Multiphase continuum models for fiber-reinforced media

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Standard homogenization finds **limitations** in many situations:

- poor scale separation
- boundary layers
- localized damage/cracks
- multiple "small" parameters: slender heterogeneities, large contrast of material properties
Introduction

Standard homogenization finds **limitations** in many situations:

- poor scale separation
- boundary layers
- localized damage/cracks
- multiple "small" parameters: slender heterogeneities, large contrast of material properties

These situations are **frequent in fiber-reinforced media**
Generalized continuum models

There are many ways of building generalized continuum models:

- **higher-grade** models (strain gradient)
- **higher-order** models: additional degrees of freedom (Cosserat, stress gradient, micromorphic)
- **non-local** kernels
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usually share a common feature: appearance of an **internal length scale**
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**Difficulties of generalized continua**

- identification of new material parameters (**homogenization** procedures)
- physical meaning of boundary conditions
- **numerical aspects**: higher regularity, increase in number of dofs, prescription of boundary conditions
Scale effects in a multilayered medium

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**Multilayered medium** : small volume fraction, inclusions much stiffer than matrix

![Diagram](image-url)
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**softening** size effect

**complex behavior** (non-local + increasing order) depending on the slenderness and stiffness contrast [Pideri and Seppecher, 1997],[Bellieud and Bouchitté, 1998]
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**complex behavior** (non-local + increasing order) depending on the slenderness and stiffness contrast [Pideri and Seppecher, 1997],[Bellieud and Bouchitté, 1998]

- shear stiffening : strain gradient effect due to fiber bending
- compression softening : boundary layer effect due to matrix/fiber relative displacement
Outline

1. A homogenization procedure from Cauchy to multiphase continua
2. Some analytical results
3. Illustrative applications
4. Conclusions
**Multiphase model: kinematics**

Two superimposed continua with different kinematics: \( u^m \) and \( u^r \)

Construction using the virtual work principle: **first-gradient** theory \( \varepsilon^i = \nabla^s u^i \)

\[
p_{\text{def}}(\varepsilon^m, \varepsilon^r, u^r - u^m) = \sigma^m : \varepsilon^m + \sigma^r : \varepsilon^r + l \cdot (u^r - u^m)
\]

**Generalized strains**
- strain of matrix displacement \( \varepsilon^m \)
- strain of reinforcement displacement \( \varepsilon^r \)
- relative displacement \( [u] = u^r - u^m \)

**Generalized stresses**
- partial matrix stress \( \sigma^m \)
- partial reinforcement stress \( \sigma^r \)
- interaction force \( l \)
Multiphase model: kinematics

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**Generalized stresses**
- partial matrix stress $\sigma^m$
- partial reinforcement stress $\sigma^r$
- interaction force $l$

**Standard Cauchy medium** by imposing $u^m = u^r = u$
One strain measure $\varepsilon = \varepsilon^m = \varepsilon^r$ associated with the Cauchy stress $\sigma = \sigma^m + \sigma^r$
Multiphase model: equilibrium equations and constitutive relations

Using the virtual work principle (same volume force $F$ for both phases):

$$\text{div } \sigma^m + l + \rho_m F = 0$$
$$\text{div } \sigma^r - l + \rho_r F = 0$$
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\text{div } \sigma^m + \mathbb{I} + \rho_m F = 0 \\
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\]

**Postulated constitutive relations** : ($\eta \ll 1$ : reinforcement volume fraction)

[Sudret, 1999], [de Buhan and Sudret, 2000]

\[
\sigma^m = C^m : \varepsilon^m \\
\sigma^r = \eta C^r : \varepsilon^r \\
I = c_I [u]
\]
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l = c_l [u]
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**going beyond** small volume fraction hypothesis: influence of strains of one phase on stresses in the other phase?
Auxiliary problem (periodic heterogeneities)

We consider an auxiliary problem with **microscopic body force** and **eigenstrains**: (perfect bonding between both phases)

\[
\text{div } \sigma + f^i = 0 \quad \forall y \in A^i
\]
\[
\sigma(y) = C^i : (\varepsilon(y) - \varepsilon^i)
\]
\[
U(y) = E \cdot y + u(y) \quad \forall y \in A
\]
\[
\sigma \cdot n \quad \mathcal{A} \text{-antiperiodic}
\]
\[
u(y) \quad \mathcal{A} \text{-periodic}
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where body forces and eigenstrains are \textbf{phase-wise uniform} and have \textbf{zero-average}:

\[
\langle f \rangle = 0 \implies f^1 = \frac{l}{\phi_1}, \quad f^2 = -\frac{l}{\phi_2} \\
\langle \varepsilon \rangle = 0 \implies \varepsilon^1 = \frac{\Delta E}{\phi_1}, \quad \varepsilon^2 = -\frac{\Delta E}{\phi_2}
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\langle \epsilon \rangle &= 0 \implies \epsilon^1 = \Delta \mathbb{E}/\phi_1, \quad \epsilon^2 = -\Delta \mathbb{E}/\phi_2
\end{align*}
\]

or \textbf{equivalently} with \( E^i = \mathbb{E} - \epsilon^i \), \( \mathbb{E} = \phi_1 \mathbb{E}^1 + \phi_2 \mathbb{E}^2 \) and \( \Delta \mathbb{E} = \phi_1 \phi_2 (\mathbb{E}^2 - \mathbb{E}^1) \)
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We consider an auxiliary problem with **microscopic body force and eigenstrains**: (perfect bonding between both phases)

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\operatorname{div} \sigma \pm \frac{I}{\phi_i} = 0 \quad \forall y \in \mathcal{A}^i
\]

\[
\sigma(y) = \mathcal{C}^i : (E^i + \nabla^s u(y))
\]

\[
U(y) = (\phi_1 E_1 + \phi_2 E_2) \cdot y + u(y) \quad \forall y \in \mathcal{A}
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\langle \epsilon \rangle = 0 \implies \epsilon^1 = \Delta E / \phi_1, \quad \epsilon^2 = -\Delta E / \phi_2
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or **equivalently** with \( E^i = E - \epsilon^i \), \( E = \phi_1 E_1 + \phi_2 E_2 \) and \( \Delta E = \phi_1 \phi_2 (E^2 - E^1) \)
Auxiliary problem (resolution)

The previous problem depends on three different loading parameters \((E^1, E^2, I)\).
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Using an extended version of Hill-Mandel’s lemma, associated dual quantities are:

\[
E^i \quad \leftrightarrow \quad \sigma^i = \frac{1}{|A|} \int_{A^i} \sigma dA = \phi_i \langle \sigma \rangle^i
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\[
I \quad \leftrightarrow \quad V = \langle U \rangle^2 - \langle U \rangle^1 = [U]
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**Solution** to the auxiliary problem depends linearly upon \((E^1, E^2, I)\):

\[
\begin{align*}
U(y) &= a^1(y) : E^1 + a^2(y) : E^2 + d(y) \cdot I \\
\varepsilon(y) &= A^1(y) : E^1 + A^2(y) : E^2 + D(y) \cdot I
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Auxiliary problem (resolution)

The previous problem depends on **three different loading parameters** \((E_1, E_2, I)\).

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\end{align*}
\]

Using the previous relations:

\[
\begin{align*}
\sigma^1 &= D^{11} : E_1 + D^{12} : E_2 + D^1 \cdot I \\
\sigma^2 &= D^{21} : E_1 + D^{22} : E_2 + D^2 \cdot I \\
V &= [a^1] : E_1 + [a^2] : E_2 + [d] \cdot I
\end{align*}
\]
Homogenized constitutive equations

Symmetry relations are obtained from the Maxwell-Betti reciprocity theorem:

\[
\sigma^1 = D^{11} : E^1 + D^{12} : E^2 + D^1 \cdot I
\]
\[
\sigma^2 = (D^{12})^T : E^1 + D^{22} : E^2 + D^2 \cdot I
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\mathbf{V} = (D^1)^T : \mathbb{E}^1 + (D^2)^T : \mathbb{E}^2 + [\mathbb{d}] \cdot I
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summing up the first two equations in the case \( E^1 = E^2 = E \) and \( I = 0 \) gives

\[ \Sigma = \sigma^1 + \sigma^2 = C^{hom} : E \]
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\[ V = (D^1)^T : E^1 + (D^2)^T : E^2 + \begin{bmatrix} d \end{bmatrix} \cdot I \]

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\[ \Sigma = \sigma^1 + \sigma^2 = C^{\text{hom}} : E \]

Unit cell with central symmetry

\( D^1 = D^2 = 0 \) so that the partial stress constitutive equations decouple from the interaction force:

\[ \sigma^1 = D^{11} : E^1 + D^{12} : E^2 \]
\[ \sigma^2 = (D^{12})^T : E^1 + D^{22} : E^2 \]
\[ I = \begin{bmatrix} d \end{bmatrix}^{-1} \cdot V \]
Link with the multiphase model

An other way to build the auxiliary problem

From the definition of $I$, interpreted as the resultant force acting on the interface:

$$I = \frac{1}{|A|} \int_{\Gamma} \sigma \cdot n \, dS$$

From [Coussy, 1998], partial stress balance at the macroscopic scale is:

$$\text{div} x \sigma_i + \rho_i F_i \pm I = 0$$

$\sigma_i$ and $I$ are indeed the pertinent generalized forces for the multiphase model.

Previous resolution gives a macroscopic complementary energy $\Psi^\star(\sigma_i, I)$.

Multiphase constitutive relations:

$$\varepsilon_i = \partial \Psi^\star / \partial \sigma_i$$

$$[u] = \partial \Psi^\star / \partial I$$

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Link with the multiphase model

An other way to build the auxiliary problem

From the definition of $I$, interpreted as the resultant force acting on the interface:

$$I = \frac{1}{|A|} \int_{\Gamma} \sigma \cdot n_{2 \rightarrow 1} dS$$

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$$\text{div} \, x \sigma_{i} + \rho_{i} F_{\pm} I = 0$$

$\sigma$ and $I$ are indeed the pertinent generalized forces for the multiphase model.

Previous resolution gives a macroscopic complementary energy $\Psi^*$:

$$\epsilon_{i} = \partial_{\Psi^*} \frac{\partial \Psi^*}{\partial \sigma_{i}}$$

$$[u] = \partial_{\Psi^*} \frac{\partial \Psi^*}{\partial I}$$
Link with the multiphase model

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$$\text{div}_x \bar{\sigma}^j + \rho_i \bar{E} \pm I = 0$$
A homogenization procedure from Cauchy to multiphase continua

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$\sigma^i$ and $I$ are indeed the pertinent generalized forces for the multiphase model.

Previous resolution gives a macroscopic complementary energy $\Psi^*(\sigma^i, I)$

Multiphase constitutive relations:

$$\varepsilon^i = \frac{\partial \Psi^*}{\partial \sigma^i} \quad \leftarrow \quad E^i$$

$$[u] = \frac{\partial \Psi^*}{\partial I} \quad \leftarrow \quad V$$
Outline

1 A homogenization procedure from Cauchy to multiphase continua

2 Some analytical results

3 Illustrative applications

4 Conclusions
Constitutive relations for a biphasic material

$D^{ij}$ can be deduced from the knowledge of $C^{hom}$:

\[
D^{11} = \phi_1 C^1 - C^1 : [C]^{-1} : \Delta C : [C]^{-1} : C^1
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D^{22} = \phi_2 C^2 - C^2 : [C]^{-1} : \Delta C : [C]^{-1} : C^2
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with $[C] = C^2 - C^1$ and $\Delta C = \langle C \rangle - C^{hom}$
Constitutive relations for a biphasic material

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Stiff linear isotropic inclusions in small volume fraction

Assumptions: $\phi_2 \ll 1$ and $\lim_{\phi_2 \to 0} \phi_2 C^2 = C^0$
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using [Hashin and Rosen, 1964]:

$$
\lim_{\phi_2 \to 0} C^{\text{hom}} = C^{\text{hom},0} = C^1 + E^0 e_1 \otimes e_1 \otimes e_1 \otimes e_1
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Some analytical results

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\begin{align*}
D^{11} &\rightarrow C^1 \\
D^{22} &\rightarrow C^{\text{hom},0} - C^1 = E^0 e_1 \otimes e_1 \otimes e_1 \otimes e_1 \\
D^{12} &\rightarrow 0
\end{align*}
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Constitutive relations for a biphasic material

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where $\mathbf{E}^0 = \lim \phi_2 E^2 = C_{1111}^0$, then

\[
\sigma^m = C^m : \varepsilon^m
\]
\[
\sigma_{11} = \mathbf{E}^0 \varepsilon_{11}, \quad \sigma_{ij} = 0
\]
\[
l = [d]^{-1} \cdot [u]
\]
Some analytical results

Determination of the interaction stiffness

Auxiliary problem with body forces only: similar to homogenization of permeability in porous media ⇒ $[d]$ will depend on the UC size $s$ as $s^2$
Determination of the interaction stiffness

Auxiliary problem with body forces only: similar to homogenization of permeability in porous media $\Rightarrow \left[ a \right]$ will depend on the UC size $s$ as $s^2$

Example for a 2D layered medium (1: matrix, 2: reinforcement, $\eta$ reinforcement volume fraction):

$$\left[ d_{11} \right] = \langle u(y) \rangle^2 - \langle u(y) \rangle^1$$
Determination of the interaction stiffness

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$$[d_{11}] = \langle u(y) \rangle^2 - \langle u(y) \rangle^1$$

$$[\mathbf{d}] = \frac{s^2}{12} \begin{bmatrix} \langle 1/\mu \rangle & 0 \\ 0 & \langle 1/(\lambda + 2\mu) \rangle \end{bmatrix}$$
Some analytical results

**Determination of the interaction stiffness**

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Example for a 2D layered medium (1: matrix, 2: reinforcement, $\eta$ reinforcement volume fraction):

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$[d] = \frac{s^2}{12} \begin{bmatrix} \langle 1/\mu \rangle & 0 \\ 0 & \langle 1/(\lambda + 2\mu) \rangle \end{bmatrix} \xrightarrow{Cr \gg Cm} \frac{s^2(1 - \eta)}{12} \begin{bmatrix} \frac{1}{\mu_m} & 0 \\ 0 & \frac{1}{\lambda_m + 2\mu_m} \end{bmatrix}$

[Sudret, 1999] estimate: pullout test on rigid inclusion

$[d_{11}] = s^2(1 - \eta)$
Determination of the interaction stiffness

Auxiliary problem with body forces only: similar to homogenization of permeability in porous media \( \Rightarrow [d] \) will depend on the UC size \( s \) as \( s^2 \)

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[d_{11}] = \langle u(y) \rangle^2 - \langle u(y) \rangle^1
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\[
[d] = \frac{s^2}{12} \begin{bmatrix}
\langle 1/\mu \rangle & 0 \\
0 & \langle 1/(\lambda + 2\mu) \rangle
\end{bmatrix} \rightarrow \frac{s^2(1-\eta)}{12} \begin{bmatrix}
\frac{1}{\mu_m} & 0 \\
0 & \frac{1}{\lambda_m + 2\mu_m}
\end{bmatrix}
\]

[Sudret, 1999] estimate: pullout test on rigid inclusion \( [d_{11}] = \frac{s^2(1-\eta)}{8\mu_m} \)
Outline

1. A homogenization procedure from Cauchy to multiphase continua

2. Some analytical results

3. Illustrative applications

4. Conclusions
A multilayered block in compression

Vertical displacement is \( u_2(y) = -\frac{\delta}{H} y \) for both phases,

\[
\begin{align*}
    u_m^m(x) &= \delta H (\nu_{hom} x + s \beta_m \sinh(x/\ell) \cosh(L/\ell)) \\
    u_m^r(x) &= \delta H (\nu_{hom} x + s \beta_r \sinh(x/\ell) \cosh(L/\ell))
\end{align*}
\]

where \( \ell \propto 1/\sqrt{c} \) and \( \nu_{hom} \) is an internal length scale arising for this problem.

First-order corrections over a distance \( \approx \ell \).
A multilayered block in compression

Vertical displacement is $u_2(y) = -\frac{\delta}{H} y$ for both phases, horizontal displacement $u_1^i(x)$ is different for each phase and satisfies:

$$\alpha_{11} \frac{d^2 u_1^m}{dx^2} + \alpha_{12} \frac{d^2 u_1^r}{dx^2} + c_I (u_1^r - u_1^m) = 0$$

$$\alpha_{12} \frac{d^2 u_1^m}{dx^2} + \alpha_{22} \frac{d^2 u_1^r}{dx^2} - c_I (u_1^r - u_1^m) = 0$$

with $\alpha_{ij} = \mathbb{D}^{ji}_{1111}$ and $c_I = [d_{11}]^{-1}$
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Vertical displacement is $u_2(y) = -\frac{\delta}{H} y$ for both phases, horizontal displacement $u_1'(x)$ is different for each phase and satisfies:

$$\alpha_{11} \frac{d^2 u_1^m}{dx^2} + \alpha_{12} \frac{d^2 u_1^r}{dx^2} + c_l (u_1^r - u_1^m) = 0$$

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with $\alpha_{ij} = D_{1111}^{ij}$ and $c_l = [d_{11}]^{-1}$

Solution is of the form:

$$u_1^m(x) = \frac{\delta}{H} \left( \nu^{hom} x + s \beta^m \frac{\sinh(x/\ell)}{\cosh(L/\ell)} \right)$$

$$u_1^r(x) = \frac{\delta}{H} \left( \nu^{hom} x + s \beta^r \frac{\sinh(x/\ell)}{\cosh(L/\ell)} \right)$$

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**first-order corrections** over a distance \( \approx \ell \)
**Comparisons**

Comparison between full heterogeneous computations, Sudret’s model and the present model: $N$ is the number of layers

<table>
<thead>
<tr>
<th></th>
<th>Matrix (phase 1)</th>
<th>Reinforcement (phase 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume fraction</td>
<td>$1 - \eta = 0.9$</td>
<td>$\eta = 0.1$</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E_1 = 10$ MPa</td>
<td>$E_2 = 1000$ MPa</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>$\nu_1 = 0.45$</td>
<td>$\nu_2 = 0.3$</td>
</tr>
</tbody>
</table>
Comparisons

Comparison between full heterogeneous computations, Sudret’s model and the present model: $N$ is the number of layers

$N = 4$

![Graph showing comparisons between different models](image_url)

- $u^m$ (heter.)
- $u^m$ [Sudret, 1999]
- $u^m$ (present)
- $u^r$ (heter.)
- $u^r$ [Sudret, 1999]
- $u^r$ (present)
- Standard homogenization

Phase displacement $u'/\delta$ vs. $x/L$

Reinforcement stress [MPa] vs. $x/L$
Comparisons

Comparison between full heterogeneous computations, Sudret’s model and the present model: $N$ is the number of layers.
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A crack-bridging problem
Appeal of such models: prescribe **different boundary conditions for each phase**
A crack-bridging problem

Appeal of such models: prescribe **different boundary conditions for each phase**

Heterogeneous FE computations with $N = 8, 16, 32$ layers (filled symbols)
A crack-bridging problem: delaminated interfaces

Modelling a region with fully delaminated interfaces (zero interaction stiffness)
A crack-bridging problem: delaminated interfaces

Modelling a region with fully delaminated interfaces (zero interaction stiffness)

Matrix displacement

Horizontal stress
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Conclusions and perspectives

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- multiphase models: phenomenological constitutive relations for fiber-reinforced materials
- homogenization procedure to identify constitutive parameters
- capture boundary effects which are important for large stiffness contrast
- some advantages over other generalized continua
- retrieves results of **shear lag models**

Perspectives

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- micromechanical estimates
- higher-grade version to include bending effects
- non-linear constitutive relations
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Thank you for your attention!
A stress-based auxiliary problem

Standard homogenization:

\[
\psi^*(\Sigma) = \min_{\sigma} \frac{1}{2|A|} \int_{A^1} \sigma : S^1 : \sigma d\Omega + \frac{1}{2|A|} \int_{A^2} \sigma : S^2 : \sigma d\Omega
\]

s.t. \( \text{div} \sigma = 0 \)

\( \sigma \cdot n \quad \mathcal{A}\text{-antiperiodic} \)

\( \langle \sigma \rangle = \Sigma \)
A stress-based auxiliary problem

Extended auxiliary problem:

\[ \Psi^*(\sigma^1, \sigma^2, I) = \min_{\sigma} \frac{1}{2|A|} \int_{A_1} \sigma : S^1 : \sigma d\Omega + \frac{1}{2|A|} \int_{A_2} \sigma : S^2 : \sigma d\Omega \]

s.t. \[ \text{div} \sigma + f(x) = 0 \]

\[ \sigma \cdot n \text{ } A \text{-antiperiodic} \]

\[ \phi_i \langle \sigma \rangle^i = \sigma^i \quad i = 1, 2 \]

\[ \frac{1}{|A|} \int_{\Gamma} \sigma \cdot n dS = I \]

with body forces at the microscopic scale.
A stress-based auxiliary problem

Extended auxiliary problem:

\[
\psi^*(\sigma^1, \sigma^2, I) = \min \quad \frac{1}{2|A|} \int_{A^1} \sigma : S^1 : \sigma d\Omega + \frac{1}{2|A|} \int_{A^2} \sigma : S^2 : \sigma d\Omega \\
\text{s.t.} \quad \text{div } \sigma + f(x) = 0 \\
\sigma \cdot n \quad \text{A-antiperiodic} \\
\phi_i \langle \sigma \rangle^i = \sigma^i \quad i = 1, 2 \\
\frac{1}{|A|} \int_{\Gamma} \sigma \cdot ndS = I
\]

with body forces at the microscopic scale. Divergence theorem on phase 2 and on phase 1 gives:

\[
\frac{1}{|A|} \int_{A^2} \left( \text{div } \sigma + f(x) \right) d\Omega = I + \phi_2 \langle f \rangle^2 \\
\frac{1}{|A|} \int_{A^1} \left( \text{div } \sigma + f(x) \right) d\Omega = -I + \phi_1 \langle f \rangle^1
\]