On the (multi)scale nature of fluid turbulence

Kolmogorov' axiomatics

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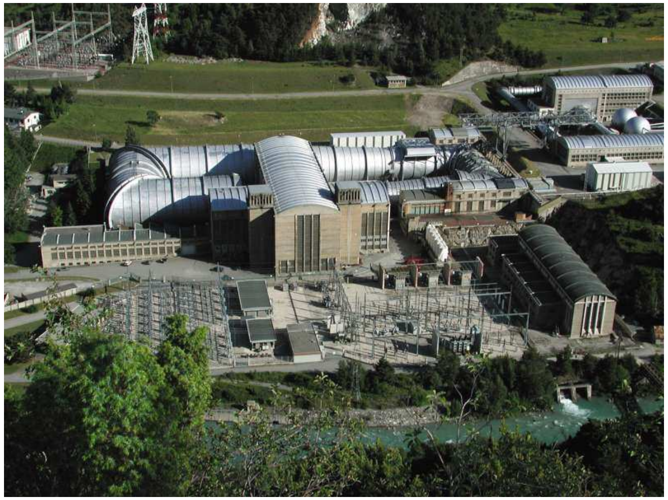
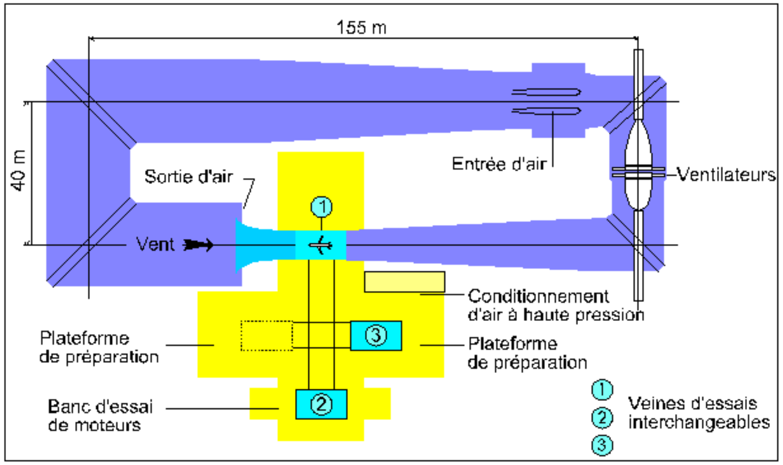
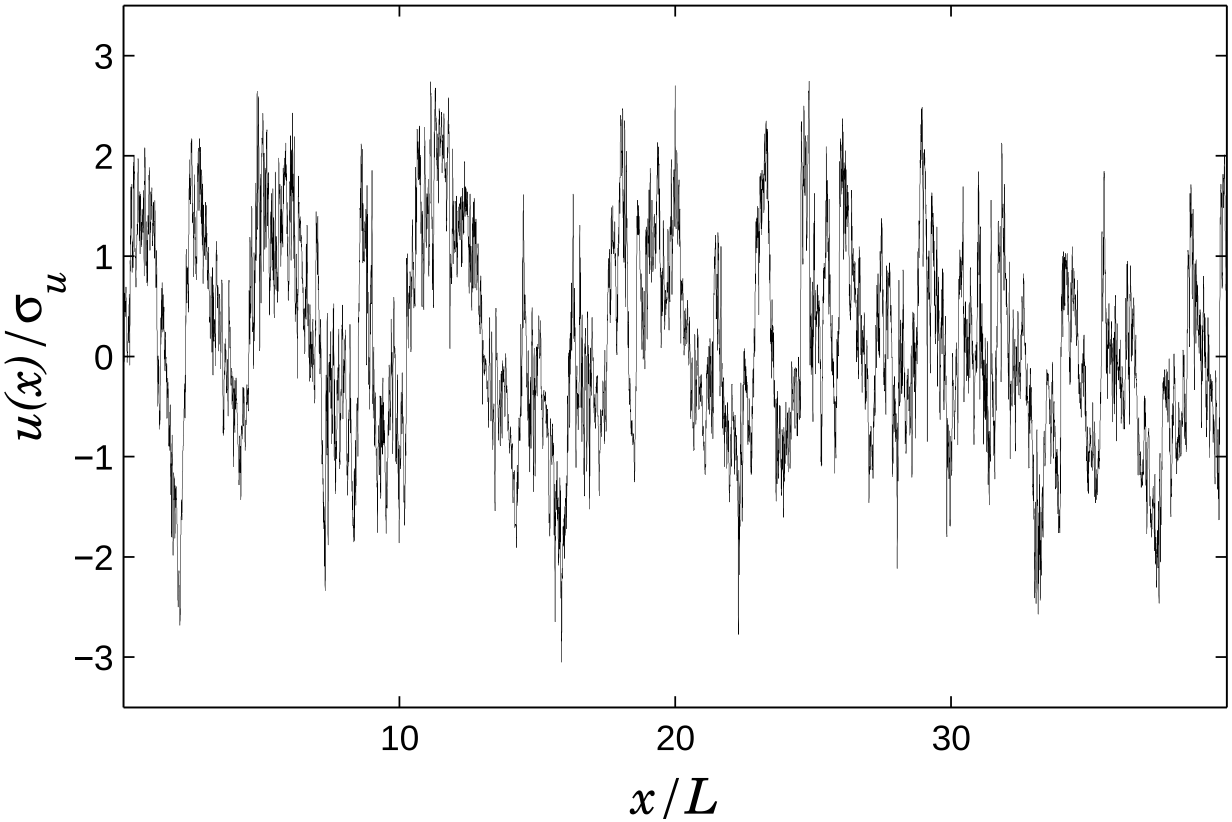
Wind tunnel at Modane







Axiomatics of Kolmogorov phenomenology

Consider (as observed) a homogeneous, isotropic **stationary** solution of the (forced over L) Navier and Stokes equations: call it $\mathbf{u}^\nu(x,t)$, with $x \in \mathbb{R}^3$.

$$\frac{\partial \mathbf{u}^\nu}{\partial t} + (\mathbf{u}^\nu.\nabla)\mathbf{u}^\nu = -\nabla p^\nu + \nu\Delta\mathbf{u}^\nu + \overbrace{f}^{\text{large-scale L forcing}}$$

- Homogeneous, isotropic velocity field, of finite variance σ^2 (independent on viscosity ν):

$$\lim_{\nu\to 0} \mathbb{E}|u^\nu|^2 = \sigma^2 < \infty.$$

- Finiteness of average dissipation

$$\lim_{\nu\to 0} \varepsilon^\nu = \lim_{\nu\to 0} \nu\mathbb{E}|\nabla u^\nu|^2 \propto \frac{\sigma^3}{L}$$

- At infinite Reynolds number, the velocity field is rough, with (consider longitudinal velocity increments) $\delta_\ell u = u(x+\ell) - u(x)$

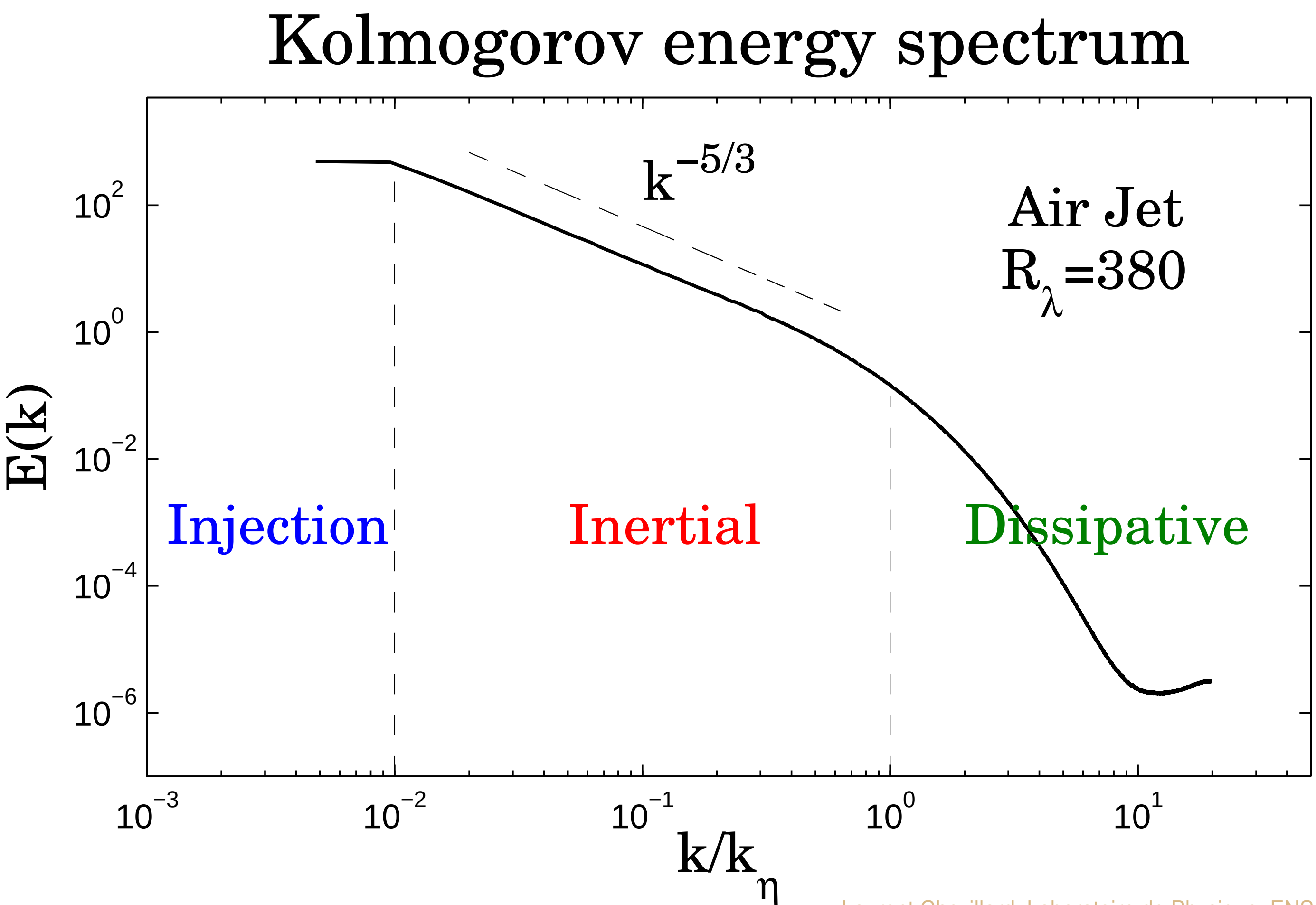
$$\mathbb{E}(\delta_\ell u)^q \underset{\ell\to 0}{\sim} C_q \ell^{\zeta_q},$$

with ζ_q a universal nonlinear function of the order q. Note $\gamma^2 = -\zeta''_0$ the intermittency coefficient, $\zeta_3 = 1$ and C_3 being exact.

Two-point statistical structure of turbulence

Define the energy spectrum (Fourier transform of the correlation) as

$$E(k) = \int e^{-2i\pi k\ell} \langle u(x)u(x+\ell)\rangle \, d\ell$$



Fractional Gaussian fields

Consider $u_\epsilon(x) = \int_{\mathbb{R}} \varphi_L(x-y) \frac{1}{|x-y|_\epsilon^{\frac{1}{2}-H}} W(dy),$

where:

- dW is a Gaussian white measure
- $0 < H < 1$ being the Hurst (i.e. Holder) exponent ($H = 1/3$ for turbulence)
- ϵ a *regularizing* length-scale (that eventually depends on viscosity)
- φ_L a large-scale (i.e. L) cut-off function (ensuring finite variance)

This field fulfills some aspects of the axiomatics of Kolmogorov:

- Convergence of the variance (whatever the regularization over ϵ)

$$\lim_{\epsilon \to 0} \mathbb{E} u_\epsilon^2 < +\infty.$$

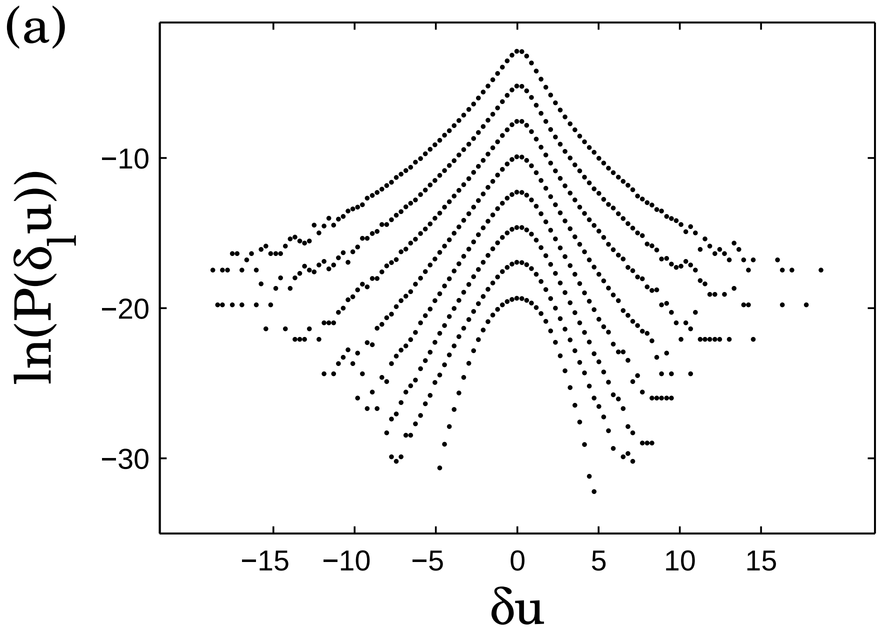
- For a proper parametrization of ϵ with viscosity, reproduces finiteness of dissipation.
- Asymptotically rough with

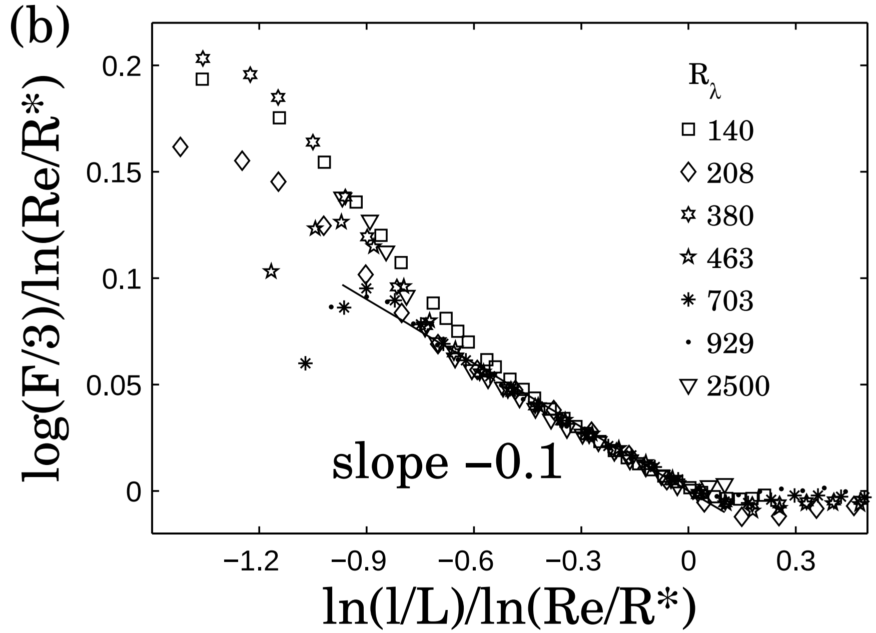
$$\mathbb{E}(\delta_\ell u)^q \underset{\ell \to 0}{\sim} C_q \ell^{qH}$$

Intermittency in Eulerian fluctuations

Eulerian longitudinal velocity increments: $\delta_\ell u(x) = u(x+\ell) - u(x)$

Flatness $F = \frac{\langle(\delta_\ell u)^4\rangle}{\langle(\delta_\ell u)^2\rangle^2}$





Temporal structure of turbulence

Consider (as observed) a homogeneous, isotropic **stationary** solution of the (forced over L) Navier and Stokes equations: call it $\mathbf{u}_\nu(x,t)$, with $x \in \mathbb{R}^3$.

- Velocity variance σ^2 is **finite** and **independent** on viscosity ν, i.e.

$$\lim_{\nu\to 0} \mathbb{E}(|\mathbf{u}_\nu|^2) = \sigma^2 < +\infty$$

- Consider the **time evolution** of the velocity field $\mathbf{u}_\nu(x_0,t)$ at a fixed position x_0. To ensure a bounded velocity variance, the flow will develop small scales:

$$\lim_{\nu\to 0} \mathbb{E}\left[|\mathbf{u}_\nu(x_0,t+\tau) - \mathbf{u}_\nu(x_0,t)|^2\right] \underset{\tau\to 0}{\propto} \tau^{2/3}$$

Regularized fractional Ornstein-Uhlenbeck processes

- Consider the following linear stochastic differential equation ($\alpha > 0$)

$$du_{\epsilon,H}(t) = -\alpha u_{\epsilon,H}(t)dt + dW_{\epsilon,H}(t)$$

where

$$dW_{\epsilon,H}(t) = \omega_{\epsilon,H}(t)dt + \epsilon^{H-\frac{1}{2}}dW(t)$$

and

$$\omega_{\epsilon,H}(t) = \left(H - \frac{1}{2}\right)\int_{-\infty}^{t} \frac{1}{(t-s+\epsilon)^{\frac{3}{2}-H}}dW(s)$$

- $u_{\epsilon,H}$ is a regularized fractional Ornstein-Uhlenbeck process of Hurst exponent H.
- 4 cases need to be discussed: (i)$1/2 < H < 1$, (ii) $H = 1/2$, (iii) $0 < H < 1/2$ and (iv) $H = 0$. (Recall that $H = 1/3$ for turbulence)
- the differential form is especially well suited for numerical simulations

Regularized fractional Ornstein-Uhlenbeck processes

For $\alpha > 0$ and a given $H \in]0,1[$, the Gaussian process $u_{\epsilon,H}(t)$ reaches a stationary regime. It is zero-average and the variance remains bounded when $\epsilon \to 0$. We note

$$\mathbb{E}u_H^2 = \lim_{\epsilon\to 0}\lim_{t\to\infty}\mathbb{E}\left[\left(u_{\epsilon,H}(t)\right)^2\right] = \frac{\alpha^{-2H}\left[\Gamma\left(H+\frac{1}{2}\right)\right]^2}{2\sin(\pi H)} < \infty,$$

where enters the Gamma function $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ defined $\forall z > 0$. Let us call then $\delta_\tau u_H$ the corresponding increment over τ, note its variance as

$$\mathbb{E}\left(\delta_\tau u_H\right)^2 = \lim_{\epsilon\to 0}\lim_{t\to\infty}\mathbb{E}\left[\left(u_{\epsilon,H}(t+\tau) - u_{\epsilon,H}(t)\right)^2\right].$$

For $H \in]0,1[$, we have the following behavior of the increment variance at small scales

$$\mathbb{E}\left(\delta_\tau u_H\right)^2 \underset{\tau\to 0}{\sim} \frac{1}{\sin(\pi H)}\frac{\left[\Gamma\left(H+\frac{1}{2}\right)\right]^2}{\Gamma(2H+1)}|\tau|^{2H}.$$

see: Phys. Rev. E 96, 033111 (2017).