On the (multi)scale nature of fluid turbulence

Kolmogorov’ axiomatics

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Consider (as observed) a homogeneous, isotropic stationary solution of the (forced over $L$) Navier and Stokes equations: call it $u^\nu(x,t)$, with $x \in \mathbb{R}^3$. 

\[
\frac{\partial u^\nu}{\partial t} + (u^\nu \cdot \nabla)u^\nu = -\nabla p^\nu + \nu \Delta u^\nu + f
\]

large-scale $L$ forcing

- Homogeneous, isotropic velocity field, of finite variance $\sigma^2$ (independent on viscosity $\nu$):
  \[
  \lim_{\nu \to 0} \mathbb{E}|u^\nu|^2 = \sigma^2 < \infty.
  \]

- Finiteness of average dissipation
  \[
  \lim_{\nu \to 0} \varepsilon^\nu = \lim_{\nu \to 0} \nu \mathbb{E}|\nabla u^\nu|^2 \propto \frac{\sigma^3}{L}
  \]

- At infinite Reynolds number, the velocity field is rough, with (consider longitudinal velocity increments) $\delta_\ell u = u(x + \ell) - u(x)$
  \[
  \mathbb{E}(\delta_\ell u)^q \sim_\ell \varepsilon \propto C_q \ell^{\zeta_q},
  \]

with $\zeta_q$ a universal nonlinear function of the order $q$. Note $\gamma^2 = -\zeta_0''$ the intermittency coefficient, $\zeta_3 = 1$ and $C_3$ being exact.
Define the energy spectrum (Fourier transform of the correlation) as

\[ E(k) = \int e^{-2\pi k\ell} \langle u(x)u(x + \ell) \rangle \, d\ell \]

\[ \text{Kolmogorov energy spectrum} \]

- \( k^{-5/3} \)
- Air Jet \( R_\lambda = 380 \)
- Injection
- Inertial
- Dissipative
**Fractional Gaussian fields**

Consider $u_\epsilon(x) = \int_{\mathbb{R}} \varphi_L(x-y) \frac{1}{|x-y|^{\frac{1}{2}-H}} W(dy)$, where:

- $dW$ is a Gaussian white measure
- $0 < H < 1$ being the Hurst (i.e. Holder) exponent ($H = 1/3$ for turbulence)
- $\epsilon$ a *regularizing* length-scale (that eventually depends on viscosity)
- $\varphi_L$ a large-scale (i.e. $L$) cut-off function (ensuring finite variance)

This field fulfills some aspects of the axiomatics of Kolmogorov:

- Convergence of the variance (whatever the regularization over $\epsilon$)

$$\lim_{\epsilon \to 0} \mathbb{E}u_\epsilon^2 < +\infty.$$ 

- For a proper parametrization of $\epsilon$ with viscosity, reproduces finiteness of dissipation.
- Asymptotically rough with

$$\mathbb{E}(\delta_\ell u)^q \sim_{\ell \to 0} C_q \ell^{qH}$$
Intermittency in Eulerian fluctuations

Eulerian longitudinal velocity increments:  \( \delta \ell u(x) = u(x + \ell) - u(x) \)

Flatness  \( F = \frac{\langle (\delta \ell u)^4 \rangle}{\langle (\delta \ell u)^2 \rangle^2} \)

![Graphs showing intermittency properties](image-url)
Consider (as observed) a homogeneous, isotropic \textbf{stationary} solution of the (forced over $L$) Navier and Stokes equations: call it $u_\nu(x, t)$, with $x \in \mathbb{R}^3$.

- Velocity variance $\sigma^2$ is \textbf{finite} and \textbf{independent} on viscosity $\nu$, i.e.

$$\lim_{\nu \to 0} \mathbb{E}(|u_\nu|^2) = \sigma^2 < +\infty$$

- Consider the \textbf{time evolution} of the velocity field $u_\nu(x_0, t)$ at a fixed position $x_0$. To ensure a bounded velocity variance, the flow will develop small scales:

$$\lim_{\nu \to 0} \mathbb{E} \left[ |u_\nu(x_0, t + \tau) - u_\nu(x_0, t)|^2 \right] \propto \tau^{2/3}$$
• Consider the following linear stochastic differential equation ($\alpha > 0$)

$$du_{\epsilon, H}(t) = -\alpha u_{\epsilon, H}(t)dt + dW_{\epsilon, H}(t)$$

where

$$dW_{\epsilon, H}(t) = \omega_{\epsilon, H}(t)dt + \epsilon^{H-\frac{1}{2}}dW(t)$$

and

$$\omega_{\epsilon, H}(t) = \left(H - \frac{1}{2}\right) \int_{-\infty}^{t} \frac{1}{(t-s+\epsilon)^{\frac{3}{2}-H}}dW(s)$$

• $u_{\epsilon, H}$ is a regularized fractional Ornstein-Uhlenbeck process of Hurst exponent $H$.

• 4 cases need to be discussed: (i) $1/2 < H < 1$, (ii) $H = 1/2$, (iii) $0 < H < 1/2$ and (iv) $H = 0$. (Recall that $H = 1/3$ for turbulence)

• the differential form is especially well suited for numerical simulations
For $\alpha > 0$ and a given $H \in ]0, 1[,$ the Gaussian process $u_{\epsilon,H}(t)$ reaches a stationary regime. It is zero-average and the variance remains bounded when $\epsilon \to 0$. We note

$$\mathbb{E} u_H^2 = \lim_{\epsilon \to 0} \lim_{t \to \infty} \mathbb{E} \left[ (u_{\epsilon,H}(t))^2 \right] = \frac{\alpha^{-2H} \left[ \Gamma \left( H + \frac{1}{2} \right) \right]^2}{2 \sin(\pi H)} < \infty,$$

where enters the Gamma function $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ defined $\forall z > 0$. Let us call then $\delta_\tau u_H$ the corresponding increment over $\tau$, note its variance as

$$\mathbb{E} (\delta_\tau u_H)^2 = \lim_{\epsilon \to 0} \lim_{t \to \infty} \mathbb{E} \left[ (u_{\epsilon,H}(t + \tau) - u_{\epsilon,H}(t))^2 \right].$$

For $H \in ]0, 1[,$ we have the following behavior of the increment variance at small scales

$$\mathbb{E} (\delta_\tau u_H)^2 \sim_{\tau \to 0} \frac{1}{\sin(\pi H)} \frac{\left[ \Gamma \left( H + \frac{1}{2} \right) \right]^2}{\Gamma(2H + 1)} |\tau|^{2H}.$$