Linear Elastic Fracture Mechanics

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The plan

• Introduction to fracture mechanics  
  – April 30, 2014

• Quasi-static instability problems (JB Leblond)  
  – May 6, 2014

• Introduction to dynamic fracture mechanics  
  – May 15, 2014

• Dynamic instabilities during fast fracture  
  – May 21, 2014
How strong is a solid? – 1. An atomic point of view

\[ F = F_{\text{max}} \sin \left( \frac{2\pi(a - a_0)}{\lambda} \right) \]

\[ \sigma = \sigma_{\text{max}} \sin \left( \frac{2\pi a_0}{\lambda} \varepsilon \right), \quad \varepsilon = \frac{a - a_0}{a_0} \]

\[ E = \left. \frac{d\sigma}{d\varepsilon} \right|_{\varepsilon \to 0} = \sigma_{\text{max}} \frac{2\pi a_0}{\lambda} \cos \left( \frac{2\pi a_0}{\lambda} \varepsilon \right) \bigg|_{\varepsilon \to 0} \]

\[ = \sigma_{\text{max}} \frac{2\pi a_0}{\lambda} \Rightarrow \sigma_{\text{max}} = \frac{E}{\pi} \frac{\lambda}{2a_0} \sim \frac{E}{\pi} \]

<table>
<thead>
<tr>
<th>Material</th>
<th>Modulus GPa</th>
<th>Strength GPa</th>
<th>Strength/Modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steels</td>
<td>200</td>
<td>0.10 – 2</td>
<td>0.0005 – 0.01</td>
</tr>
<tr>
<td>Glass</td>
<td>70</td>
<td>0.17</td>
<td>0.0025</td>
</tr>
<tr>
<td>Carbon fibers</td>
<td>400</td>
<td>4</td>
<td>0.01</td>
</tr>
<tr>
<td>Glass fibers</td>
<td>70</td>
<td>11</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Macroscopic strength is significantly smaller than the theoretical strength
How strong is a solid? – 2. The role of defects

\[ \sigma_{22}(a, 0) = \sigma \left( 1 + 2 \frac{a}{b} \right) \Rightarrow \sigma_{22}(a, 0) \approx 2\sigma \left( \frac{a}{\rho} \right)^{\frac{1}{2}} \text{ where } \rho = \frac{b^2}{a} \]

\[ \sigma_f \sqrt{a} \approx \frac{1}{2} \sigma_{\text{max}} \sqrt{\rho} = \text{constant} \]

\[ \rho \sim 1E-10 \text{ m} \]
\[ a \sim 1E-6 \text{ m} \] \Rightarrow \sigma_f \sim 0.005\sigma_{\text{max}}

Griffith’s experiments

1. Used experiments on glass tubes and glass bulbs loaded under internal pressure to show that \( \sigma \sqrt{a} \) was constant

2. Manufactured fresh glass fibers with diameters in the range of 1 mm to 3 microns to show that small fibers had strength of about 11 GPa
The continuum view of fracture

- Process of fracture can be cleavage, intergranular/ transgranular fracture (polycrystalline materials), cavitation (ductile metals), disentanglement ( polymers), microcracking (glasses,...), fiber breakage,...
- Details of processes within $L_p$ are not important; only the total energy needed for the fracture process is assumed to play a role in the development of the fracture
- $L_p$ is small – “small-scale process zone” – what does this mean?
The energy balance – a continuum view

Total energy of the system: \( E = \Pi + U_s \)

where \( U_s \) … surface energy (or fracture energy)

\[
\Pi = -W_{\partial R} + U_R \quad \text{… potential energy of the body}
\]

\( W_{\partial R} \) … work done by the external forces on the body

\( U_R \) … strain energy stored in the body

\[
G(a) \equiv - \frac{d\Pi}{da} \quad \text{… Energy Release Rate}
\]

define

\[
R \equiv \frac{dU_s}{da} \quad \text{… fracture resistance}
\]

At equilibrium, \( E'(a) = 0 \Rightarrow \)

\[
G\left(a_c\right) = R \quad \text{… Fracture criterion}
\]

Stable if \( E''(a_c) > 0 \)
Remarks 1: \[ G(a_c) = R \ldots \text{Fracture criterion} \]

- \( a_c \) is the equilibrium crack length (reversible)
- Fracture resistance \( R(a) \) includes the effect of all dissipative fracture processes and is typically calibrated from experiments.
- For Linearly Elastic Fracture Mechanics (LEFM), the region outside \( L_p \) must exhibit linear elastic behavior, but this is not a general requirement.
- Other than \( L_p \) being “small”, there is no length scale here! The theory works at length scales from the atomic to the tectonic.
- Need methods to calculate \( G(a) \) for specific crack problems
Remarks 2: \[ G(a_c) = R \] Fracture criterion

- \( R \) varies over several orders of magnitude
  - True surface energy is \( \sim O(1) \) J/m\(^2\)
  - Glasses and ceramics \( \sim 10 \) J/m\(^2\)
  - Polymers \( \sim 1 \) kJ/m\(^2\)
  - Metals \( \sim 100 \) kJ/m\(^2\)
- Differences arise due to different mechanisms of deformation and failure
- Must be determined through calibration experiments, such as the pioneering work of Obreimoff (1930)
Calculation of $G(a)$

$C_M$ … compliance of the loading system

- $C_M = 0 \implies$ fixed displacement
- $C_M \to \infty \implies$ fixed load

$C(a) = \Delta / P$ … compliance of the specimen

$\Delta_T$ … total displacement (fixed)

$\Delta_T = \Delta + C_M P = \left[ C(a) + C_M \right] P$

Strain Energy: $U_R = \frac{1}{2} C_M P^2 + \frac{1}{2} C(a) P^2$

$$- \left. \frac{d\Pi}{da} \right|_{\Delta_T} = G(a) = \frac{1}{2} P^2 C'(a)$$

$$\frac{1}{2} P^2 C'(a) = R$$

...equilibrium crack length

Stable if $E''(a_c) > 0 \implies$

$$\left[ C'(a) \right]^2 > C''(a) \left( C(a) + C_M \right)$$

Stability depends on $C_M$!

Can cause stick-slip and other unstable crack growth effects
Example 1: $G(a)$ for a double cantilever beam

\[
C(a) = \frac{8a^3}{Ed^3} \ ; \ C'(a) = \frac{24a^2}{Ed^3}
\]

\[
G\left( a_c \right) = R \Rightarrow a^2P^2 = \frac{ER}{12d^3}
\]

unstable $\iff \left[ C'(a) \right]^2 > C''(a) \left( C(a) + C_M \right)$ $\Rightarrow$ stable

\[
C(a) = \frac{4a^3}{Ed^3} \ ; \ C'(a) = \frac{12a^2}{Ed^3}
\]

\[
G\left( a_c \right) = R \Rightarrow a = \left( \frac{3Ed^3\Delta^2}{8R} \right)^{\frac{1}{4}}
\]
Example 2: $G(a)$ for an infinite strip specimen

$$G(a) = \frac{E\Delta}{\sqrt{(1-\nu^2)}h}$$

$\implies$ stable
Fracture mechanics – the global point of view

• The global point of view works quite well for a number of problems.
• It circumvents detailed calculation of stress/strain states in the vicinity of the crack.
• Has been applied successfully in a number of structural applications
• Difficulty in calculating the compliance, $C(a)$
• Difficulty in calibrating the fracture energy, $R$
• Difficulty in selecting/identifying fracture path
• Modern numerical simulations incorporate the energy approach through the phase-field methodology.
Fracture mechanics – why a local point of view?

- Provides a systematic way of calculating $G(a)$
- Provides a method for analyzing different loading symmetries
- Local approach based on stress and strain fields permits decoupling of path selection from failure characterization
Loading symmetries

Mode I or Opening mode

Mode II or In-plane shear

Mode III or Anti-plane shear
Linear elasticity

\[ y(x) = x + u(x) \]

\[ \varepsilon(x) = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right] \]

\[ \sigma(x) = \lambda \varepsilon_{kk} \mathbf{1} + 2\mu \varepsilon \]

\[ \nabla \cdot \sigma + f = 0 \]

- boundary conditions

\[ u(x) = u^*(x) \quad \text{on} \quad \partial_1 R \]

\[ s(x) = \sigma(x)n = s^*(x) \quad \text{on} \quad \partial_2 R \]

Anti-plane shear

\[ u_\alpha = 0; \quad u_3 = u_3(x_1, x_2) \]

\[ \nabla^2 u_3 = 0 \]

Plane strain

\[ u_\alpha = u_\alpha(x_1, x_2); \quad u_3 \propto x_3 \]

\[ \varepsilon_{\alpha\beta}(x_1, x_2) = \frac{1}{2} \left( u_{\alpha,\beta} + u_{\beta,\alpha} \right) \]

\[ \varepsilon_{33} = \text{const}; \quad \varepsilon_{3\alpha} = 0 \]

\[ \sigma_{\alpha\beta}(x_1, x_2) = \lambda \varepsilon_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta} \]

\[ \sigma_{33}(x_1, x_2) = \nu \sigma_{\gamma\gamma} \]

\[ \sigma_{3\alpha}(x_1, x_2) = 0 \]

If \( \sigma_{11} = \phi_{,22}; \sigma_{22} = \phi_{,11}; \sigma_{12} = -\phi_{,12} \)

then \( \nabla^4 \phi = 0 \)
The J-integral

\[ J = \int_{\Gamma} \left( U_R n_1 - \sigma_{\alpha\beta} n_\beta \frac{\partial u_\alpha}{\partial x_1} \right) ds \]

1. The integral is zero if contour is closed inside the body without enclosing singularities

2. If the contour goes from below to above the crack surface as indicated, the integral is independent of the path

3. This integral can be interpreted in terms of the energy release rate:

\[ J = G(a) = -d\Pi/da \]

4. Path independence implies that \( \sigma : \varepsilon \sim r^{-1} \) and therefore,

\[ \sigma \sim r^{-1/2} \tilde{\sigma}(\theta); \quad \varepsilon \sim r^{-1/2} \tilde{\varepsilon}(\theta) \]
Anti-plane shear

\[ \sigma_{\alpha 3} = \frac{K_{III}^\infty}{\sqrt{2\pi r}} h_{\alpha 3}(\theta) \]

\[ K_{III} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{32}(r, 0^\pm) \]

1. Anti-plane shear can exist only in specimens without bounding planes – in axisymmetric geometries or infinitely thick plates

2. Free surfaces in finite thickness plates introduce coupling to mode II

3. Connection to \( J \) and \( G \) obtained by using the path independent integral: \[ J = G(a) = \frac{1}{2\mu} K_{III}^2 \]

4. Failure criterion for mode III is still being debated (more on this later!)

Mode III or Anti-plane shear
In-plane loading symmetries

Mode I or Opening mode

Mode II or In-plane shear

mode I stress intensity factor

\[ K_I = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{22} (r, 0^\pm) \]

mode II stress intensity factor

\[ K_{II} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{12} (r, 0^\pm) \]

\[ \sigma_{\alpha\beta}(r, \theta) = \frac{K_I}{\sqrt{2\pi r}} F^I_{\alpha\beta}(\theta) + T \delta_{1\alpha} \delta_{1\beta} + \frac{K_{II}}{\sqrt{2\pi r}} F^{II}_{\alpha\beta}(\theta) + \cdots \]

\[ T \ldots \text{represents nonsingular stress} \]

and plays a role in crack path stability
Calculation of the stress intensity factors

- Elastic boundary value problem to be solved
  - Numerous examples exist in handbooks
  - Robust numerical methods based on FEM, BEM, available
  - Considered a solved problem: Given a geometry, loading, etc, there is no difficulty in determining $K_I$, $K_{II}$, and $K_{III}$.

  $$K_I = K_I \left( \text{load, crack length, geometry} \right)$$

- Example: Single-edge-notched specimen

  $$K_I = \frac{P}{tW} \sqrt{\pi a f} \left( \frac{a}{W} \right)$$

  $$\Delta = \frac{P}{tWE} \ g \left( \frac{a}{W} \right) \ldots \text{displacement}$$
Fracture criterion for in-plane loading

1. Connection to $J$ and $G$ obtained by using the path independent integral:

$$ J = G(a) = \frac{1-\nu^2}{E} \left( K_I^2 + K_{II}^2 \right) $$

2. For pure mode I loading, the crack grows along the line of symmetry and the energy based fracture criterion can be restated in terms of the stress intensity factor

$$ K_I = K_{IC} = \sqrt{\frac{ER}{1-\nu^2}} $$

3. Residual strength (load carrying capacity) can be determined for structural applications

4. Stability of structures can be evaluated
Snap-back instability

\[ K_I = \frac{P}{tW} \sqrt{\pi a f \left( \frac{a}{W} \right)} \]

\[ \Delta = \frac{P}{tWE} g \left( \frac{a}{W} \right) \ldots \text{displacement} \]

Crack is unstable in load control and displacement control.
1. For combined modes I and II, we need other criterion (criteria?) that dictates the crack path selection
   a. Maximize energy release rate
   b. Maximum “hoop” stress
   c. Principle of local symmetry:

\[
K_I = K_{IC}, \quad K_{II} = 0
\]

2. Maximum hoop stress criterion is simplest to use

3. Experimental scatter is large and unable to discriminate between the different criteria

**Principle of Local Symmetry:** Goldstein and Salganik, Int J Fract, 1974
Crack path evolution under Mode I + II

Principle of Local Symmetry works very well for this problem

Photograph Courtesy of Dov Bahat
Ben Gurion University
Tectonofractography, Springer

Possible fracture criteria for mixed mode I + III

**Criterion I:** Goldstein and Salganik,  
Int J Fract, 1974

\[
K_{II} = 0 \\
J(K_I, K_{III}) = 0
\]

**Criterion II:** Lin, Mear and Ravi-Chandar, Int J Fract, 2010

\[
K_I = K_{IC}, \quad K_{II} = 0, \quad K_{III} = 0
\]

\[
\left(\frac{1}{2} - \nu\right)\tan 2\phi = \frac{K_{III}^{\infty}}{K_I^{\infty}}
\]

Hull, Int J Fract, 1995  
Cooke and Pollard, J Geophy Res, 1996
Mixed mode I + III crack problem

Below a threshold of $K_{III}/K_I$, the crack front twists
Above the threshold, crack front fragments

$\phi_{cr} = 3.3^\circ$

Sommer, Eng Frac Mech, 1970
Mixed mode I + III crack problem

Figure 1. Geometry of test specimen.

$L^* = 5\frac{1}{4}$ Inches for test then shortened to $1/4$ inch for insertion of pins.

Knauss, Int J Fract, 1971
Summary and plan

• Energy based method can provide a simple way of analyzing fracture problem (with some residual difficulty regarding the path selection)

• Stress-intensity factor based method provides an effective way of designing fracture critical structures – residual strength diagram