Multiphase continuum models for fiber-reinforced media

Jérémy Bleyer

Laboratoire Navier,
Ecole des Ponts ParisTech-IFSTTAR-CNRS (UMR 8205), Université Paris-Est

GdR MePhy, June 5th 2018

Introduction

Standard homogenization finds **limitations** in many situations:

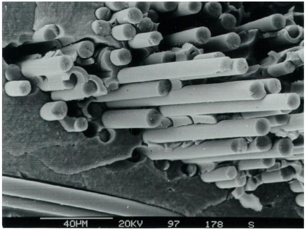
- poor scale separation
- boundary layers
- localized damage/cracks
- multiple "small" parameters: slender heterogeneities, large contrast of material properties

Introduction

Standard homogenization finds **limitations** in many situations:

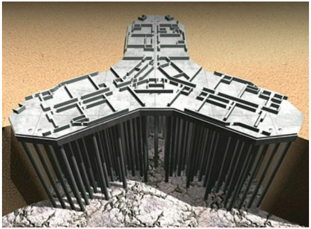
- poor scale separation
- boundary layers
- localized damage/cracks
- multiple "small" parameters: slender heterogeneities, large contrast of material properties

These situations are **frequent in fiber-reinforced media**



Wikimedia Commons

theconstructor.org



oobject.com

Generalized continuum models

There are **many ways of building generalized continuum models** :

- **higher-grade** models (strain gradient)
- **higher-order** models : additional degrees of freedom (Cosserat, stress gradient, micromorphic)
- **non-local** kernels

Generalized continuum models

There are **many ways of building generalized continuum models** :

- **higher-grade** models (strain gradient)
- **higher-order** models : additional degrees of freedom (Cosserat, stress gradient, micromorphic)
- **non-local** kernels

usually share a common feature : appearance of an **internal length scale**

Generalized continuum models

There are **many ways of building generalized continuum models** :

- **higher-grade** models (strain gradient)
- **higher-order** models : additional degrees of freedom (Cosserat, stress gradient, micromorphic)
- **non-local** kernels

usually share a common feature : appearance of an **internal length scale**

Difficulties of generalized continua

- identification of new material parameters (**homogenization** procedures)
- physical meaning of boundary conditions
- *numerical aspects* : higher regularity, increase in number of dofs, prescription of boundary conditions

Scale effects in a multilayered medium

strain gradient models usually induce a **stiffening** size-effect

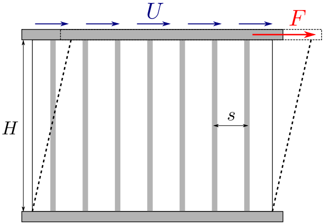
Scale effects in a multilayered medium

strain gradient models usually induce a **stiffening** size-effect
(stress gradient has been shown to induce a **softening** size-effect [Tran, 2016])

Scale effects in a multilayered medium

strain gradient models usually induce a **stiffening** size-effect
(stress gradient has been shown to induce a **softening** size-effect [Tran, 2016])

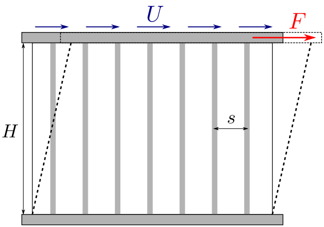
Multilayered medium : small volume fraction, inclusions much stiffer than matrix

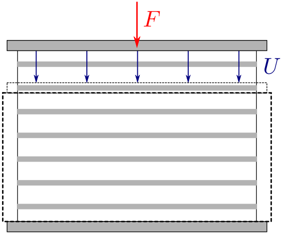


Scale effects in a multilayered medium

strain gradient models usually induce a **stiffening** size-effect
(stress gradient has been shown to induce a **softening** size-effect [Tran, 2016])

Multilayered medium : small volume fraction, inclusions much stiffer than matrix

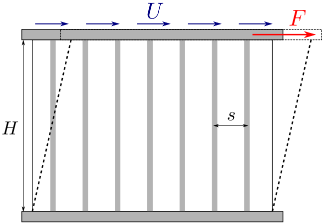




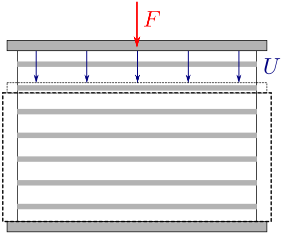
Scale effects in a multilayered medium

strain gradient models usually induce a **stiffening** size-effect
(stress gradient has been shown to induce a **softening** size-effect [Tran, 2016])

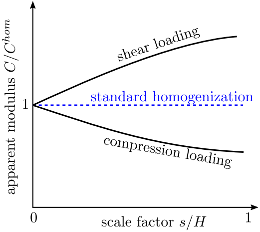
Multilayered medium : small volume fraction, inclusions much stiffer than matrix



stiffening size effect



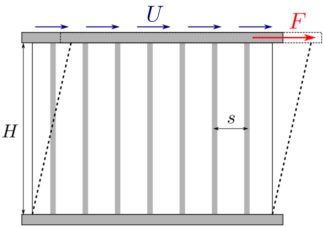
softening size effect



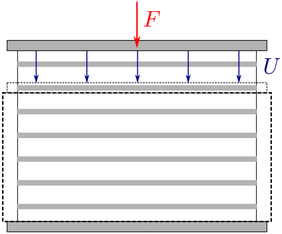
Scale effects in a multilayered medium

strain gradient models usually induce a **stiffening** size-effect
(stress gradient has been shown to induce a **softening** size-effect [Tran, 2016])

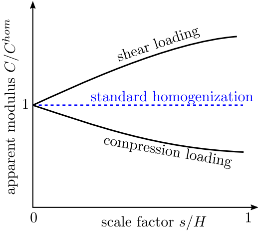
Multilayered medium : small volume fraction, inclusions much stiffer than matrix



stiffening size effect



softening size effect

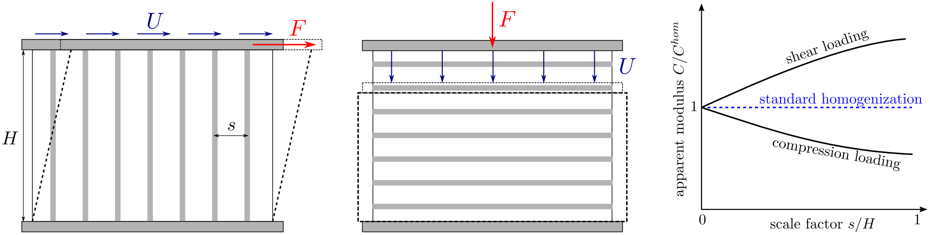


complex behavior (non-local + increasing order) depending on the slenderness and stiffness contrast [Pideri and Seppecher, 1997],[Bellieud and Bouchitté, 1998]

Scale effects in a multilayered medium

strain gradient models usually induce a **stiffening** size-effect
(stress gradient has been shown to induce a **softening** size-effect [Tran, 2016])

Multilayered medium : small volume fraction, inclusions much stiffer than matrix



stiffening size effect | **softening** size effect

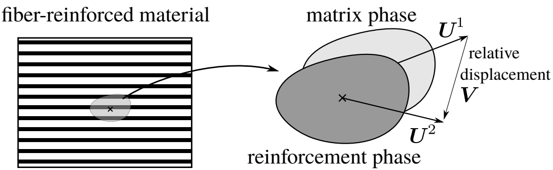
complex behavior (non-local + increasing order) depending on the slenderness and stiffness contrast [Pideri and Seppecher, 1997],[Bellieud and Bouchitté, 1998]

- shear stiffening : strain gradient effect due to fiber bending
- compression softening : boundary layer effect due to matrix/fiber relative displacement

Outline

1. A homogenization procedure from Cauchy to multiphase continua
2. Some analytical results
3. Illustrative applications
4. Conclusions

Multiphase model : kinematics



Two superimposed continua with different kinematics : $\underline{u}^m$ and $\underline{u}^r$

Construction using the virtual work principle : **first-gradient** theory $\underline{\underline{\varepsilon}}^i = \underline{\underline{\nabla^s u^i}}$

$$p_{def}(\underline{\underline{\varepsilon}}^m, \underline{\underline{\varepsilon}}^r, \underline{u}^r - \underline{u}^m) = \underline{\underline{\sigma}}^m : \underline{\underline{\varepsilon}}^m + \underline{\underline{\sigma}}^r : \underline{\underline{\varepsilon}}^r + \underline{I} \cdot (\underline{u}^r - \underline{u}^m)$$

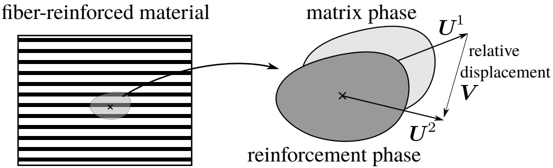
Generalized strains

- strain of matrix displacement $\underline{\underline{\varepsilon}}^m$
- strain of reinforcement displacement $\underline{\underline{\varepsilon}}^r$
- relative displacement $[\![\underline{u}]\!] = \underline{u}^r - \underline{u}^m$

Generalized stresses

- partial matrix stress $\underline{\underline{\sigma}}^m$
- partial reinforcement stress $\underline{\underline{\sigma}}^r$
- interaction force $\underline{I}$

Multiphase model : kinematics



Two superimposed continua with different kinematics : $\underline{u}^m$ and $\underline{u}^r$

Construction using the virtual work principle : **first-gradient** theory $\underline{\underline{\varepsilon}}^i = \underline{\underline{\nabla^s u^i}}$

$$p_{def}(\underline{\underline{\varepsilon}}^m, \underline{\underline{\varepsilon}}^r, \underline{u}^r - \underline{u}^m) = \underline{\underline{\sigma}}^m : \underline{\underline{\varepsilon}}^m + \underline{\underline{\sigma}}^r : \underline{\underline{\varepsilon}}^r + \underline{I} \cdot (\underline{u}^r - \underline{u}^m)$$

Generalized strains

- strain of matrix displacement $\underline{\underline{\varepsilon}}^m$
- strain of reinforcement displacement $\underline{\underline{\varepsilon}}^r$
- relative displacement $[\![\underline{u}]\!] = \underline{u}^r - \underline{u}^m$

Generalized stresses

- partial matrix stress $\underline{\underline{\sigma}}^m$
- partial reinforcement stress $\underline{\underline{\sigma}}^r$
- interaction force $\underline{I}$

Standard Cauchy medium by imposing $\underline{u}^m = \underline{u}^r = \underline{u}$
One strain measure $\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^m = \underline{\underline{\varepsilon}}^r$ associated with the Cauchy stress $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^m + \underline{\underline{\sigma}}^r$

Multiphase model : equilibrium equations and constitutive relations

Using the virtual work principle (**same** volume force $\underline{F}$ for both phases):

$$\begin{aligned}
\operatorname{div} \underline{\underline{\sigma}}^m + \underline{I} + \rho_m \underline{F} &= \underline{0} \\
\operatorname{div} \underline{\underline{\sigma}}^r - \underline{I} + \rho_r \underline{F} &= \underline{0}
\end{aligned}$$

Multiphase model : equilibrium equations and constitutive relations

Using the virtual work principle (**same** volume force $\underline{F}$ for both phases):

$$\begin{aligned} \operatorname{div} \underline{\underline{\sigma}}^m + \underline{I} + \rho_m \underline{F} &= \underline{0} \\ \operatorname{div} \underline{\underline{\sigma}}^r - \underline{I} + \rho_r \underline{F} &= \underline{0} \end{aligned}$$

Postulated constitutive relations : ($\eta \ll 1$: reinforcement volume fraction)

[Sudret, 1999], [de Buhan and Sudret, 2000]

$$\begin{aligned} \underline{\underline{\sigma}}^m &= \mathbb{C}^m : \underline{\underline{\varepsilon}}^m \\ \underline{\underline{\sigma}}^r &= \eta \mathbb{C}^r : \underline{\underline{\varepsilon}}^r \\ \underline{I} &= \underline{\underline{c}}_I [\![\underline{u}]\!] \end{aligned}$$

Multiphase model : equilibrium equations and constitutive relations

Using the virtual work principle (**same** volume force $\underline{F}$ for both phases):

$$\begin{aligned} \operatorname{div} \underline{\underline{\sigma}}^m + \underline{I} + \rho_m \underline{F} &= \underline{0} \\ \operatorname{div} \underline{\underline{\sigma}}^r - \underline{I} + \rho_r \underline{F} &= \underline{0} \end{aligned}$$

Postulated constitutive relations : ($\eta \ll 1$: reinforcement volume fraction)

[Sudret, 1999], [de Buhan and Sudret, 2000]

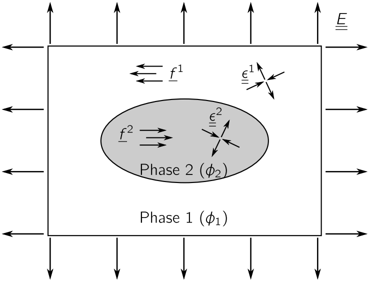
$$\begin{aligned} \underline{\underline{\sigma}}^m &= \mathbb{C}^m : \underline{\underline{\varepsilon}}^m \\ \underline{\underline{\sigma}}^r &= \eta \mathbb{C}^r : \underline{\underline{\varepsilon}}^r \\ \underline{I} &= \underline{\underline{c}}_I [\![\underline{u}]\!] \end{aligned}$$

going beyond small volume fraction hypothesis : influence of strains of one phase on stresses in the other phase ?

Auxiliary problem (periodic heterogeneities)

We consider an auxiliary problem with **microscopic body force** and **eigenstrains**:
(perfect bonding between both phases)

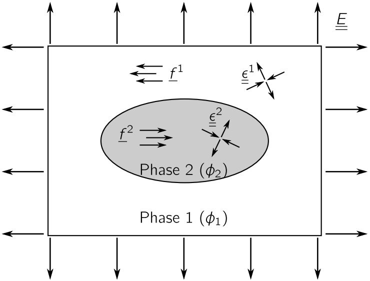
$$
\begin{aligned}
\operatorname{div}\underline{\underline{\sigma}} + \underline{f}^i &= \underline{0} && \forall \underline{y} \in \mathcal{A}^i \\
\underline{\underline{\sigma}}(\underline{y}) &= \mathbb{C}^i : (\underline{\underline{\varepsilon}}(\underline{y}) - \underline{\underline{\epsilon}}^i) && \\
\underline{U}(\underline{y}) &= \underline{\underline{E}} \cdot \underline{y} + \underline{u}(\underline{y}) && \forall \underline{y} \in \mathcal{A} \\
\underline{\underline{\sigma}} \cdot \underline{n} & && \mathcal{A}\text{-antiperiodic} \\
\underline{u}(\underline{y}) & && \mathcal{A}\text{-periodic}
\end{aligned}
$$



Auxiliary problem (periodic heterogeneities)

We consider an auxiliary problem with **microscopic body force** and **eigenstrains**: (perfect bonding between both phases)

$$
\begin{aligned}
\operatorname{div} \underline{\underline{\sigma}} + \underline{f}^i &= \underline{0} && \forall \underline{y} \in \mathcal{A}^i \\
\underline{\underline{\sigma}}(\underline{y}) &= \mathbb{C}^i : (\underline{\underline{\varepsilon}}(\underline{y}) - \underline{\underline{\epsilon}}^i) && \\
\underline{U}(\underline{y}) &= \underline{\underline{E}} \cdot \underline{y} + \underline{u}(\underline{y}) && \forall \underline{y} \in \mathcal{A} \\
\underline{\underline{\sigma}} \cdot \underline{n} & && \mathcal{A}\text{-antiperiodic} \\
\underline{u}(\underline{y}) & && \mathcal{A}\text{-periodic}
\end{aligned}
$$



where body forces and eigenstrains are **phase-wise uniform** and have **zero-average**:

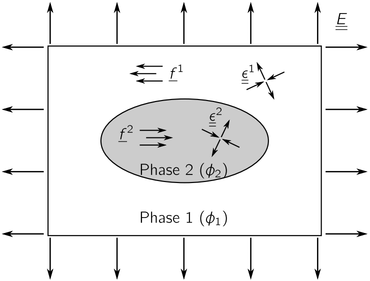
$$
\langle \underline{f} \rangle = 0 \implies \underline{f}^1 = \underline{I}/\phi_1, \qquad \underline{f}^2 = -\underline{I}/\phi_2
$$

$$
\langle \underline{\underline{\epsilon}} \rangle = 0 \implies \underline{\underline{\epsilon}}^1 = \underline{\underline{\Delta E}}/\phi_1, \quad \underline{\underline{\epsilon}}^2 = -\underline{\underline{\Delta E}}/\phi_2
$$

Auxiliary problem (periodic heterogeneities)

We consider an auxiliary problem with **microscopic body force** and **eigenstrains**: (perfect bonding between both phases)

$$
\begin{aligned}
\operatorname{div} \underline{\sigma} + \underline{f}^i &= \underline{0} && \forall \underline{y} \in \mathcal{A}^i \\
\underline{\underline{\sigma}}(\underline{y}) &= \mathbb{C}^i : (\underline{\underline{\varepsilon}}(\underline{y}) - \underline{\underline{\epsilon}}^i) && \\
\underline{U}(\underline{y}) &= \underline{\underline{E}} \cdot \underline{y} + \underline{u}(\underline{y}) && \forall \underline{y} \in \mathcal{A} \\
\underline{\underline{\sigma}} \cdot \underline{n} & && \mathcal{A}\text{-antiperiodic} \\
\underline{u}(\underline{y}) & && \mathcal{A}\text{-periodic}
\end{aligned}
$$



where body forces and eigenstrains are **phase-wise uniform** and have **zero-average**:

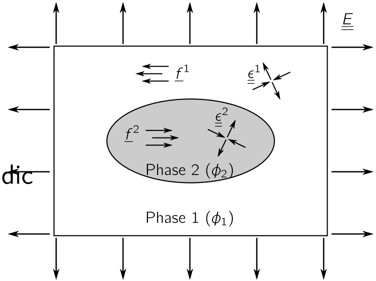
$$
\begin{aligned}
\langle \underline{f} \rangle = 0 &\implies \underline{f}^1 = \underline{I}/\phi_1, \quad \underline{f}^2 = -\underline{I}/\phi_2 \\
\langle \underline{\underline{\epsilon}} \rangle = 0 &\implies \underline{\underline{\epsilon}}^1 = \underline{\underline{\Delta E}}/\phi_1, \quad \underline{\underline{\epsilon}}^2 = -\underline{\underline{\Delta E}}/\phi_2
\end{aligned}
$$

or **equivalently** with $\underline{\underline{E}}^i = \underline{\underline{E}} - \underline{\underline{\epsilon}}^i$, $\underline{\underline{E}} = \phi_1 \underline{\underline{E}}^1 + \phi_2 \underline{\underline{E}}^2$ and $\underline{\underline{\Delta E}} = \phi_1\phi_2(\underline{\underline{E}}^2 - \underline{\underline{E}}^1)$

Auxiliary problem (periodic heterogeneities)

We consider an auxiliary problem with **microscopic body force** and **eigenstrains**: (perfect bonding between both phases)

$$
\begin{aligned}
\operatorname{div}\underline{\underline{\sigma}} \pm \underline{I}/\phi_i &= \underline{0} && \forall \underline{y} \in \mathcal{A}^i \\
\underline{\underline{\sigma}}(\underline{y}) &= \mathbb{C}^i : (\underline{\underline{E}}^i + \underline{\underline{\nabla^s \underline{u}}}(\underline{y})) && \\
\underline{U}(\underline{y}) &= (\phi_1 \underline{\underline{E}}_1 + \phi_2 \underline{\underline{E}}_2) \cdot \underline{y} + \underline{u}(\underline{y}) && \forall \underline{y} \in \mathcal{A} \\
\underline{\underline{\sigma}} \cdot \underline{n} & && \mathcal{A}\text{-antiperiodic} \\
\underline{u}(\underline{y}) & && \mathcal{A}\text{-periodic}
\end{aligned}
$$



where body forces and eigenstrains are **phase-wise uniform** and have **zero-average**:

$$
\begin{aligned}
\langle \underline{f} \rangle = 0 &\implies \underline{f}^1 = \underline{I}/\phi_1, \qquad \underline{f}^2 = -\underline{I}/\phi_2 \\
\langle \underline{\underline{\epsilon}} \rangle = 0 &\implies \underline{\underline{\epsilon}}^1 = \underline{\underline{\Delta E}}/\phi_1, \quad \underline{\underline{\epsilon}}^2 = -\underline{\underline{\Delta E}}/\phi_2
\end{aligned}
$$

or **equivalently** with $\underline{\underline{E}}^i = \underline{\underline{E}} - \underline{\underline{\epsilon}}^i$, $\underline{\underline{E}} = \phi_1 \underline{\underline{E}}^1 + \phi_2 \underline{\underline{E}}^2$ and $\underline{\underline{\Delta E}} = \phi_1 \phi_2 (\underline{\underline{E}}^2 - \underline{\underline{E}}^1)$

Auxiliary problem (resolution)

The previous problem depends on **three different loading parameters** $(\underline{\underline{E}}^1, \underline{\underline{E}}^2, \underline{I})$

Auxiliary problem (resolution)

The previous problem depends on **three different loading parameters** $(\underline{\underline{E}}^1, \underline{\underline{E}}^2, \underline{I})$
Using an extended version of Hill-Mandel's lemma, associated dual quantities are:

$$\begin{array}{ccccl} \underline{\underline{E}}^i & \longleftrightarrow & \underline{\underline{\sigma}}^i & = & \dfrac{1}{|\mathcal{A}|}\displaystyle\int_{\mathcal{A}^i} \underline{\underline{\sigma}} dA = \phi_i \langle \underline{\underline{\sigma}} \rangle^i \\ \underline{I} & \longleftrightarrow & \underline{V} & = & \langle \underline{U} \rangle^2 - \langle \underline{U} \rangle^1 = [\![\underline{U}]\!] \end{array}$$

Auxiliary problem (resolution)

The previous problem depends on **three different loading parameters** $(\underline{\underline{E}}^1, \underline{\underline{E}}^2, \underline{I})$
Using an extended version of Hill-Mandel's lemma, associated dual quantities are:

$$\begin{array}{rclcl} \underline{\underline{E}}^i & \longleftrightarrow & \underline{\underline{\sigma}}^i & = & \dfrac{1}{|\mathcal{A}|}\displaystyle\int_{\mathcal{A}^i} \underline{\underline{\sigma}} dA = \phi_i \langle \underline{\underline{\sigma}} \rangle^i \\ \underline{I} & \longleftrightarrow & \underline{V} & = & \langle \underline{U} \rangle^2 - \langle \underline{U} \rangle^1 = [\![\underline{U}]\!] \end{array}$$

Solution to the auxiliary problem depends linearly upon $(\underline{\underline{E}}^1, \underline{\underline{E}}^2, \underline{I})$:

$$\begin{array}{rcl} \underline{U}(\underline{y}) & = & \boldsymbol{a}^1(\underline{y}) : \underline{\underline{E}}^1 + \boldsymbol{a}^2(\underline{y}) : \underline{\underline{E}}^2 + \underline{\underline{d}}(\underline{y}) \cdot \underline{I} \\ \underline{\underline{\varepsilon}}(\underline{y}) & = & \mathbb{A}^1(\underline{y}) : \underline{\underline{E}}^1 + \mathbb{A}^2(\underline{y}) : \underline{\underline{E}}^2 + \boldsymbol{D}(\underline{y}) \cdot \underline{I} \end{array}$$

Auxiliary problem (resolution)

The previous problem depends on **three different loading parameters** $(\underline{\underline{E}}^1, \underline{\underline{E}}^2, \underline{I})$
Using an extended version of Hill-Mandel's lemma, associated dual quantities are:

$$\begin{aligned} \underline{\underline{E}}^i \quad &\longleftrightarrow \quad \underline{\underline{\sigma}}^i \quad = \quad \frac{1}{|\mathcal{A}|}\int_{\mathcal{A}^i} \underline{\underline{\sigma}} dA = \phi_i \langle \underline{\underline{\sigma}} \rangle^i \\ \underline{I} \quad &\longleftrightarrow \quad \underline{V} \quad = \quad \langle \underline{U} \rangle^2 - \langle \underline{U} \rangle^1 = [\![\underline{U}]\!] \end{aligned}$$

Solution to the auxiliary problem depends linearly upon $(\underline{\underline{E}}^1, \underline{\underline{E}}^2, \underline{I})$:

$$\begin{aligned} \underline{U}(\underline{y}) \quad &= \quad \boldsymbol{a}^1(\underline{y}) : \underline{\underline{E}}^1 + \boldsymbol{a}^2(\underline{y}) : \underline{\underline{E}}^2 + \underline{\underline{d}}(\underline{y}) \cdot \underline{I} \\ \underline{\underline{\varepsilon}}(\underline{y}) \quad &= \quad \mathbb{A}^1(\underline{y}) : \underline{\underline{E}}^1 + \mathbb{A}^2(\underline{y}) : \underline{\underline{E}}^2 + \boldsymbol{D}(\underline{y}) \cdot \underline{I} \end{aligned}$$

Using the previous relations:

$$\begin{aligned} \underline{\underline{\sigma}}^1 &= \mathbb{D}^{11} : \underline{\underline{E}}^1 + \mathbb{D}^{12} : \underline{\underline{E}}^2 + \boldsymbol{D}^1 \cdot \underline{I} \\ \underline{\underline{\sigma}}^2 &= \mathbb{D}^{21} : \underline{\underline{E}}^1 + \mathbb{D}^{22} : \underline{\underline{E}}^2 + \boldsymbol{D}^2 \cdot \underline{I} \\ \underline{V} &= [\![\boldsymbol{a}^1]\!] : \underline{\underline{E}}^1 + [\![\boldsymbol{a}^2]\!] : \underline{\underline{E}}^2 + [\![\underline{\underline{d}}]\!] \cdot \underline{I} \end{aligned}$$

Homogenized constitutive equations

Symmetry relations are obtained from the Maxwell-Betti reciprocity theorem:

$$\begin{aligned}
\underline{\underline{\sigma}}^1 &= \mathbb{D}^{11} : \underline{\underline{E}}^1 + \mathbb{D}^{12} : \underline{\underline{E}}^2 + \boldsymbol{D}^1 \cdot \underline{I} \\
\underline{\underline{\sigma}}^2 &= (\mathbb{D}^{12})^\mathsf{T} : \underline{\underline{E}}^1 + \mathbb{D}^{22} : \underline{\underline{E}}^2 + \boldsymbol{D}^2 \cdot \underline{I} \\
\underline{V} &= (\boldsymbol{D}^1)^\mathsf{T} : \underline{\underline{E}}^1 + (\boldsymbol{D}^2)^\mathsf{T} : \underline{\underline{E}}^2 + [\![\underline{\underline{d}}]\!] \cdot \underline{I}
\end{aligned}$$

Homogenized constitutive equations

Symmetry relations are obtained from the Maxwell-Betti reciprocity theorem:

$$\begin{aligned}
\underline{\underline{\sigma}}^1 &= \mathbb{D}^{11} : \underline{\underline{E}}^1 + \mathbb{D}^{12} : \underline{\underline{E}}^2 + \boldsymbol{D}^1 \cdot \underline{I} \\
\underline{\underline{\sigma}}^2 &= (\mathbb{D}^{12})^\mathsf{T} : \underline{\underline{E}}^1 + \mathbb{D}^{22} : \underline{\underline{E}}^2 + \boldsymbol{D}^2 \cdot \underline{I} \\
\underline{V} &= (\boldsymbol{D}^1)^\mathsf{T} : \underline{\underline{E}}^1 + (\boldsymbol{D}^2)^\mathsf{T} : \underline{\underline{E}}^2 + [\![\underline{\underline{d}}]\!] \cdot \underline{I}
\end{aligned}$$

summing up the first two equations in the case $\underline{\underline{E}}^1 = \underline{\underline{E}}^2 = \underline{\underline{E}}$ and $\underline{I} = \underline{0}$ gives $\underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^1 + \underline{\underline{\sigma}}^2 = \mathbb{C}^{hom} : \underline{\underline{E}}$

Homogenized constitutive equations

Symmetry relations are obtained from the Maxwell-Betti reciprocity theorem:

$$\begin{aligned}
\underline{\underline{\sigma}}^1 &= \mathbb{D}^{11} : \underline{\underline{E}}^1 + \mathbb{D}^{12} : \underline{\underline{E}}^2 + \boldsymbol{D}^1 \cdot \underline{I} \\
\underline{\underline{\sigma}}^2 &= (\mathbb{D}^{12})^\mathsf{T} : \underline{\underline{E}}^1 + \mathbb{D}^{22} : \underline{\underline{E}}^2 + \boldsymbol{D}^2 \cdot \underline{I} \\
\underline{V} &= (\boldsymbol{D}^1)^\mathsf{T} : \underline{\underline{E}}^1 + (\boldsymbol{D}^2)^\mathsf{T} : \underline{\underline{E}}^2 + [\![\underline{\underline{d}}]\!] \cdot \underline{I}
\end{aligned}$$

summing up the first two equations in the case $\underline{\underline{E}}^1 = \underline{\underline{E}}^2 = \underline{\underline{E}}$ and $\underline{I} = \underline{0}$ gives $\underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^1 + \underline{\underline{\sigma}}^2 = \mathbb{C}^{hom} : \underline{\underline{E}}$

Unit cell with central symmetry

$\boldsymbol{D}^1 = \boldsymbol{D}^2 = \mathbf{0}$ so that the partial stress constitutive equations decouple from the interaction force :

$$\begin{aligned}
\underline{\underline{\sigma}}^1 &= \mathbb{D}^{11} : \underline{\underline{E}}^1 + \mathbb{D}^{12} : \underline{\underline{E}}^2 \\
\underline{\underline{\sigma}}^2 &= (\mathbb{D}^{12})^\mathsf{T} : \underline{\underline{E}}^1 + \mathbb{D}^{22} : \underline{\underline{E}}^2 \\
\underline{I} &= [\![\underline{\underline{d}}]\!]^{-1} \cdot \underline{V}
\end{aligned}$$

Link with the multiphase model

An other way to build the auxiliary problem

Link with the multiphase model

An other way to build the auxiliary problem

From the definition of $\underline{I}$, interpreted as the resultant force acting on the interface:

$$\underline{I} = \frac{1}{|\mathcal{A}|}\int_{\Gamma} \underline{\underline{\sigma}} \cdot \underline{n}_{2\to 1} dS$$

Link with the multiphase model

An other way to build the auxiliary problem

From the definition of $\underline{I}$, interpreted as the resultant force acting on the interface:

$$\underline{I} = \frac{1}{|\mathcal{A}|} \int_\Gamma \underline{\underline{\sigma}} \cdot \underline{n}_{2\to 1} dS$$

From [Coussy, 1998], partial stress balance at the **macroscopic scale** is:

$$\mathrm{div}_x \, \underline{\underline{\sigma}}^i + \rho_i \underline{F} \pm \underline{I} = \underline{0}$$

Link with the multiphase model

An other way to build the auxiliary problem

From the definition of $\underline{I}$, interpreted as the resultant force acting on the interface:

$$\underline{I} = \frac{1}{|\mathcal{A}|} \int_\Gamma \underline{\underline{\sigma}} \cdot \underline{n}_{2\to 1} dS$$

From [Coussy, 1998], partial stress balance at the **macroscopic scale** is:

$$\mathrm{div}_x \, \underline{\underline{\sigma}}^i + \rho_i \underline{F} \pm \underline{I} = \underline{0}$$

$\underline{\underline{\sigma}}^i$ and $\underline{I}$ are indeed the **pertinent generalized forces** for the multiphase model

Link with the multiphase model

An other way to build the auxiliary problem

From the definition of $\underline{I}$, interpreted as the resultant force acting on the interface:

$$\underline{I} = \frac{1}{|\mathcal{A}|}\int_\Gamma \underline{\underline{\sigma}} \cdot \underline{n}_{2\to 1} dS$$

From [Coussy, 1998], partial stress balance at the **macroscopic scale** is:

$$\operatorname{div}_x \underline{\underline{\sigma}}^i + \rho_i \underline{F} \pm \underline{I} = \underline{0}$$

$\underline{\underline{\sigma}}^i$ and $\underline{I}$ are indeed the **pertinent generalized forces** for the multiphase model

Previous resolution gives a **macroscopic complementary energy** $\Psi^*(\underline{\underline{\sigma}}^i, \underline{I})$

Multiphase constitutive relations:

$$\underline{\underline{\varepsilon}}^i = \frac{\partial \Psi^*}{\partial \underline{\underline{\sigma}}^i} \quad \longleftarrow \underline{\underline{E}}^i$$

$$[\![\underline{u}]\!] = \frac{\partial \Psi^*}{\partial \underline{I}} \quad \longleftarrow \underline{V}$$

Outline

1. A homogenization procedure from Cauchy to multiphase continua
2. **Some analytical results**
3. Illustrative applications
4. Conclusions

Constitutive relations for a biphasic material

$\mathbb{D}^{ij}$ can be deduced from the knowledge of $\mathbb{C}^{hom}$:

$$\begin{aligned}
\mathbb{D}^{11} &= \phi_1 \mathbb{C}^1 - \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^1 \\
\mathbb{D}^{22} &= \phi_2 \mathbb{C}^2 - \mathbb{C}^2 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2 \\
\mathbb{D}^{12} &= \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2
\end{aligned}$$

with $[\![\mathbb{C}]\!] = \mathbb{C}^2 - \mathbb{C}^1$ and $\Delta\mathbb{C} = \langle\mathbb{C}\rangle - \mathbb{C}^{hom}$

Constitutive relations for a biphasic material

$\mathbb{D}^{ij}$ can be deduced from the knowledge of $\mathbb{C}^{hom}$:

$$\begin{aligned}
\mathbb{D}^{11} &= \phi_1\mathbb{C}^1 - \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^1 \\
\mathbb{D}^{22} &= \phi_2\mathbb{C}^2 - \mathbb{C}^2 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2 \\
\mathbb{D}^{12} &= \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2
\end{aligned}$$

with $[\![\mathbb{C}]\!] = \mathbb{C}^2 - \mathbb{C}^1$ and $\Delta\mathbb{C} = \langle\mathbb{C}\rangle - \mathbb{C}^{hom}$

Stiff linear isotropic inclusions in small volume fraction

Assumptions: $\phi_2 \ll 1$ and $\lim\limits_{\phi_2\to 0} \phi_2\mathbb{C}^2 = \mathbb{C}^0$

Constitutive relations for a biphasic material

$\mathbb{D}^{ij}$ can be deduced from the knowledge of $\mathbb{C}^{hom}$:

$$\begin{aligned}
\mathbb{D}^{11} &= \phi_1\mathbb{C}^1 - \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^1 \\
\mathbb{D}^{22} &= \phi_2\mathbb{C}^2 - \mathbb{C}^2 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2 \\
\mathbb{D}^{12} &= \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2
\end{aligned}$$

with $[\![\mathbb{C}]\!] = \mathbb{C}^2 - \mathbb{C}^1$ and $\Delta\mathbb{C} = \langle\mathbb{C}\rangle - \mathbb{C}^{hom}$

Stiff linear isotropic inclusions in small volume fraction

Assumptions: $\phi_2 \ll 1$ and $\lim\limits_{\phi_2\to 0} \phi_2\mathbb{C}^2 = \mathbb{C}^0$

using [Hashin and Rosen, 1964]:

$$\lim_{\phi_2\to 0} \mathbb{C}^{hom} = \mathbb{C}^{hom,0} = \mathbb{C}^1 + \mathsf{E}^0 \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1$$

where $\mathsf{E}^0 = \lim \phi_2 \mathsf{E}^2 = \mathbb{C}^0_{1111}$

Constitutive relations for a biphasic material

$\mathbb{D}^{ij}$ can be deduced from the knowledge of $\mathbb{C}^{hom}$:

$$\begin{aligned}
\mathbb{D}^{11} &= \phi_1\mathbb{C}^1 - \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^1 \\
\mathbb{D}^{22} &= \phi_2\mathbb{C}^2 - \mathbb{C}^2 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2 \\
\mathbb{D}^{12} &= \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2
\end{aligned}$$

with $[\![\mathbb{C}]\!] = \mathbb{C}^2 - \mathbb{C}^1$ and $\Delta\mathbb{C} = \langle\mathbb{C}\rangle - \mathbb{C}^{hom}$

Stiff linear isotropic inclusions in small volume fraction

Assumptions: $\phi_2 \ll 1$ and $\lim\limits_{\phi_2\to 0} \phi_2\mathbb{C}^2 = \mathbb{C}^0$

using [Hashin and Rosen, 1964]:

$$\lim_{\phi_2\to 0} \mathbb{C}^{hom} = \mathbb{C}^{hom,0} = \mathbb{C}^1 + \mathsf{E}^0 \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1$$

where $\mathsf{E}^0 = \lim \phi_2 \mathsf{E}^2 = \mathbb{C}^0_{1111}$, then

$$\begin{aligned}
\mathbb{D}^{11} &\to \mathbb{C}^1 \\
\mathbb{D}^{22} &\to \mathbb{C}^{hom,0} - \mathbb{C}^1 = \mathsf{E}^0 \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1 \\
\mathbb{D}^{12} &\to 0
\end{aligned}$$

Constitutive relations for a biphasic material

$\mathbb{D}^{ij}$ can be deduced from the knowledge of $\mathbb{C}^{hom}$:

$$\begin{aligned}
\mathbb{D}^{11} &= \phi_1\mathbb{C}^1 - \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^1 \\
\mathbb{D}^{22} &= \phi_2\mathbb{C}^2 - \mathbb{C}^2 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2 \\
\mathbb{D}^{12} &= \mathbb{C}^1 : [\![\mathbb{C}]\!]^{-1} : \Delta\mathbb{C} : [\![\mathbb{C}]\!]^{-1} : \mathbb{C}^2
\end{aligned}$$

with $[\![\mathbb{C}]\!] = \mathbb{C}^2 - \mathbb{C}^1$ and $\Delta\mathbb{C} = \langle\mathbb{C}\rangle - \mathbb{C}^{hom}$

Stiff linear isotropic inclusions in small volume fraction

Assumptions: $\phi_2 \ll 1$ and $\lim\limits_{\phi_2\to 0} \phi_2\mathbb{C}^2 = \mathbb{C}^0$

using [Hashin and Rosen, 1964]:

$$\lim_{\phi_2\to 0} \mathbb{C}^{hom} = \mathbb{C}^{hom,0} = \mathbb{C}^1 + \mathsf{E}^0 \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1$$

where $\mathsf{E}^0 = \lim \phi_2 \mathsf{E}^2 = \mathbb{C}^0_{1111}$, then

$$\begin{aligned}
\underline{\underline{\sigma}}^m &= \mathbb{C}^m : \underline{\underline{\varepsilon}}^m \\
\sigma^r_{11} &= \mathsf{E}^0 \varepsilon^r_{11}, \quad \sigma^r_{ij} = 0 \\
\underline{I} &= [\![\underline{\underline{d}}]\!]^{-1} \cdot [\![\underline{u}]\!]
\end{aligned}$$

Determination of the interaction stiffness

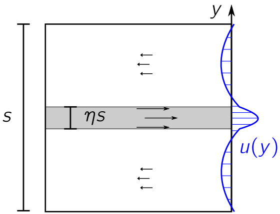
Auxiliary problem with body forces only: similar to homogenization of **permeability in porous media** $\Rightarrow [\![\underline{\underline{d}}]\!]$ **will depend on the UC size** s as s^2

Determination of the interaction stiffness

Auxiliary problem with body forces only: similar to homogenization of **permeability in porous media** $\Rightarrow [\![\underline{\underline{d}}]\!]$ **will depend on the UC size** s as s^2

Example for a 2D layered medium (1: matrix, 2: reinforcement, η reinforcement volume fraction):

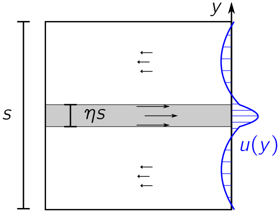
$$[\![d_{11}]\!] = \langle u(y) \rangle^2 - \langle u(y) \rangle^1$$



Determination of the interaction stiffness

Auxiliary problem with body forces only: similar to homogenization of **permeability in porous media** $\Rightarrow$ $[\![\underline{\underline{d}}]\!]$ **will depend on the UC size** s as s^2

Example for a 2D layered medium (1: matrix, 2: reinforcement, η reinforcement volume fraction):



$$[\![d_{11}]\!] = \langle u(y) \rangle^2 - \langle u(y) \rangle^1$$

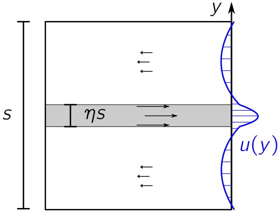
$$[\![\underline{\underline{d}}]\!] = \frac{s^2}{12} \begin{bmatrix} \langle 1/\mu \rangle & 0 \\ 0 & \langle 1/(\lambda + 2\mu) \rangle \end{bmatrix}$$

Determination of the interaction stiffness

Auxiliary problem with body forces only: similar to homogenization of **permeability in porous media** $\Rightarrow [\![\underline{\underline{d}}]\!]$ **will depend on the UC size** s as s^2

Example for a 2D layered medium (1: matrix, 2: reinforcement, η reinforcement volume fraction):

$$[\![d_{11}]\!] = \langle u(y)\rangle^2 - \langle u(y)\rangle^1$$



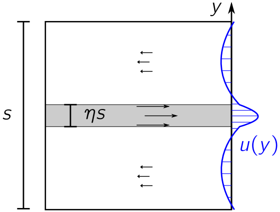
$$[\![\underline{\underline{d}}]\!] = \frac{s^2}{12}\begin{bmatrix}\langle 1/\mu\rangle & 0 \\ 0 & \langle 1/(\lambda+2\mu)\rangle\end{bmatrix} \xrightarrow[\mathbb{C}^r \gg \mathbb{C}^m]{} \frac{s^2(1-\eta)}{12}\begin{bmatrix}\dfrac{1}{\mu_m} & 0 \\ 0 & \dfrac{1}{\lambda_m+2\mu_m}\end{bmatrix}$$

Determination of the interaction stiffness

Auxiliary problem with body forces only: similar to homogenization of **permeability in porous media** $\Rightarrow [\![\underline{\underline{d}}]\!]$ **will depend on the UC size** s as s^2

Example for a 2D layered medium (1: matrix, 2: reinforcement, η reinforcement volume fraction):

$$[\![d_{11}]\!] = \langle u(y) \rangle^2 - \langle u(y) \rangle^1$$



$$[\![\underline{\underline{d}}]\!] = \frac{s^2}{12}\begin{bmatrix} \langle 1/\mu \rangle & 0 \\ 0 & \langle 1/(\lambda+2\mu) \rangle \end{bmatrix} \xrightarrow[\mathbb{C}^r \gg \mathbb{C}^m]{} \frac{s^2(1-\eta)}{12}\begin{bmatrix} \dfrac{1}{\mu_m} & 0 \\ 0 & \dfrac{1}{\lambda_m + 2\mu_m} \end{bmatrix}$$

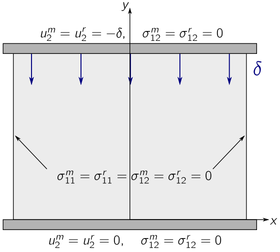
[Sudret, 1999] estimate : pullout test on **rigid** inclusion $[\![d_{11}]\!] = \dfrac{s^2(1-\eta)}{8\mu_m}$

Outline

1. A homogenization procedure from Cauchy to multiphase continua
2. Some analytical results
3. **Illustrative applications**
4. Conclusions

A multilayered block in compression

Vertical displacement is $u_2(y) = -\dfrac{\delta}{H}y$ for both phases,

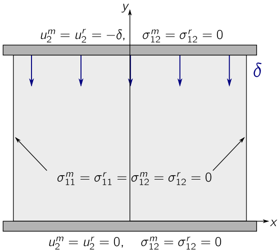


A multilayered block in compression

Vertical displacement is $u_2(y) = -\dfrac{\delta}{H}y$ for both phases, horizontal displacement $u_1^i(x)$ is different for each phase and satisfies:

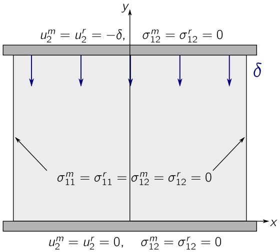
$$\alpha_{11}\frac{d^2u_1^m}{dx^2} + \alpha_{12}\frac{d^2u_1^r}{dx^2} + c_I(u_1^r - u_1^m) = 0$$
$$\alpha_{12}\frac{d^2u_1^m}{dx^2} + \alpha_{22}\frac{d^2u_1^r}{dx^2} - c_I(u_1^r - u_1^m) = 0$$

with $\alpha_{ij} = \mathbb{D}^{ij}_{1111}$ and $c_I = [\![d_{11}]\!]^{-1}$



A multilayered block in compression

Vertical displacement is $u_2(y) = -\dfrac{\delta}{H}y$ for both phases, horizontal displacement $u_1^i(x)$ is different for each phase and satisfies:



$$\alpha_{11}\frac{d^2u_1^m}{dx^2} + \alpha_{12}\frac{d^2u_1^r}{dx^2} + c_I(u_1^r - u_1^m) = 0$$
$$\alpha_{12}\frac{d^2u_1^m}{dx^2} + \alpha_{22}\frac{d^2u_1^r}{dx^2} - c_I(u_1^r - u_1^m) = 0$$

with $\alpha_{ij} = \mathbb{D}_{1111}^{ij}$ and $c_I = [\![d_{11}]\!]^{-1}$
Solution is of the form:

$$u^m(x) = \frac{\delta}{H}\left(\nu^{hom}x + s\beta^m\frac{\sinh(x/\ell)}{\cosh(L/\ell)}\right)$$
$$u^r(x) = \frac{\delta}{H}\left(\nu^{hom}x + s\beta^r\frac{\sinh(x/\ell)}{\cosh(L/\ell)}\right)$$

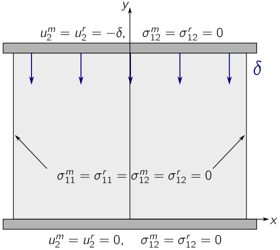
where $\ell \propto 1/\sqrt{c_I} \propto s$ is an internal length scale arising for this problem

A multilayered block in compression

Vertical displacement is $u_2(y) = -\dfrac{\delta}{H} y$ for both phases,

horizontal displacement $u_1^i(x)$ is different for each phase and satisfies:

$$\alpha_{11} \frac{d^2 u_1^m}{dx^2} + \alpha_{12} \frac{d^2 u_1^r}{dx^2} + c_I (u_1^r - u_1^m) = 0$$
$$\alpha_{12} \frac{d^2 u_1^m}{dx^2} + \alpha_{22} \frac{d^2 u_1^r}{dx^2} - c_I (u_1^r - u_1^m) = 0$$



with $\alpha_{ij} = \mathbb{D}_{1111}^{ij}$ and $c_I = [\![d_{11}]\!]^{-1}$

Solution is of the form:

$$u^m(x) = \frac{\delta}{H} \left(\nu^{hom} x + s\beta^m \frac{\sinh(x/\ell)}{\cosh(L/\ell)} \right)$$
$$u^r(x) = \frac{\delta}{H} \left(\nu^{hom} x + s\beta^r \frac{\sinh(x/\ell)}{\cosh(L/\ell)} \right)$$

where $\ell \propto 1/\sqrt{c_I} \propto s$ is an internal length scale arising for this problem

first-order corrections over a distance $\approx \ell$

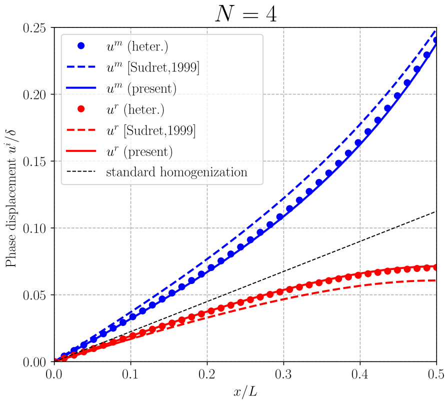
Comparisons

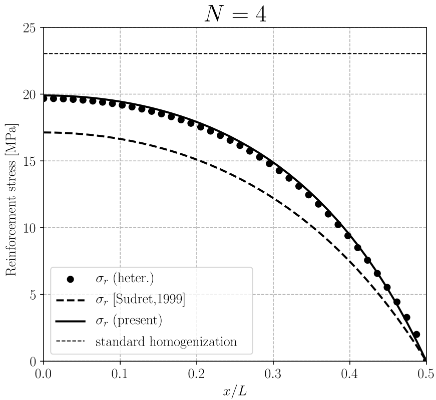
Comparison between full heterogeneous computations, Sudret's model and the present model : N is the number of layers

	Matrix (phase 1)	Reinforcement (phase 2)
Volume fraction	$1-\eta = 0.9$	$\eta = 0.1$
Young's modulus	$E_1 = 10$ MPa	$E_2 = 1000$ MPa
Poisson ratio	$\nu_1 = 0.45$	$\nu_2 = 0.3$

Comparisons

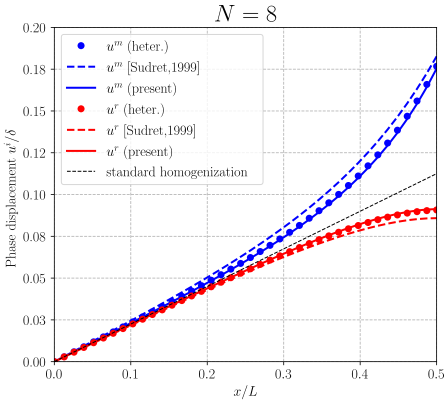
Comparison between full heterogeneous computations, Sudret's model and the present model : N is the number of layers

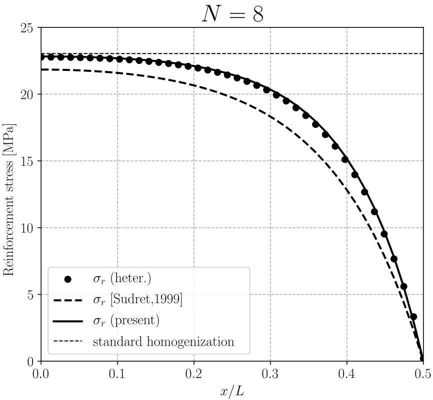




Comparisons

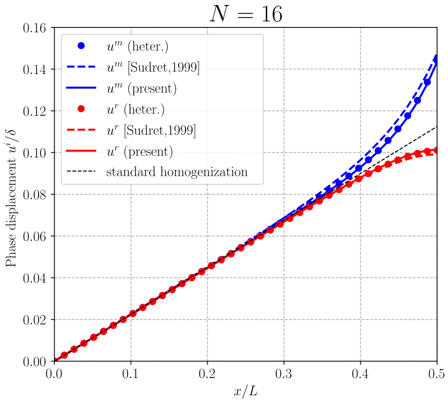
Comparison between full heterogeneous computations, Sudret's model and the present model : N is the number of layers

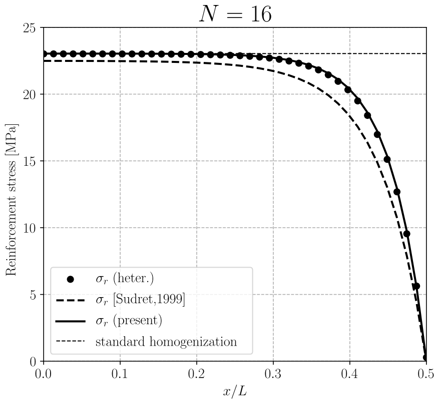




Comparisons

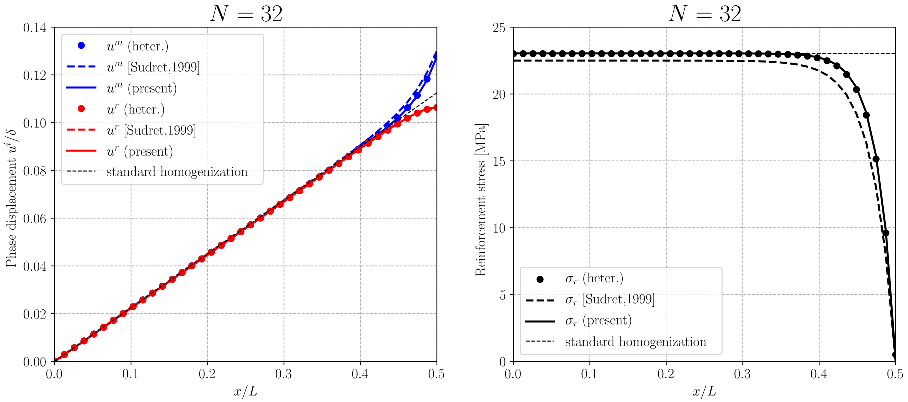
Comparison between full heterogeneous computations, Sudret's model and the present model : N is the number of layers





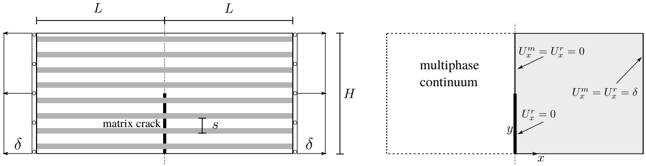
Comparisons

Comparison between full heterogeneous computations, Sudret's model and the present model : N is the number of layers



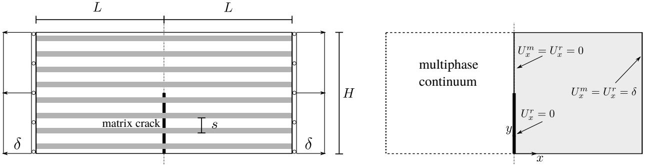
A crack-bridging problem

Appeal of such models: prescribe **different boundary conditions for each phase**

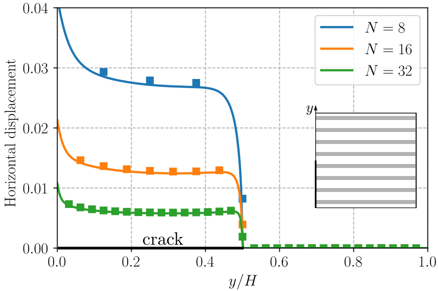


A crack-bridging problem

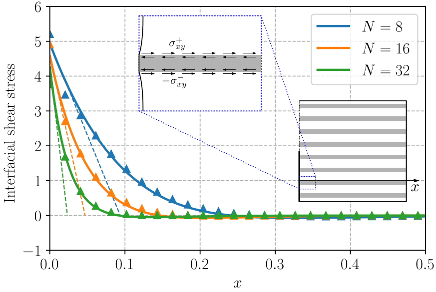
Appeal of such models: prescribe **different boundary conditions for each phase**



Heterogeneous FE computations with $N = 8, 16, 32$ layers (filled symbols)



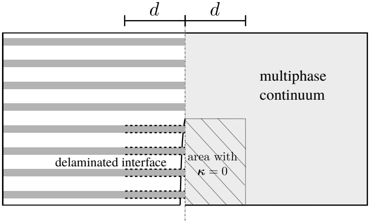
Matrix displacement



Interfacial shear stress

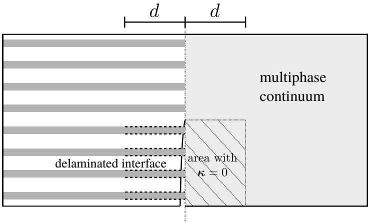
A crack-bridging problem: delaminated interfaces

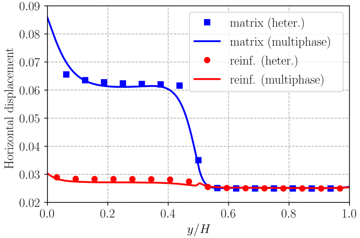
Modelling a region with fully delaminated interfaces (zero interaction stiffness)



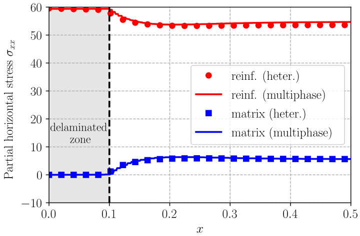
A crack-bridging problem: delaminated interfaces

Modelling a region with fully delaminated interfaces (zero interaction stiffness)





Matrix displacement



Horizontal stress

Outline

1 A homogenization procedure from Cauchy to multiphase continua

2 Some analytical results

3 Illustrative applications

4 Conclusions

Conclusions and perspectives

Conclusions

- multiphase models : phenomenological constitutive relations for fiber-reinforced materials
- homogenization procedure to identify constitutive parameters
- capture boundary effects which are important for large stiffness contrast
- some advantages over other generalized continua
- retrieves results of **shear lag models**

Conclusions and perspectives

Conclusions

- multiphase models : phenomenological constitutive relations for fiber-reinforced materials
- homogenization procedure to identify constitutive parameters
- capture boundary effects which are important for large stiffness contrast
- some advantages over other generalized continua
- retrieves results of **shear lag models**

Perspectives

- justification through asymptotic analysis, range of application ?
- micromechanical estimates
- higher-grade version to **include bending effects**
- **non-linear** constitutive relations

Conclusions and perspectives

Conclusions

- multiphase models : phenomenological constitutive relations for fiber-reinforced materials
- homogenization procedure to identify constitutive parameters
- capture boundary effects which are important for large stiffness contrast
- some advantages over other generalized continua
- retrieves results of **shear lag models**

Perspectives

- justification through asymptotic analysis, range of application ?
- micromechanical estimates
- higher-grade version to **include bending effects**
- **non-linear** constitutive relations

Thank you for your attention!

A stress-based auxiliary problem

Standard homogenization:

$$
\begin{aligned}
\Psi^*(\underline{\underline{\Sigma}}) = \min_{\underline{\underline{\sigma}}} \quad & \frac{1}{2|\mathcal{A}|}\int_{\mathcal{A}^1} \underline{\underline{\sigma}} : \mathbb{S}^1 : \underline{\underline{\sigma}} d\Omega + \frac{1}{2|\mathcal{A}|}\int_{\mathcal{A}^2} \underline{\underline{\sigma}} : \mathbb{S}^2 : \underline{\underline{\sigma}} d\Omega \\
\text{s.t.} \quad & \operatorname{div} \underline{\underline{\sigma}} = \underline{0} \\
& \underline{\underline{\sigma}} \cdot \underline{n} \quad \mathcal{A}\text{-antiperiodic} \\
& \langle \underline{\underline{\sigma}} \rangle = \underline{\underline{\Sigma}}
\end{aligned}
$$

A stress-based auxiliary problem

Extended auxiliary problem:

$$\begin{aligned}
\Psi^*(\underline{\underline{\sigma}}^1, \underline{\underline{\sigma}}^2, \underline{I}) = \min_{\underline{\underline{\sigma}}} \quad & \frac{1}{2|\mathcal{A}|}\int_{\mathcal{A}^1} \underline{\underline{\sigma}} : \mathbb{S}^1 : \underline{\underline{\sigma}} d\Omega + \frac{1}{2|\mathcal{A}|}\int_{\mathcal{A}^2} \underline{\underline{\sigma}} : \mathbb{S}^2 : \underline{\underline{\sigma}} d\Omega \\
\text{s.t.} \quad & \operatorname{div} \underline{\underline{\sigma}} + \underline{f}(\underline{x}) = \underline{0} \\
& \underline{\underline{\sigma}} \cdot \underline{n} \quad \mathcal{A}\text{-antiperiodic} \\
& \phi_i \langle \underline{\underline{\sigma}} \rangle^i = \underline{\underline{\sigma}}^i \quad i = 1, 2 \\
& \frac{1}{|\mathcal{A}|}\int_{\Gamma} \underline{\underline{\sigma}} \cdot \underline{n} dS = \underline{I}
\end{aligned}$$

with body forces at the microscopic scale.

A stress-based auxiliary problem

Extended auxiliary problem:

$$\begin{aligned}\Psi^*(\underline{\underline{\sigma}}^1, \underline{\underline{\sigma}}^2, \underline{I}) = \min_{\underline{\underline{\sigma}}} \quad & \frac{1}{2|\mathcal{A}|}\int_{\mathcal{A}^1} \underline{\underline{\sigma}} : \mathbb{S}^1 : \underline{\underline{\sigma}} d\Omega + \frac{1}{2|\mathcal{A}|}\int_{\mathcal{A}^2} \underline{\underline{\sigma}} : \mathbb{S}^2 : \underline{\underline{\sigma}} d\Omega \\ \text{s.t.} \quad & \operatorname{div} \underline{\underline{\sigma}} + \underline{f}(\underline{x}) = \underline{0} \\ & \underline{\underline{\sigma}} \cdot \underline{n} \quad \mathcal{A}\text{-antiperiodic} \\ & \phi_i \langle \underline{\underline{\sigma}} \rangle^i = \underline{\underline{\sigma}}^i \quad i = 1, 2 \\ & \frac{1}{|\mathcal{A}|}\int_{\Gamma} \underline{\underline{\sigma}} \cdot \underline{n} dS = \underline{I}\end{aligned}$$

with body forces at the microscopic scale. Divergence theorem on phase 2 and on phase 1 gives :

$$\frac{1}{|\mathcal{A}|}\int_{\mathcal{A}^2} \left(\operatorname{div} \underline{\underline{\sigma}} + \underline{f}(\underline{x})\right) d\Omega = \underline{I} + \phi_2 \langle \underline{f} \rangle^2$$

$$\frac{1}{|\mathcal{A}|}\int_{\mathcal{A}^1} \left(\operatorname{div} \underline{\underline{\sigma}} + \underline{f}(\underline{x})\right) d\Omega = -\underline{I} + \phi_1 \langle \underline{f} \rangle^1$$

Return