\beta} - \rho_{cg} V_\alpha^{cg} V_\beta^{cg}\right)$$

$$\sigma_{\alpha\beta} = \left(\sum_{ij\neq i}^{N} \sum_{\gamma=1}^{n+m} K_{\alpha\gamma}^{ij} q_\gamma^j \hat{r}_\beta^{ji} \int_0^1 \Phi(\mathbf{r} - \hat{\mathbf{r}}^i + s\hat{\mathbf{r}}^{ji})ds - \sum_{ij}^{N} \sum_{\gamma=n+1}^{n+m} K_{\alpha\gamma}^{ij} q_\gamma^j g_\beta^j\right) - \sum_i^N m^i \tilde{v}_\alpha^i \tilde{v}_\beta^i \Phi^i$$



CG Elastic modulus :

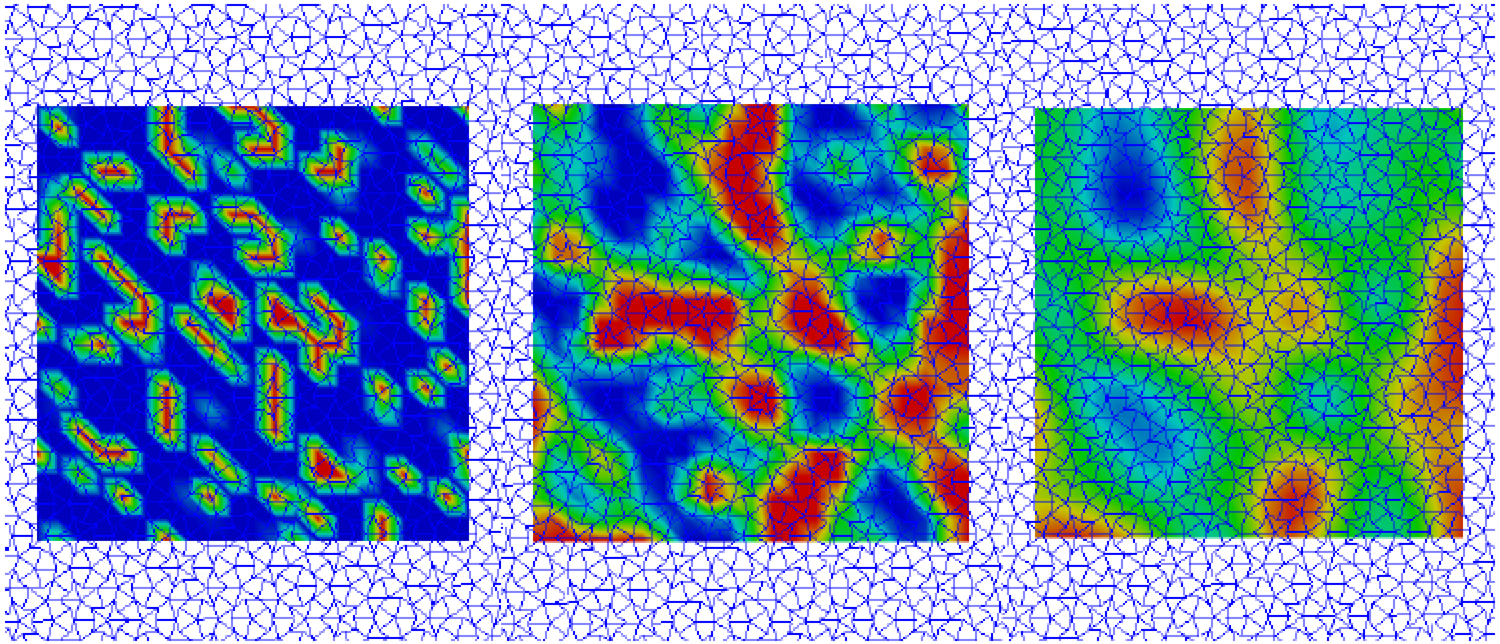
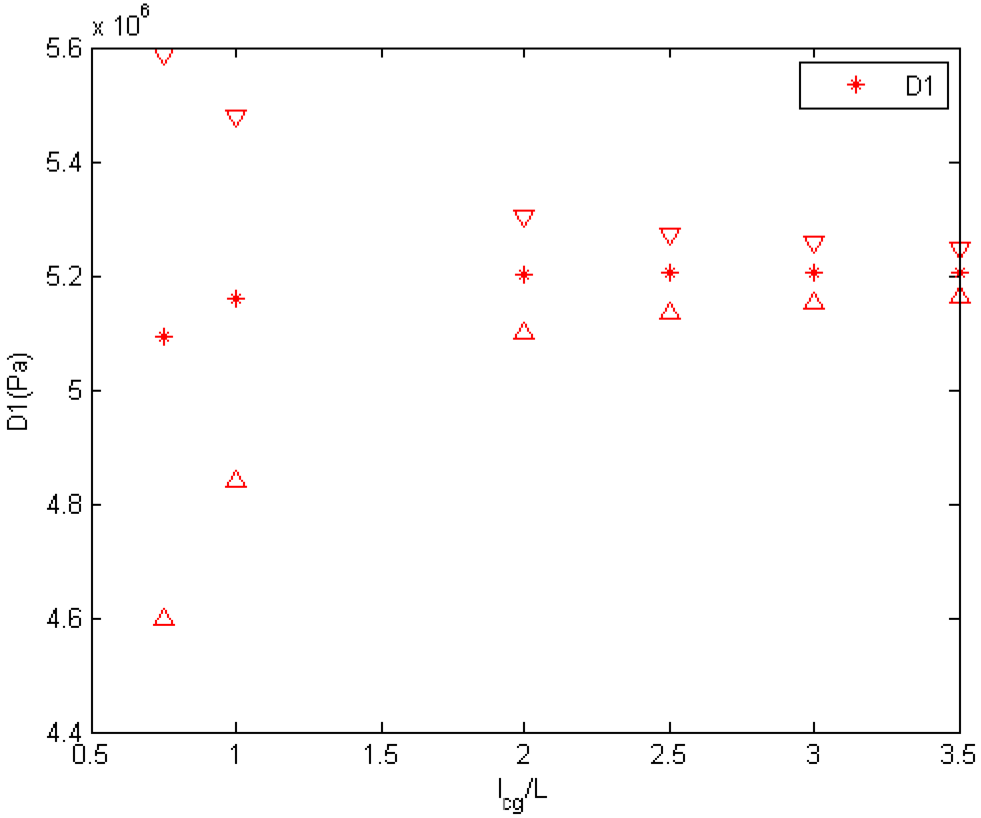


FIGURE 3.11 – spacial distribution of D1 in kyte & dart for different CG length (for left to right) 1L , 2.5L and 3.5L.



Antisymmetric stresses

$$\sum_{\alpha\beta} \varepsilon_{\alpha\beta}\sigma_{\alpha\beta} = -\frac{1}{2}\sum_{ij}^{N}\sum_{\gamma=1}^{n+m} \widehat{\mathbf{r}}^{ji}\times\mathbf{f}^{ij}\int_0^1 \Phi(\mathbf{r}-\widehat{\mathbf{r}}^i + s\widehat{\mathbf{r}}^{ji})ds$$

→ Cosserat medium

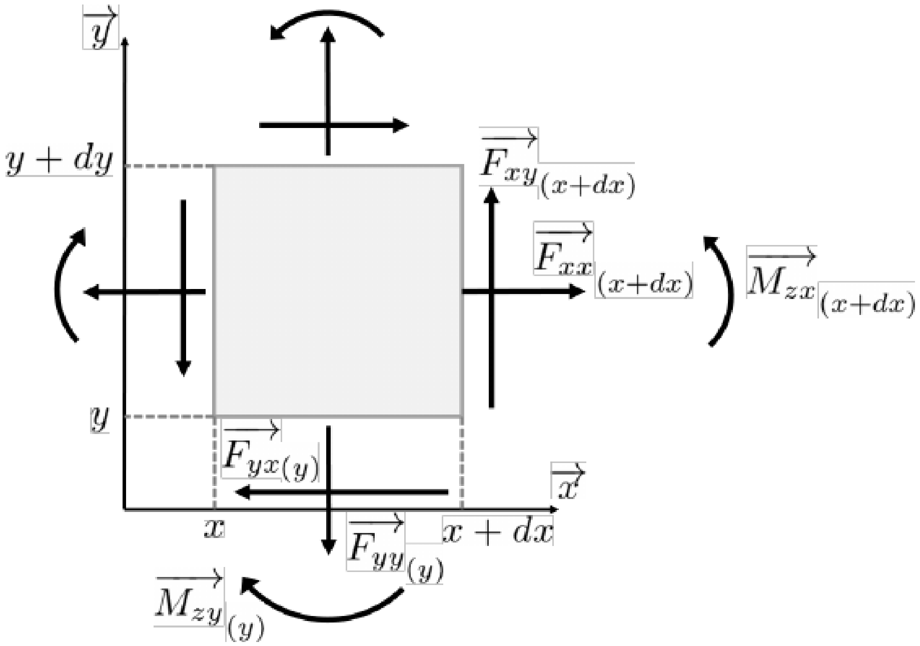


FIGURE 1.13 – Forces and torques definition.

→

$$\mathbf{div}(\boldsymbol{\sigma}) = \rho\ddot{\mathbf{u}}$$

$$div(\mathbf{m}) - (\sigma_{xy} - \sigma_{yx}) = I\ddot{\theta}$$

CG Rotation, Curvatures and Torques :

$$\theta_{\alpha}^{cg} = \frac{\sum_i \omega_{\alpha}^{i} \Phi_i}{I^{cg}}$$

$$k_{\alpha\beta}^{cg} = \frac{\partial \theta_{\alpha}^{cg}}{\partial r_{\beta}} = \frac{1}{I_{cg}^{2}} \sum_{ij}^{N} \left(\omega^{i} m^{j} - \omega^{j} m^{i} \right) \frac{\partial \Phi^{i}}{\partial r_{\beta}} \Phi^{j}$$

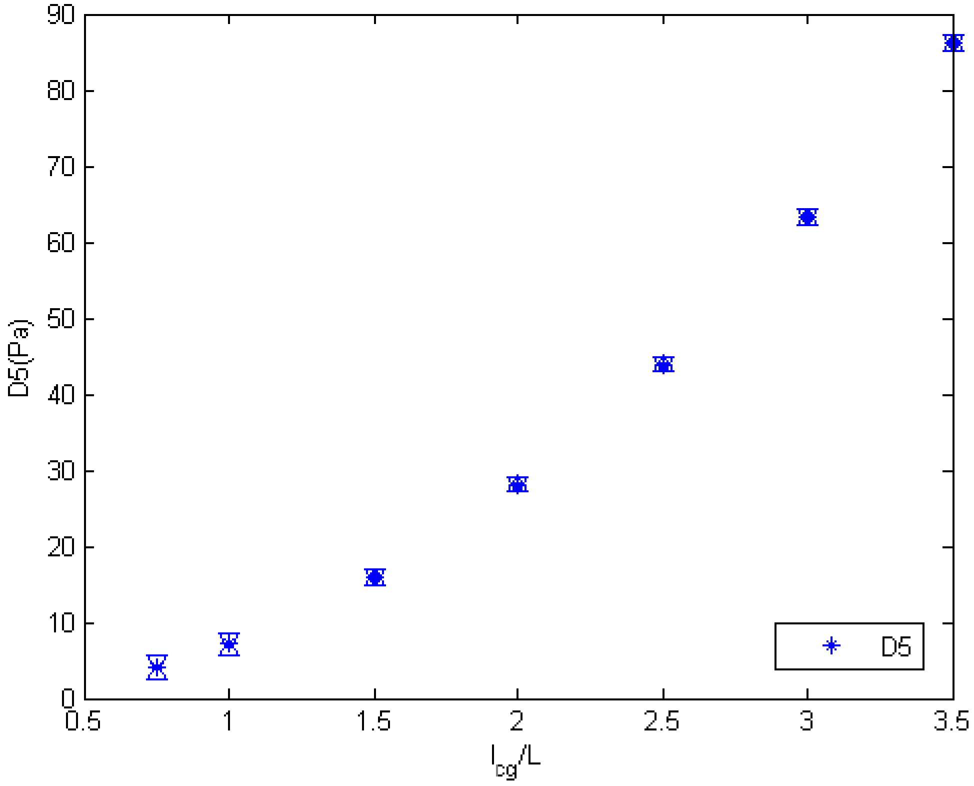
$$J_{\alpha}^{cg}(\mathbf{r}) = \sum_{i} \left(J_{\alpha}^{i} + \sum_{\varphi\xi}^{n} \varepsilon_{\varphi\xi} (r_{\varphi}^{i} - r_{\varphi}) p_{\xi}^{i}\right) \Phi^{i}(\mathbf{r})$$

$$\frac{\partial J^{cg}}{\partial t} = \sum_{\beta}^{n} \frac{\partial}{\partial r_{\beta}} \left(m_{\beta} - V_{\beta}^{cg} J^{cg} \right) - \sum_{\beta\gamma}^{n} \varepsilon_{\beta\gamma} \sigma_{\beta\gamma}$$

$$\begin{aligned} m_{\beta} = & \sum_{ij \neq i}^{N} \sum_{\varphi\xi}^{n} \sum_{\gamma=1}^{n+m} \left(\varepsilon_{\varphi\xi} (r_{\varphi}^{i} - r_{\varphi}) K_{\xi\gamma}^{ij} q_{\gamma}^{j} + K_{\theta\gamma}^{ij} q_{\gamma}^{j} \right) . \widehat{r}_{\beta}^{ji} \int_{0}^{1} \Phi(\mathbf{r} - \widehat{\mathbf{r}}^{i} + s \widehat{\mathbf{r}}^{ji}) ds \\ & - \sum_{ij}^{N} \sum_{\varphi\xi}^{n} \sum_{\gamma=n+1}^{n+m} \varepsilon_{\varphi\xi} (r_{\varphi}^{j} - r_{\varphi}) K_{\xi\gamma}^{ij} q_{\gamma}^{j} g_{\beta}^{j} - \sum_{i}^{N} \sum_{\varphi\xi}^{n} \left(\varepsilon_{\varphi\xi} (r_{\varphi}^{i} - r_{\varphi}) p_{\xi}^{i} + J^{i} \right) \tilde{v}_{\beta}^{i} \Phi^{i} \end{aligned}$$

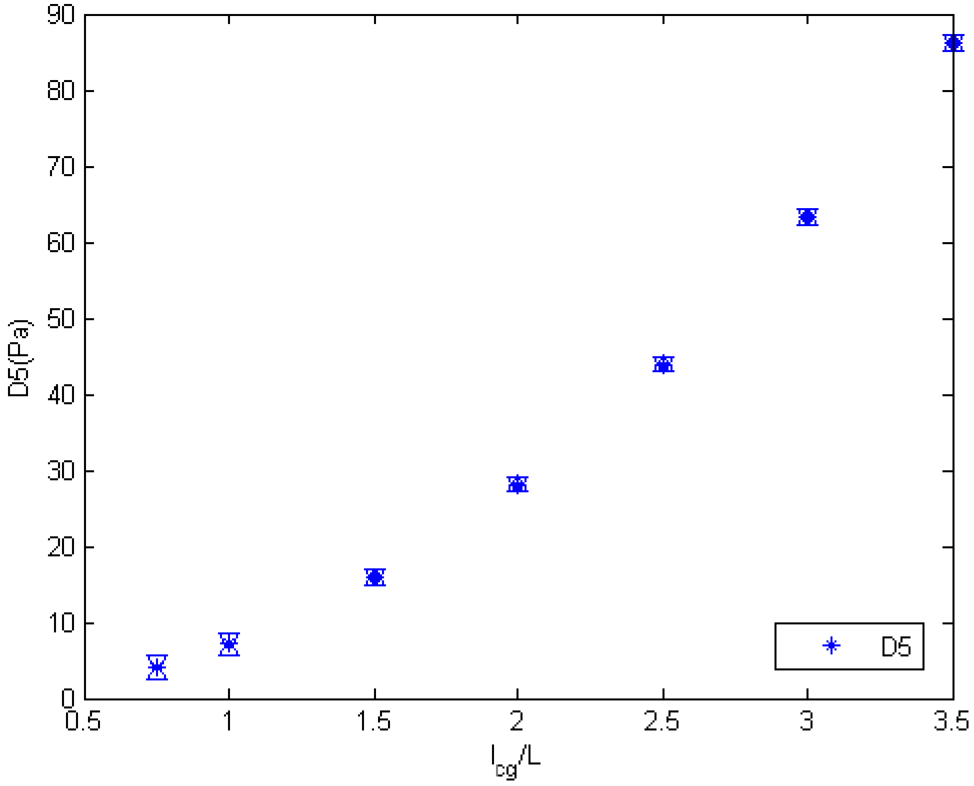
CG Rotation and Torques :

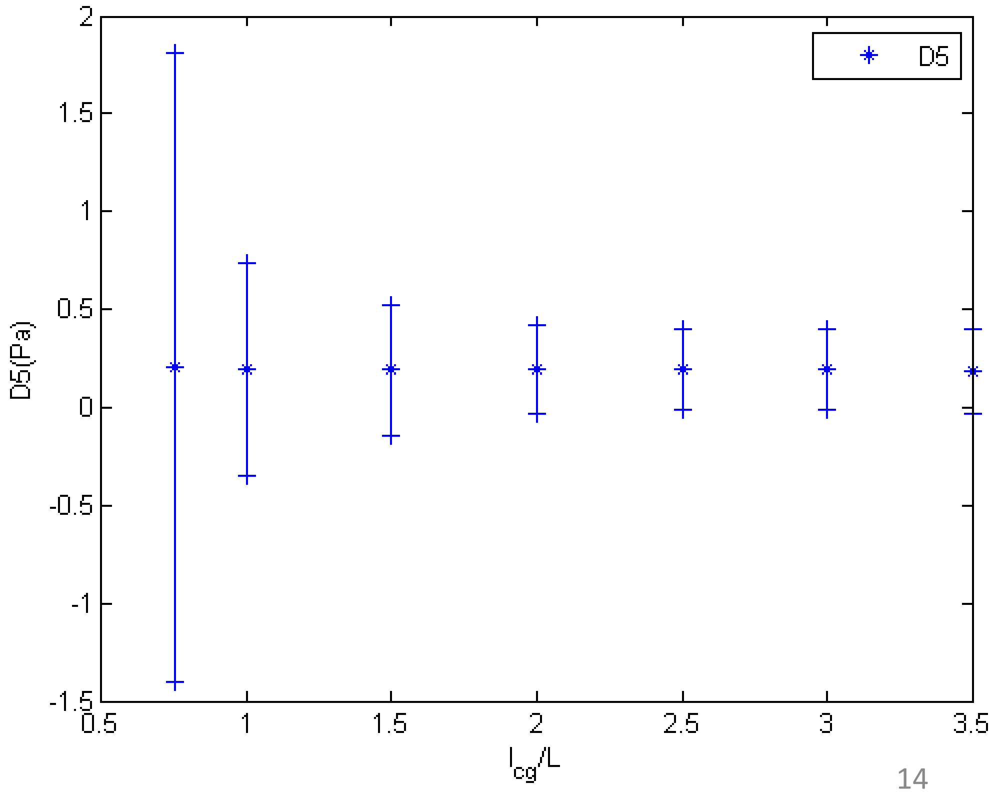
$$m_\beta = \sum_{ij\neq i}^{N}\sum_{\varphi\xi}^{n}\sum_{\gamma=1}^{n+m}\left(\varepsilon_{\varphi\xi}(r^i_\varphi - r_\varphi)K^{ij}_{\xi\gamma}q^j_\gamma + K^{ij}_{\theta\gamma}q^j_\gamma\right).\widehat{r}^{ji}_\beta \int_0^1 \Phi(\mathbf{r} - \widehat{\mathbf{r}}^i + s\widehat{\mathbf{r}}^{ji})ds$$
$$-\sum_{ij}^{N}\sum_{\varphi\xi}^{n}\sum_{\gamma=n+1}^{n+m}\varepsilon_{\varphi\xi}(r^j_\varphi - r_\varphi)K^{ij}_{\xi\gamma}q^j_\gamma g^j_\beta - \sum_{i}^{N}\sum_{\varphi\xi}^{n}\left(\varepsilon_{\varphi\xi}(r^i_\varphi - r_\varphi)p^i_\xi + J^i\right)\tilde{v}^i_\beta \Phi^i$$



CG Rotation and Torques :

$$
\begin{aligned}
m_\beta = {} & \sum_{ij\neq i}^{N}\sum_{\varphi\xi}^{n}\sum_{\gamma=1}^{n+m}\left(\varepsilon_{\varphi\xi}(r_\varphi^i - r_\varphi)K_{\xi\gamma}^{ij}q_\gamma^j + K_{\theta\gamma}^{ij}q_\gamma^j\right).\widehat{r}_\beta^{ji}\int_0^1 \Phi(\mathbf{r}-\widehat{\mathbf{r}}^i + s\widehat{\mathbf{r}}^{ji})ds \\
& -\sum_{ij}^{N}\sum_{\varphi\xi}^{n}\sum_{\gamma=n+1}^{n+m}\varepsilon_{\varphi\xi}(r_\varphi^j - r_\varphi)K_{\xi\gamma}^{ij}q_\gamma^j g_\beta^j - \sum_{i}^{N}\sum_{\varphi\xi}^{n}\left(\varepsilon_{\varphi\xi}(r_\varphi^i - r_\varphi)p_\xi^i + J^i\right)\tilde{v}_\beta^i\Phi^i
\end{aligned}
$$





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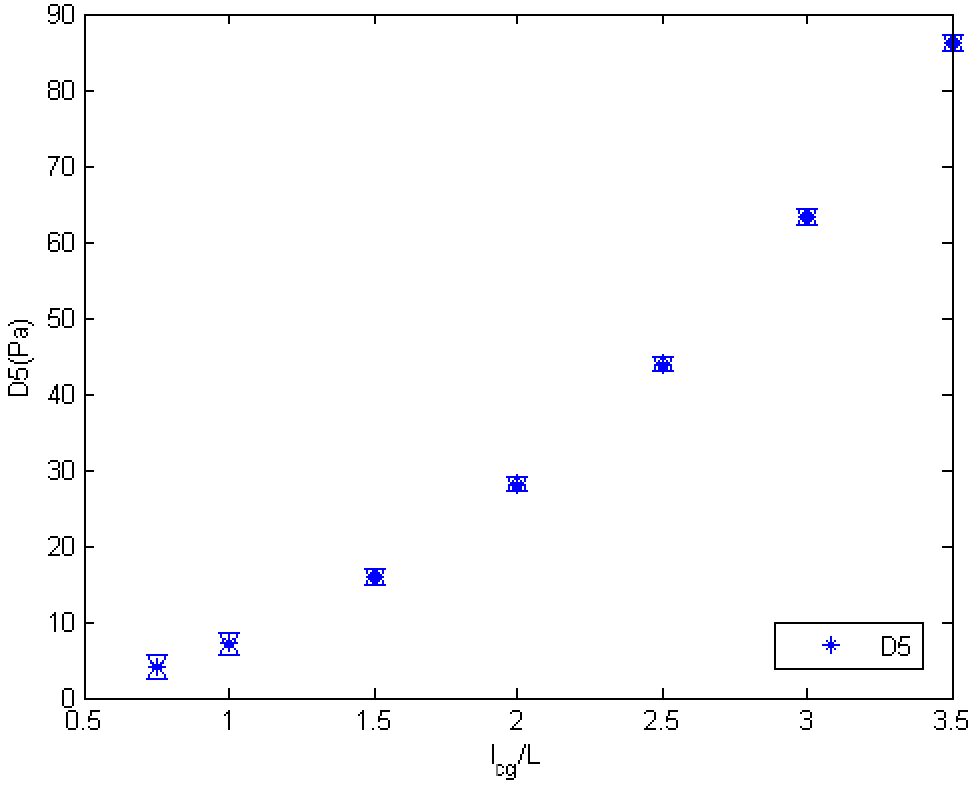

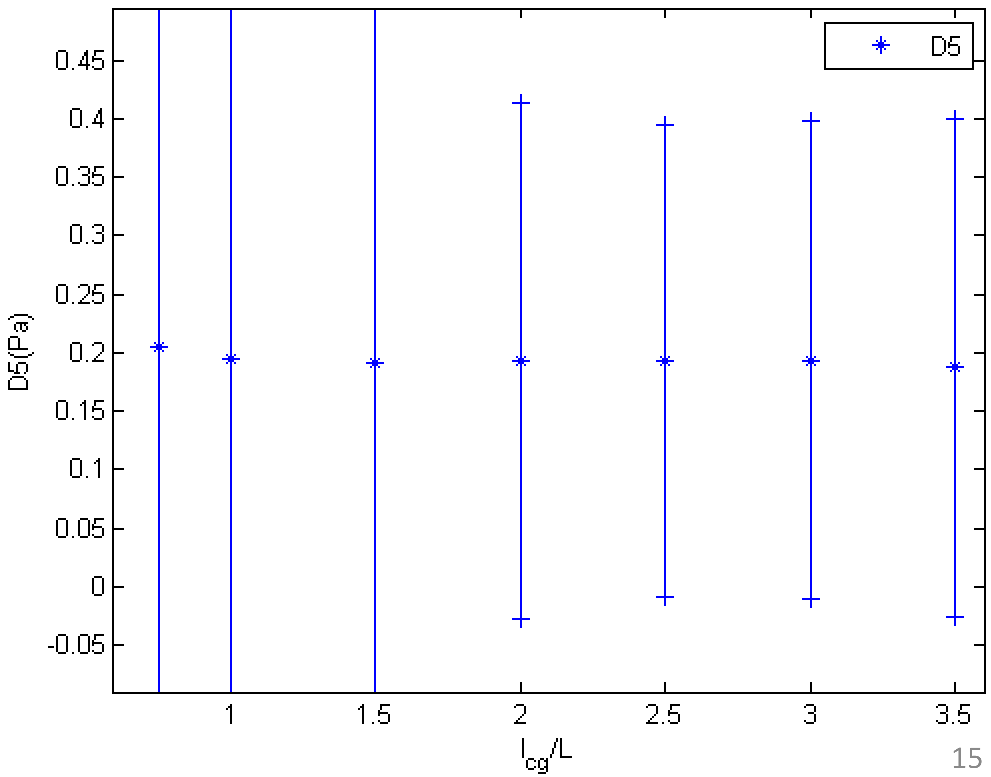

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