"Measure what is measurable, and make measurable what is not so." - *Galileo GALILEI*

WHAT STARTS HERE CHANGES THE WORLD

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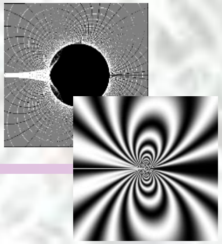

Linear Elastic Fracture Mechanics

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Lecture presented at the

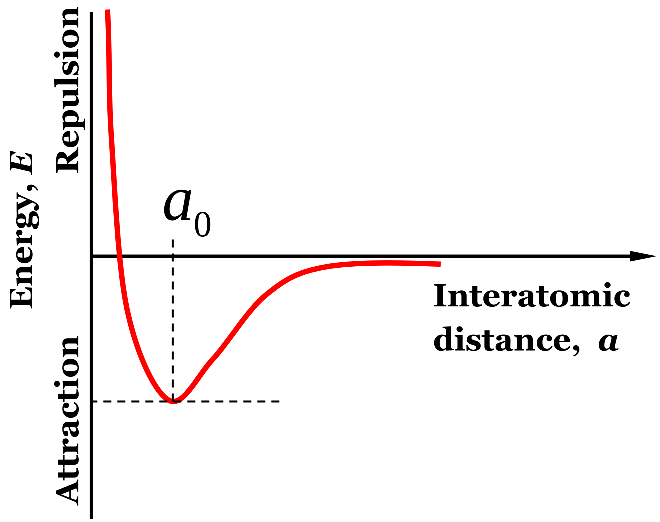
University of Pierre and Marie Curie

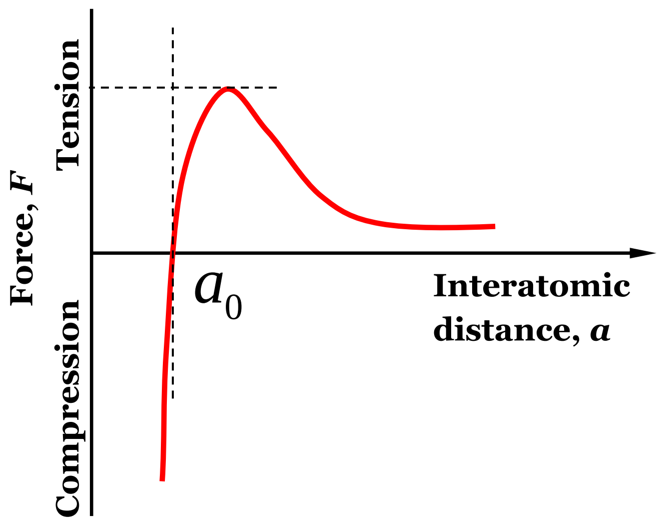
April 30, 2014

The plan

- Introduction to fracture mechanics
  - April 30, 2014
- Quasi-static instability problems (JB Leblond)
  - May 6, 2014
- Introduction to dynamic fracture mechanics
  - May 15, 2014
- Dynamic instabilities during fast fracture
  - May 21, 2014

How strong is a solid? – 1. An atomic point of view





$$F = F_{\max} \sin\left(\frac{2\pi(a-a_0)}{\lambda}\right) \Rightarrow$$

$$\sigma = \sigma_{\max} \sin\left(\frac{2\pi a_0}{\lambda}\varepsilon\right), \quad \varepsilon = \frac{(a-a_0)}{a_0}$$

$$E = \left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon\to 0} = \sigma_{\max}\frac{2\pi a_0}{\lambda}\cos\left(\frac{2\pi a_0}{\lambda}\varepsilon\right)\Bigg|_{\varepsilon\to 0}$$

$$= \sigma_{\max}\frac{2\pi a_0}{\lambda} \Rightarrow \boxed{\sigma_{\max} = \frac{E}{\pi}\frac{\lambda}{2a_0} \sim \frac{E}{\pi}}$$

Material	Modulus GPa	Strength GPa	Strength/Modulus
Steels	200	0.10 – 2	0.0005 – 0.01
Glass	70	0.17	0.0025
Carbon fibers	400	4	0.01
Glass fibers	70	11	0.16

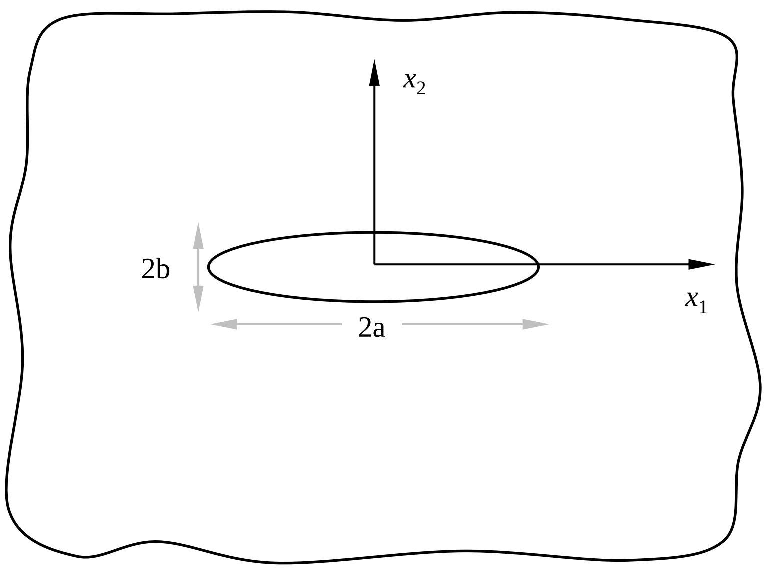
Macroscopic strength is significantly smaller than the theoretical strength

How strong is a solid? – 2. The role of defects

$$\sigma_{22}(a,0) = \sigma\left(1 + 2\frac{a}{b}\right) \Rightarrow \sigma_{22}(a,0) \approx 2\sigma\left(\frac{a}{\rho}\right)^{\frac{1}{2}} \quad \text{where } \rho = b^2 / a$$

$$\boxed{\sigma_f \sqrt{a} \approx \frac{1}{2}\sigma_{\max}\sqrt{\rho} = \text{constant}}$$

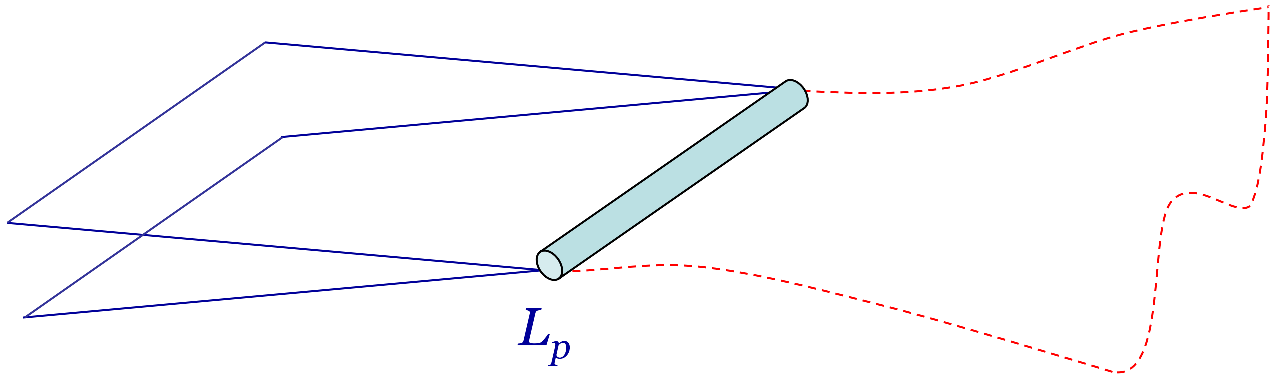
$$\left.\begin{array}{l} \rho \sim 1E-10 \text{ m} \\ a \sim 1E-6 \text{ m} \end{array}\right\} \Rightarrow \quad \sigma_f \sim 0.005\sigma_{\max}$$



Griffith's experiments

1. Used experiments on glass tubes and glass bulbs loaded under internal pressure to show that $\sigma\sqrt{a}$ was constant
2. Manufactured fresh glass fibers with diameters in the range of 1 mm to 3 microns to show that small fibers had strength of about 11 GPa

The continuum view of fracture



- Process of fracture can be cleavage, intergranular/ transgranular fracture (polycrystalline materials), cavitation (ductile metals), disentanglement (polymers), microcracking (glasses,…), fiber breakage,…
- Details of processes within L_p are not important; only the total energy needed for the fracture process is assumed to play a role in the development of the fracture
- L_p is small – “small-scale process zone” – what does this mean?

The energy balance – a continuum view

Total energy of the system: $E = \Pi + U_s$

where $U_s \ldots$surface energy (or fracture energy)

$$\Pi = -W_{\partial R} + U_R \ldots \text{potential energy of the body}$$

$W_{\partial R} \ldots$ work done by the external forces on the body

$U_R \ldots$ strain energy stored in the body

define

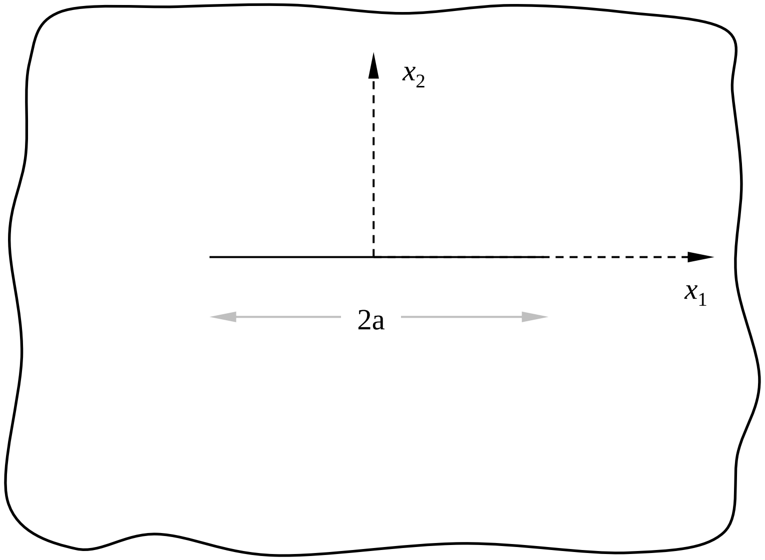
$$G(a) \equiv -\frac{d\Pi}{da} \ldots \text{Energy Release Rate}$$

$$R \equiv \frac{dU_s}{da} \ldots \text{fracture resistance}$$

At equilibrium, $E'(a) = 0 \Rightarrow$

$$\boxed{G(a_c) = R \ldots \text{Fracture criterion}}$$

Stable if $E''(a_c) > 0$



Remarks 1: $\boxed{G(a_c) = R \ldots \text{Fracture criterion}}$

- a_c is the equilibrium crack length (reversible)
- Fracture resistance *R(a)* includes the effect of all dissipative *fracture* processes and is typically calibrated from experiments.
- For Linearly Elastic Fracture Mechanics (LEFM), the region outside L_p must exhibit linear elastic behavior, but this is not a general requirement.
- Other than L_p being "small", there is no length scale here! The theory works at length scales from the atomic to the tectonic.
- Need methods to calculate *G(a)* for specific crack problems

Remarks 2: $G(a_c) = R \ldots$ Fracture criterion

- R varies over several orders of magnitude
  - True surface energy is ~ O(1) J/m^2
  - Glasses and ceramics~ 10 J/m^2
  - Polymers ~1 kJ/m^2
  - Metals ~ 100 kJ/m^2
- Differences arise due to different mechanisms of deformation and failure
- Must be determined through calibration experiments, such as the pioneering work of Obreimoff (1930)

Calculation of *G(a)*

C_M …compliance of the loading system

$C_M = 0 \Rightarrow$ fixed displacement

$C_M \to \infty \Rightarrow$ fixed load

$C(a) = \Delta / P$ …compliance of the specimen

Δ_T …total displacement (fixed)

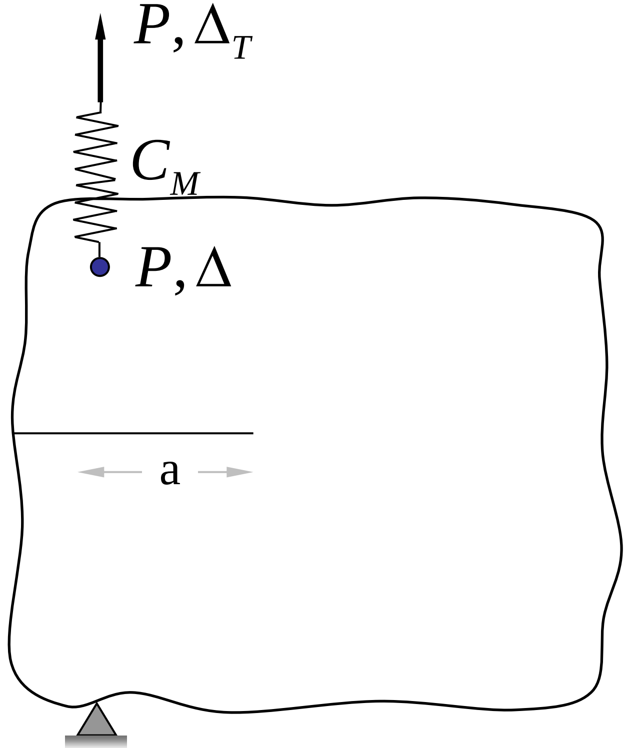
$$\Delta_T = \Delta + C_M P = \left[C(a) + C_M \right] P$$

Strain Energy: $U_R = \frac{1}{2} C_M P^2 + \frac{1}{2} C(a) P^2$

$$-\left.\frac{d\Pi}{da}\right|_{\Delta_T} = G(a) = \frac{1}{2} P^2 C'(a)$$

$$\boxed{\frac{1}{2} P^2 C'(a) = R}$$

…equilibrium crack length



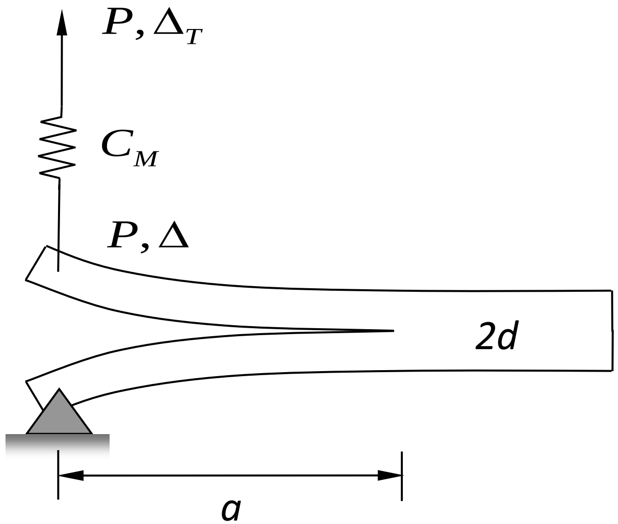
Stable if $E''(a_c) > 0 \Rightarrow$

$$\boxed{\left[C'(a) \right]^2 > C''(a) \left(C(a) + C_M \right)}$$

Stability depends on C_M!

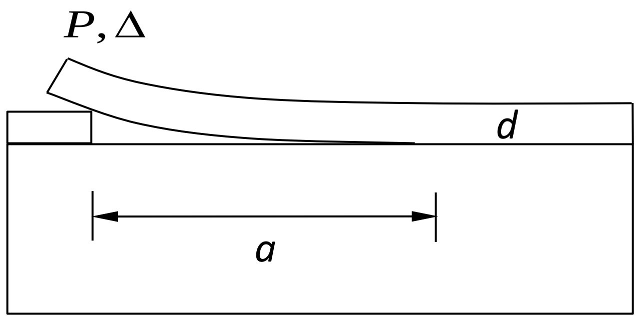
Can cause stick-slip and other unstable crack growth effects

Example 1: *G(a)* for a double cantilever beam



$$C(a) = \frac{8a^3}{Ed^3}; C'(a) = \frac{24a^2}{Ed^3}$$

$$G(a_c) = R \Rightarrow a^2 P^2 = \frac{ER}{12d^3}$$



$$C(a) = \frac{4a^3}{Ed^3}; C'(a) = \frac{12a^2}{Ed^3}$$

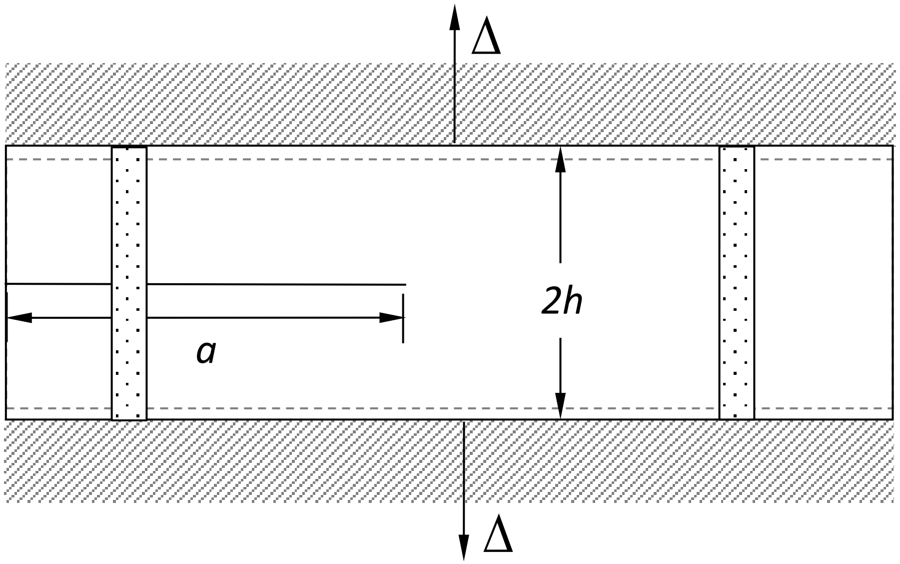
$$G(a_c) = R \Rightarrow a = \left(\frac{3Ed^3\Delta^2}{8R}\right)^{\frac{1}{4}}$$

unstable $\Leftarrow$ $$[C'(a)]^2 > C''(a)\left(C(a) + C_M\right)$$ $\Rightarrow$ stable

Example 2: *G(a)* for an infinite strip specimen

$$G(a) = \frac{E\Delta}{\sqrt{(1-\nu^2)h}}$$

$\Rightarrow$ stable



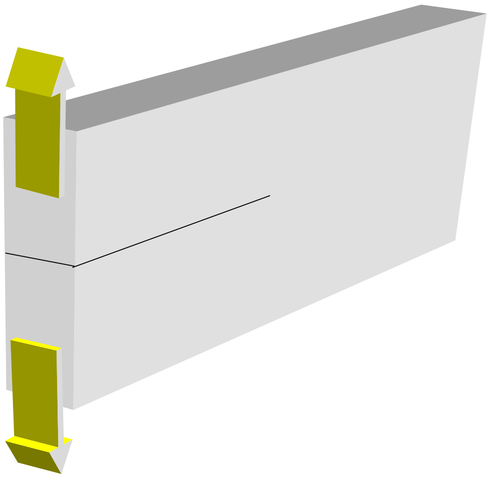
Fracture mechanics – the global point of view

- The global point of view works quite well for a number of problems.
- It circumvents detailed calculation of stress/strain states in the vicinity of the crack.
- Has been applied successfully in a number of structural applications
- Difficulty in calculating the compliance, $C(a)$
- Difficulty in calibrating the fracture energy, R
- Difficulty in selecting/identifying fracture path
- Modern numerical simulations incorporate the energy approach through the phase-field methodology.

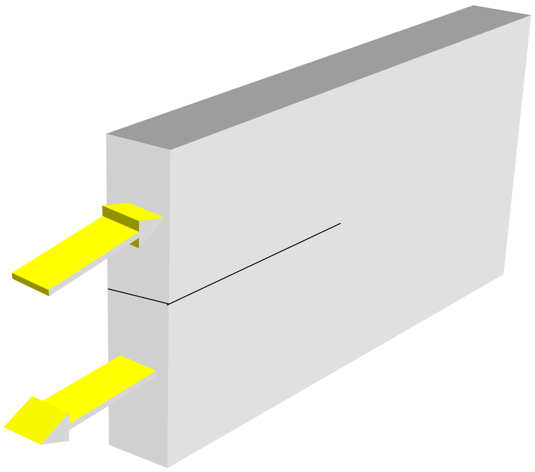
Fracture mechanics – why a local point of view?

- Provides a systematic way of calculating $G(a)$
- Provides a method for analyzing different loading symmetries
- Local approach based on stress and strain fields permits decoupling of path selection from failure characterization

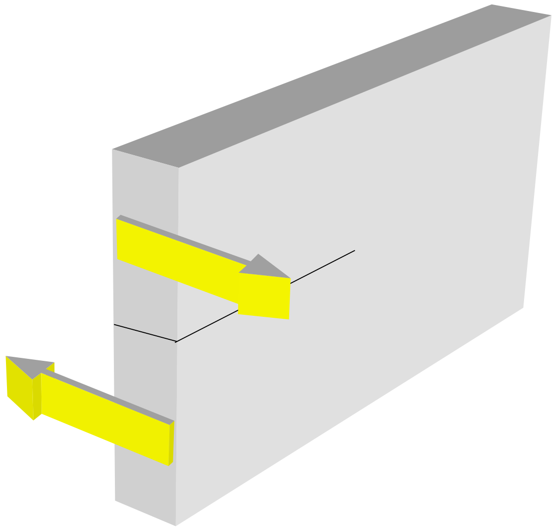
Loading symmetries



Mode I or
Opening mode



Mode II or
In-plane shear



Mode III or
Anti-plane shear

Linear elasticity

$$\mathbf{y}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$$

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \lambda\varepsilon_{kk}\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}$$

- boundary conditions

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^*(\mathbf{x}) \qquad \text{on } \partial_1 R$$

$$\mathbf{s}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})\mathbf{n} = \mathbf{s}^*(\mathbf{x}) \qquad \text{on } \partial_2 R$$

Anti-plane shear

$$u_\alpha = 0; \quad u_3 = u_3(x_1, x_2)$$

$$\nabla^2 u_3 = 0$$

Plane strain

$$u_\alpha = u_\alpha(x_1, x_2); \quad u_3 \propto x_3$$

$$\begin{cases} \varepsilon_{\alpha\beta}(x_1, x_2) = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \\ \varepsilon_{33} = \text{const}; \; \varepsilon_{3\alpha} = 0 \end{cases}$$

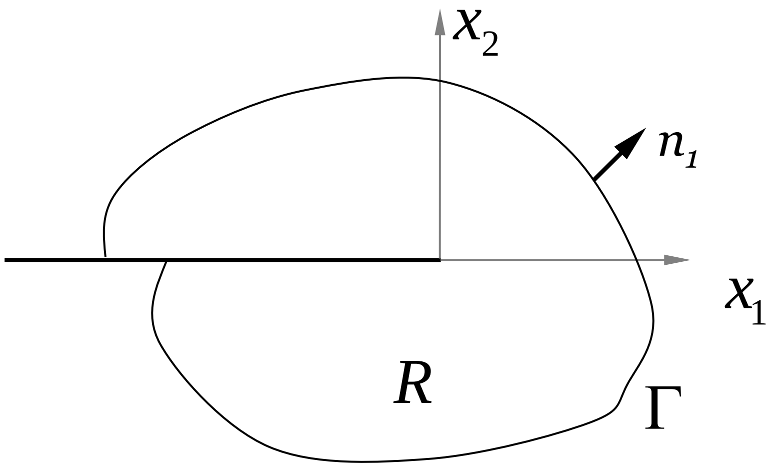
$$\begin{cases} \sigma_{\alpha\beta}(x_1, x_2) = \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu\varepsilon_{\alpha\beta} \\ \sigma_{33}(x_1, x_2) = \nu\sigma_{\gamma\gamma} \\ \sigma_{3\alpha}(x_1, x_2) = 0 \end{cases}$$

If $\sigma_{11} = \phi_{,22}; \sigma_{22} = \phi_{,11}; \sigma_{12} = -\phi_{,12}$

then $\nabla^4\phi = 0$

The J-integral

$$J = \int_{\Gamma} \left(U_R n_1 - \sigma_{\alpha\beta} n_\beta \frac{\partial u_\alpha}{\partial x_1} \right) ds$$



1. The integral is zero if contour is closed inside the body without enclosing singularities
2. If the contour goes from below to above the crack surface as indicated, the integral is independent of the path
3. This integral can be interpreted in terms of the energy release rate:

$$J = G(a) = -d\Pi / da$$

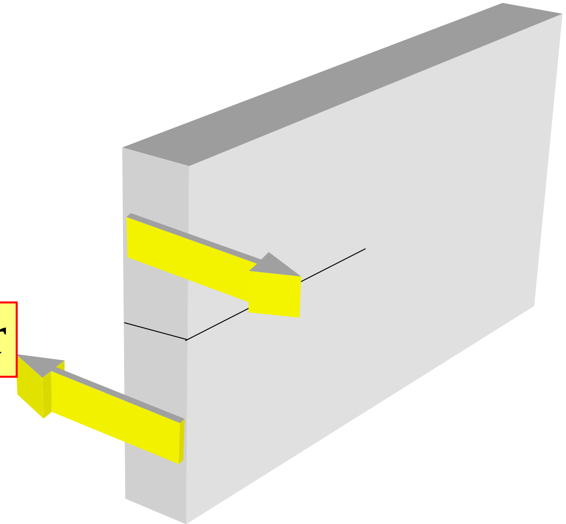
4. Path independence implies that $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} \sim r^{-1}$ and therefore,

$$\boldsymbol{\sigma} \sim r^{-1/2} \tilde{\boldsymbol{\sigma}}(\theta); \ \boldsymbol{\varepsilon} \sim r^{-1/2} \tilde{\boldsymbol{\varepsilon}}(\theta)$$

Anti-plane shear

$$\boxed{\sigma_{\alpha 3} = \frac{K_{III}^{\infty}}{\sqrt{2\pi r}} h_{\alpha 3}(\theta)}$$

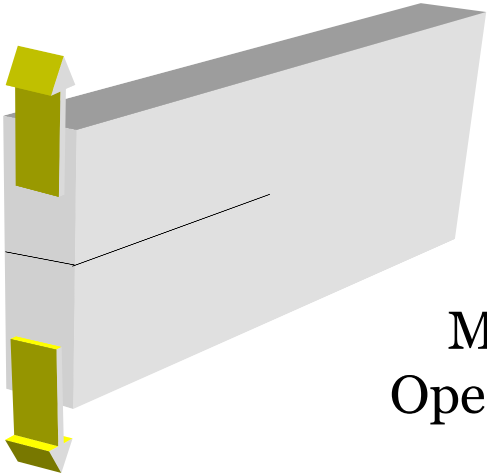
$$K_{III} = \lim_{r \to 0} \sqrt{2\pi r}\, \sigma_{32}\left(r, 0^{\pm}\right)$$ mode III stress intensity factor



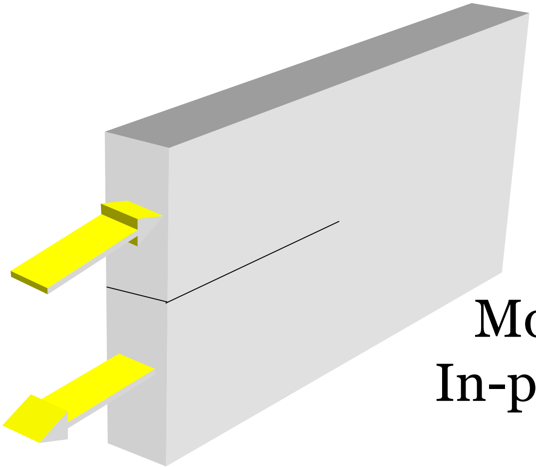
Mode III or
Anti-plane shear

1. Anti-plane shear can exist only in specimens without bounding planes – in axisymmetric geometries or infinitely thick plates
2. Free surfaces in finite thickness plates introduce coupling to mode II
3. Connection to J and G obtained by using the path independent integral: $J = G(a) = \dfrac{1}{2\mu} K_{III}^2$
4. Failure criterion for mode III is still being debated (more on this later!)

In-plane loading symmetries



Mode I or
Opening mode



Mode II or
In-plane shear

mode I stress intensity factor

$$K_I = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{22}\left(r,0^{\pm}\right)$$

mode II stress intensity factor

$$K_{II} = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{12}\left(r,0^{\pm}\right)$$

$$\sigma_{\alpha\beta}\left(r,\theta\right) = \frac{K_I}{\sqrt{2\pi r}} F^{I}_{\alpha\beta}\left(\theta\right) + T\delta_{1\alpha}\delta_{1\beta} + \frac{K_{II}}{\sqrt{2\pi r}} F^{II}_{\alpha\beta}\left(\theta\right) + \cdots$$

$T\ldots$represents nonsingular stress and plays a role in crack path stability

Calculation of the stress intensity factors

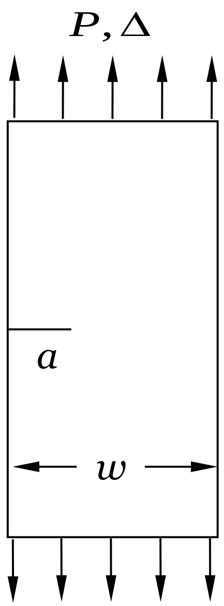
- Elastic boundary value problem to be solved
  - Numerous examples exist in handbooks
  - Robust numerical methods based on FEM, BEM, available
  - Considered a solved problem: Given a geometry, loading, etc, there is no difficulty in determining K_I, K_{II}, and K_{III}.

$$K_I = K_I\left(\text{load, crack length, geometry}\right)$$

- Example: Single-edge-notched specimen

$$K_I = \frac{P}{tW}\sqrt{\pi a}\, f\left(\frac{a}{W}\right)$$

$$\Delta = \frac{P}{tWE}\, g\left(\frac{a}{W}\right) \ldots \text{displacement}$$



Fracture criterion for in-plane loading

1. Connection to *J* and *G* obtained by using the path independent integral:

$$J = G(a) = \frac{1-\nu^2}{E}\left(K_I^2 + K_{II}^2\right)$$

2. For pure mode I loading, the crack grows along the line of symmetry and the energy based fracture criterion can be restated in terms of the stress intensity factor

$$\boxed{K_I = K_{IC} = \sqrt{\frac{ER}{1-\nu^2}}}$$

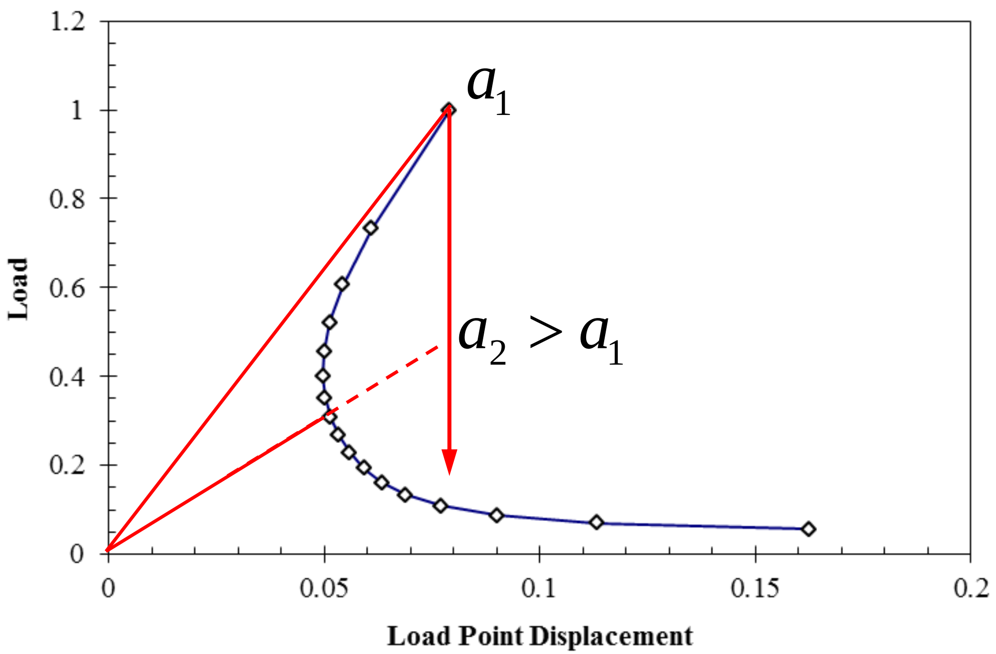
3. Residual strength (load carrying capacity) can be determined for structural applications

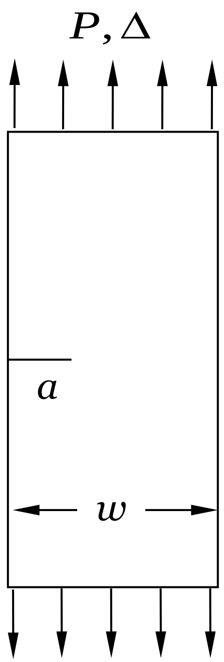
4. Stability of structures can be evaluated

Snap-back instability

$$K_I = \frac{P}{tW}\sqrt{\pi a}\, f\left(\frac{a}{W}\right)$$

$$\Delta = \frac{P}{tWE}\, g\left(\frac{a}{W}\right) \ldots \text{displacement}$$



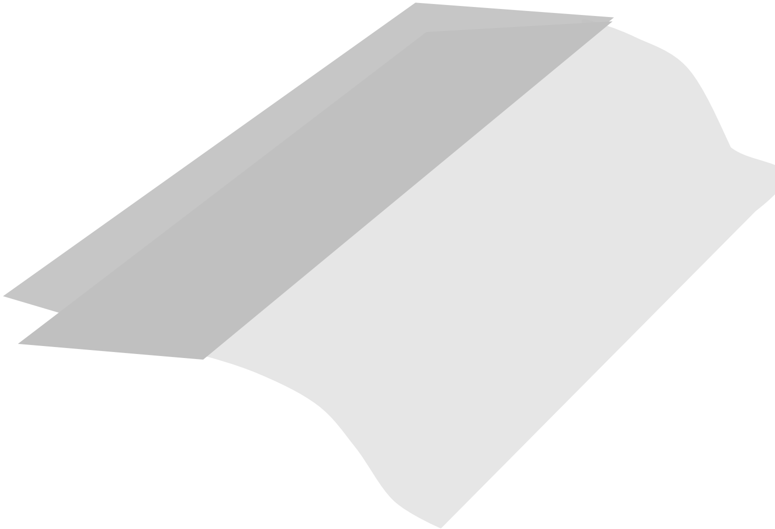


Crack is unstable in load control and displacement control

Fracture criterion for mixed mode I + II crack

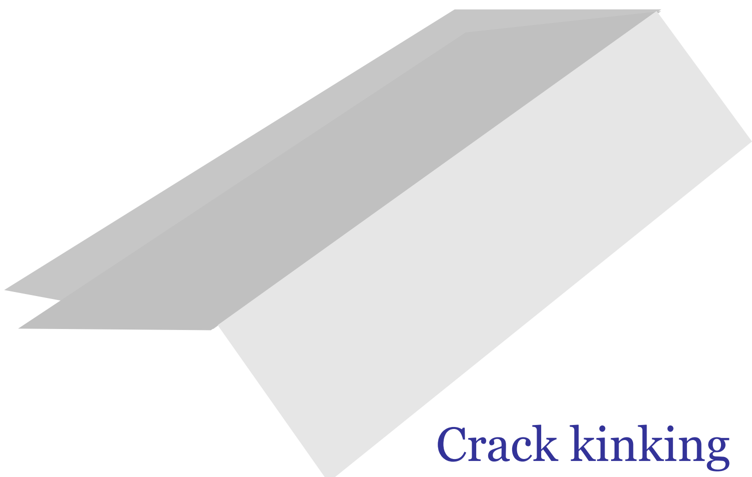
1. For combined modes I and II, we need other criterion (criteria?) that dictates the crack path selection
   a. Maximize energy release rate
   b. Maximum "hoop" stress
   c. Principle of local symmetry:

$$\boxed{K_I = K_{IC}, \quad K_{II} = 0}$$



Crack tilting

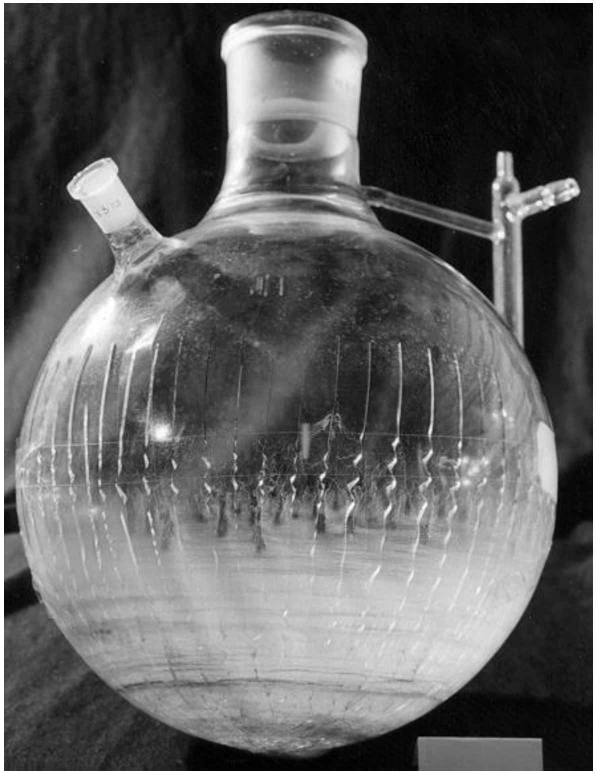
2. Maximum hoop stress criterion is simplest to use
3. Experimental scatter is large and unable to discriminate between the different criteria



Crack kinking

Principle of Local Symmetry: Goldstein and Salganik, Int J Fract, 1974

Crack path evolution under Mode I + II



Photograph Courtesy of Dov Bahat
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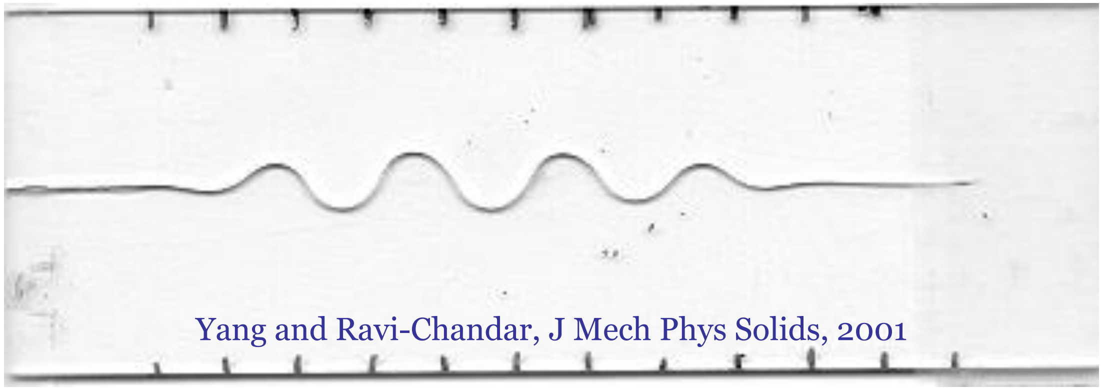
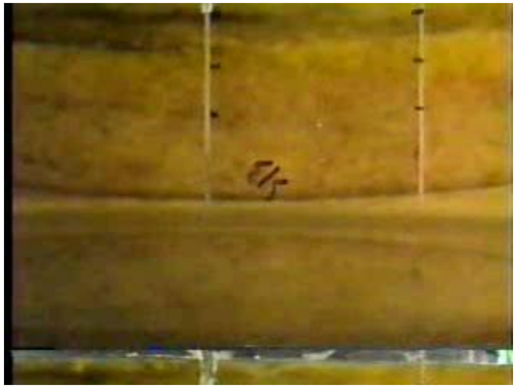
***Tectonofractography**, Springer*



Yang and Ravi-Chandar, J Mech Phys Solids, 2001

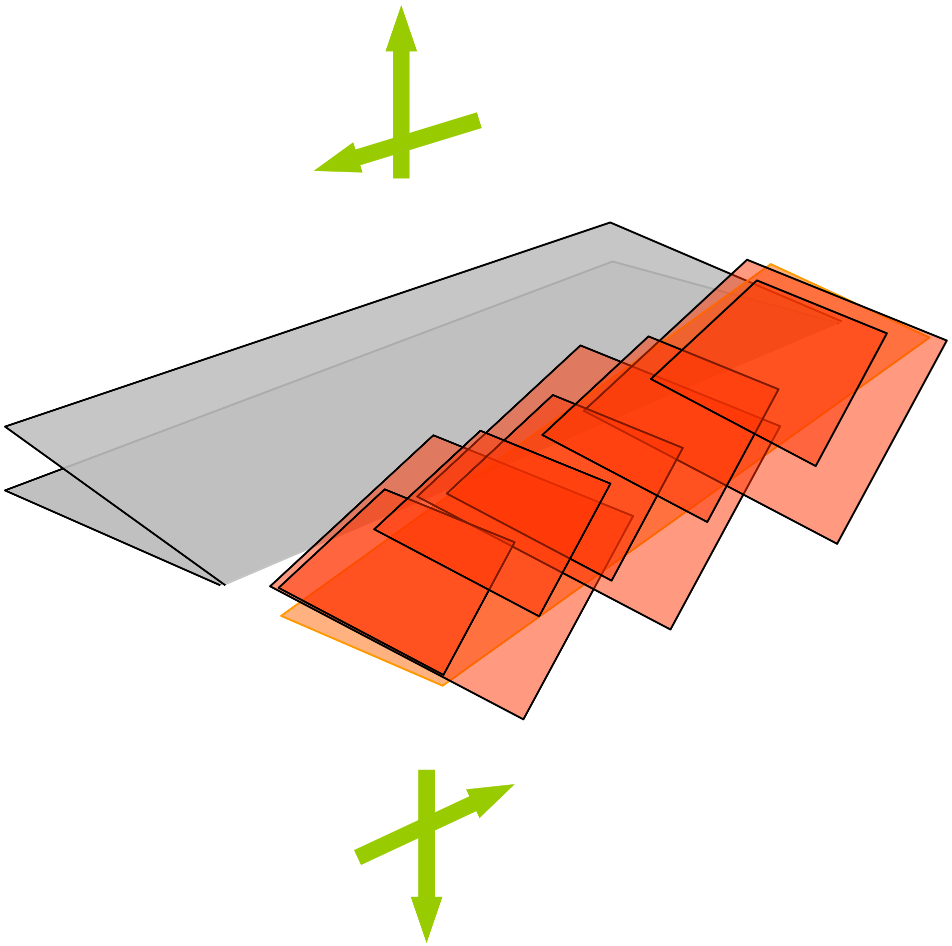


Principle of Local Symmetry works very well for this problem

Possible fracture criteria for mixed mode I + III

Criterion I: Goldstein and Salganik, Int J Fract, 1974

$$K_{II} = 0$$
$$f\left(K_I, K_{III}\right) = 0$$



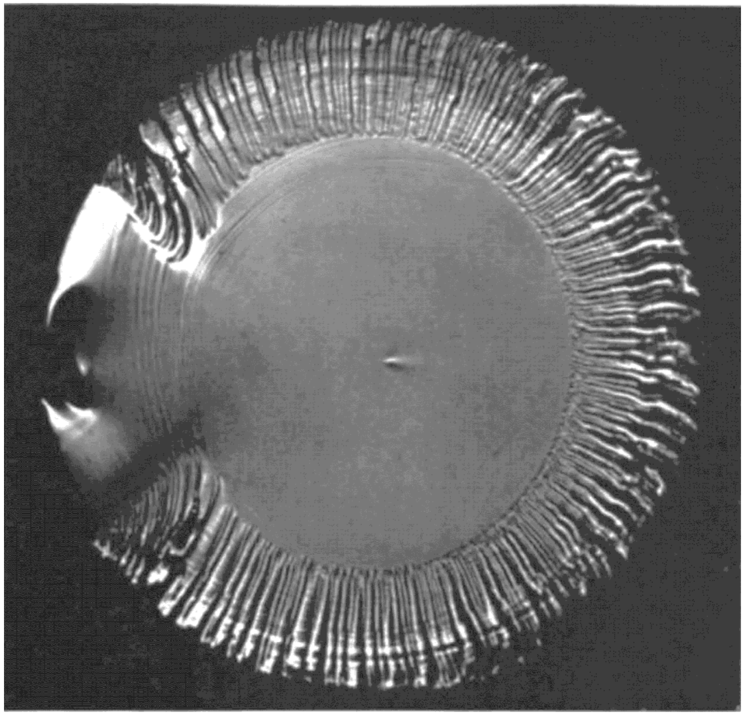
Criterion II: Lin, Mear and Ravi-Chandar, Int J Fract, 2010

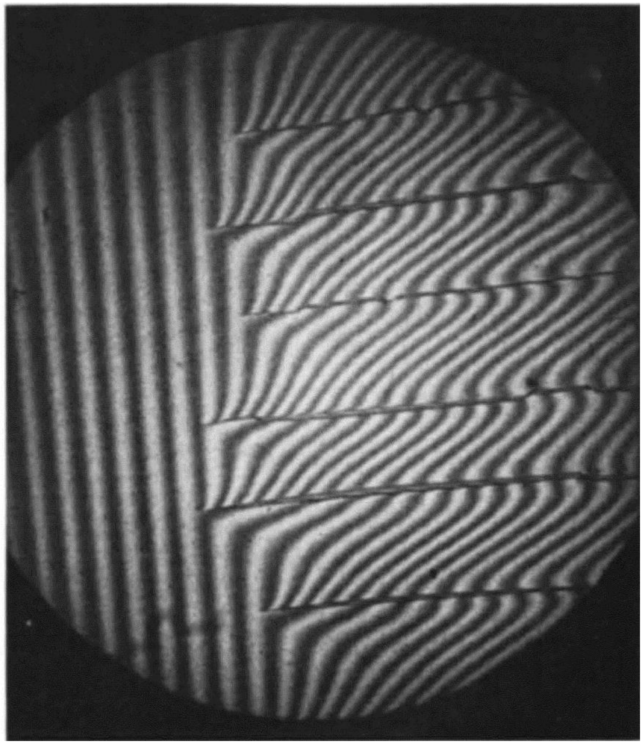
$$K_I = K_{IC}, \quad K_{II} = 0, \quad K_{III} = 0$$

$$\left(\frac{1}{2} - \nu\right) \tan 2\phi = \frac{K_{III}^{\infty}}{K_I^{\infty}}$$

Hull, Int J Fract, 1995
Cooke and Pollard, J Geophy Res, 1996

Mixed mode I + III crack problem





Below a threshold of K_{III}/K_I, the crack front twists

Above the threshold, crack front fragments

$\phi_{cr} = 3.3^{\circ}$

Sommer, Eng Frac Mech, 1970

Mixed mode I + III crack problem

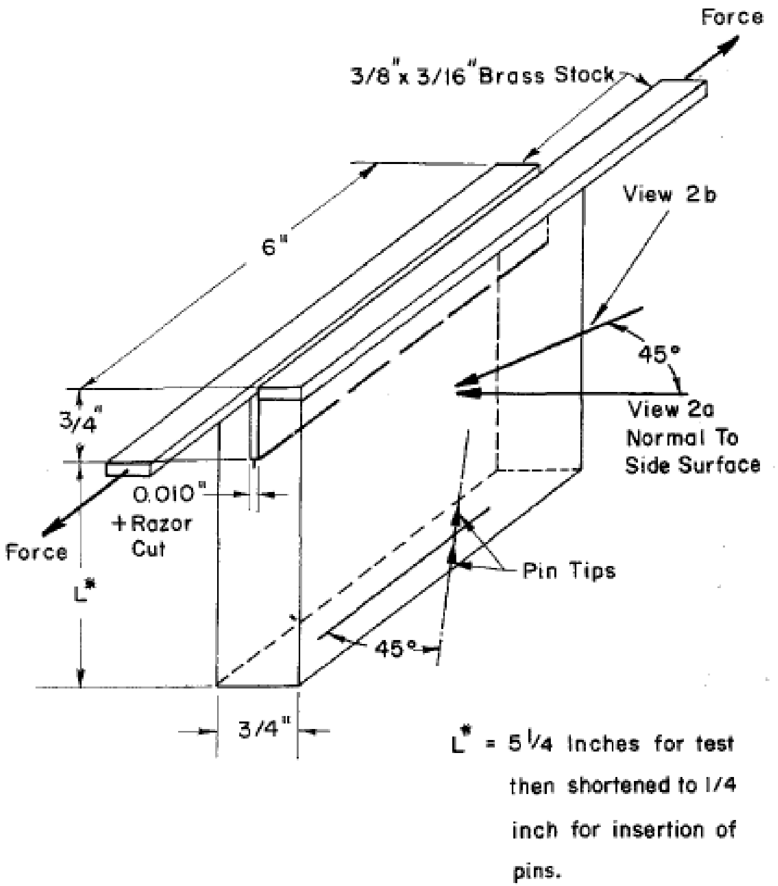
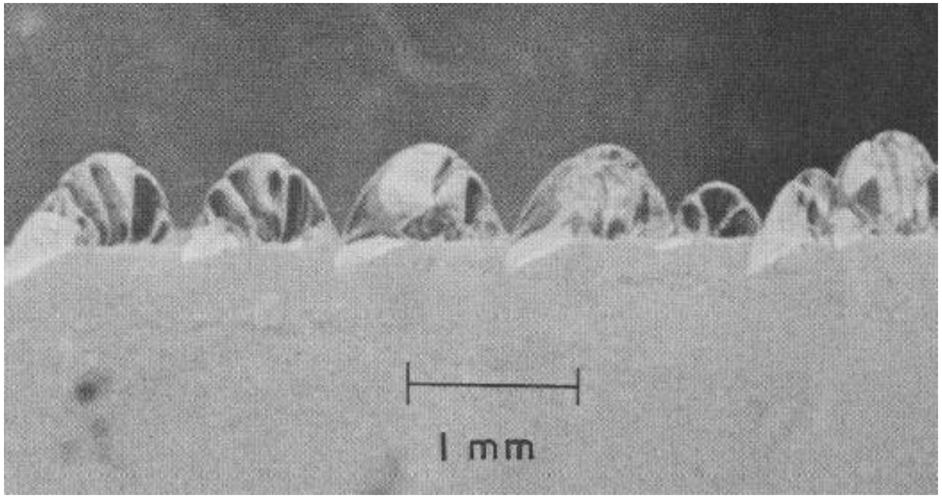
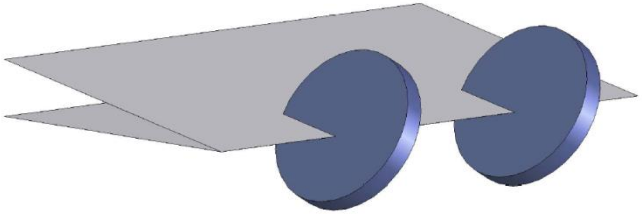


Figure 1. Geometry of test specimen.





Knauss, Int J Fract, 1971

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Summary and plan

- Energy based method can provide a simple way of analyzing fracture problem (with some residual difficulty regarding the path selection)
- Stress-intensity factor based method provides an effective way of designing fracture critical structures – residual strength diagram