Lecture 4 Introduction to Large Deformation Elastic Fracture Mechanics

Since there is no literature on 3 dimensional theory of cracks in hyper-elastic solids, I will focus on plane theories where the deformation state is completely characterized by two in-plane material coordinates. Because of time limitations, I restrict this lecture to plane strain deformation only. As in lecture 3, the material is assumed to be isotropic, homogeneous, incompressible and Hyper-elastic. The results below are from the paper by

(*) R. A. Stephenson: J. of Elasticity, 12, 1, (1982), 65-99 with slight modifications (I have corrected typographical errors and added comments of my own).

Notation: Stephenson's $\overset{\circ}{W} = \Phi$ in my notation. Also, his $\vec{n}$ is denoted by $\vec{m}$ in my notation.

Preliminary: Plane strain theory in large deformation theory:

In plane strain deformation, the out of plane displacement u_3 is assumed to be identically zero. The in plane displacements u_α are assumed to depend only on the in-plane material coordinates x_α , $\alpha = 1,2$.

For incompressible Hyperelastic solids, $\lambda_3 = 1$, this implies that $I_1 = \lambda_1^2 + \lambda_1^{-2} + 1 \Rightarrow I_1 = I_2$ so the *work function W is a function of I_1 only*. The deformation gradient tensor, $\tilde{F}$ with respect to an orthornormal basis, can be represented by the matrix

$$F_{ij}\vec{e}_i\vec{e}_j \leftrightarrow \begin{bmatrix} y_{1,1} & y_{1,2} & 0 \\ y_{2,1} & y_{2,2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow F_{ij}^T\vec{e}_i\vec{e}_j \leftrightarrow \begin{bmatrix} y_{1,1} & y_{2,1} & 0 \\ y_{1,2} & y_{2,2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow F_{ij}^{-T}\vec{e}_i\vec{e}_j = \begin{bmatrix} y_{2,2} & -y_{2,1} & 0 \\ -y_{1,2} & y_{1,1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.1}$$

In (4.1), I have used the incompressibility condition

$$J = det\,\tilde{F} = y_{1,1}\,y_{2,2} - y_{1,2}\,y_{2,1} = 1 \tag{4.2}$$

According to Lecture 3, when the work function is independent of I_2 , the 1st Piola stress is related to the deformation gradient by

$$\tilde{\sigma} = 2\Phi'(I)\tilde{F} - p\tilde{F}^{-T} \qquad \text{where } \Phi'(I) = d\Phi / dI \tag{4.3}$$

where *p* is the pressure require to enforce incompressibility (4.2). Since we are mostly interested in the in-plane fields, we define the 2D deformation gradient and 1st Piola stress tensors by

$$\tilde{F} = F_{\alpha\beta}\vec{e}_\alpha\vec{e}_\beta, \quad \alpha,\beta=1,2 \qquad \tilde{F} \leftrightarrow \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} \tag{4.4a}$$

$$\sigma_{\alpha\beta} = 2\Phi'(I)F_{\alpha\beta} - pF_{\alpha\beta}^{-T} \tag{4.4b}$$

where

$$I \equiv tr\left[\tilde{F}^T \tilde{F}\right] = F_{\alpha\beta} F_{\alpha\beta} \tag{4.4c}$$

Finally, (4.1) and (4.3) imply that the only non-vanishing out of plane component of the 1st Piola stress is

$$\sigma_{33} = 2\Phi'(I) - p\,. \tag{4.5}$$

Since σ_{33} is independent of the out of plane coordinates, is easy to show that the only non-trivial equilibrium equations are the in-plane ones, and these are, in the reference configuration:

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} = 0 \tag{4.6}$$

Note that σ_{33} can be found once the in-plane deformation and the pressure p are determined. In the following, we treat $\tilde{F}$ and $\tilde{\sigma}$ as 2D tensors defined by (4.4a-c).

In lecture 3, I have chosen the origin of the coordinate system to be the same for the reference and deformed configuration. In Stephenson's paper, he shifted the origin of his deformed coordinate system to the deformed crack tip position as shown in figure below. Such a translation does not affect the deformation gradient and all relevant quantities in the calculation.

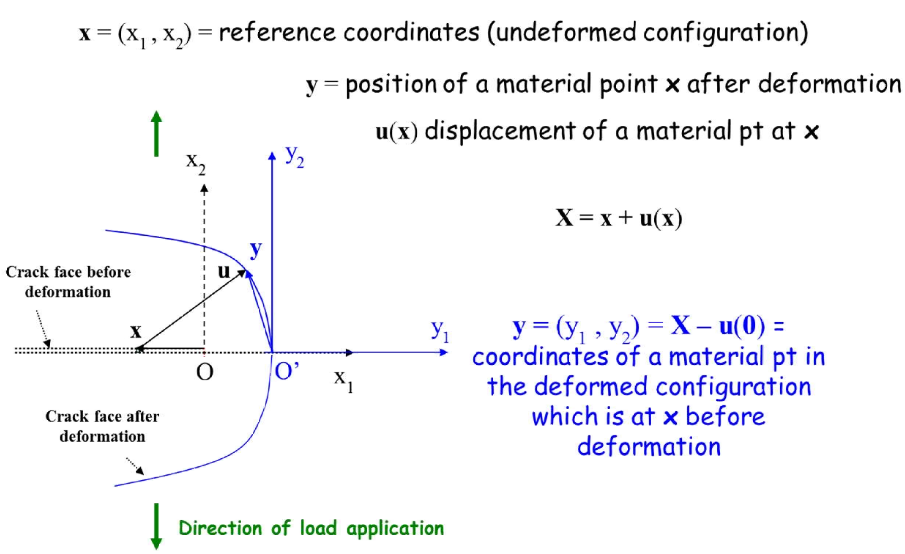

Figure 1: Schematic figure showing the reference and the deformed configuration of a Mode I plane strain crack. Bold letters denote vectors. Note the origin of the deformed coordinate system (y_1, y_2) is located at the deformed crack tip O'.

Governing equations and boundary conditions

Since we are interested in the behavior near the crack tip, the crack can be modeled as semi-infinite with its tip at the origin where $\vec{x}=\vec{0}$. The traction free boundary condition on the crack faces in the reference configuration is:

$$\sigma_{12}(r,\theta=\pm\pi)=\sigma_{22}(r,\theta=\pm\pi)=0 \qquad (4.7)$$

where $(r,\theta=\pm\pi)$ is a polar coordinate system with the origin at $\vec{x}=\vec{0}$, that is, $x_1=r\cos\theta, x_2=r\sin\theta,$ where $-\pi<\theta\leq\pi$. In the following, we make use of the standard formulae:

$$\frac{\partial}{\partial x_1}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial x_2}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \qquad (4.8)$$

Using (4.8), the incompressiblity constraint (4.2) in polar coordinates is:

$$J=\frac{1}{r}\left(\frac{\partial y_1}{\partial r}\frac{\partial y_2}{\partial\theta}-\frac{\partial y_2}{\partial r}\frac{\partial y_1}{\partial\theta}\right)=1 \qquad (4.9)$$

Substituting (4.4b) into the equilibrium equations (4.6) leads to

$$\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}=0\Leftrightarrow\frac{\partial\left[2\Phi'(I)F_{\alpha\beta}-pF_{\alpha\beta}^{-T}\right]}{\partial x_\beta}=0 \qquad (4.10a)$$

Using (4.8), (4.4b) and

$$\tilde{F}^{-T}\leftrightarrow\begin{bmatrix} y_{2,2} & -y_{2,1} \\ -y_{1,2} & y_{1,1}\end{bmatrix} \qquad (4.10b)$$

and after some calculations, the equilibrium equations (4.10a) can be written as two coupled nonlinear PDE's (partial differential equations) involving the unknown pressure *p* and the deformed coordinates y_1, y_2:

$$\begin{aligned}\frac{\partial p}{\partial r}&=2\Phi'(I)H_r+2\Phi''(I)\left[g_{rr}\frac{\partial I}{\partial r}+\frac{g_{r\theta}}{r^2}\frac{\partial I}{\partial\theta}\right]\\ \frac{\partial p}{\partial\theta}&=2\Phi'(I)H_\theta+2\Phi''(I)\left[g_{r\theta}\frac{\partial I}{\partial r}+\frac{g_{\theta\theta}}{r^2}\frac{\partial I}{\partial\theta}\right]\end{aligned} \qquad (4.11a,b)$$

where $\Phi''(I)=d^2\Phi/dI^2$ and

$$g_{rr}=\frac{\partial y_\alpha}{\partial r}\frac{\partial y_\alpha}{\partial r}, \quad g_{r\theta}=\frac{\partial y_\alpha}{\partial r}\frac{\partial y_\alpha}{\partial\theta}, \quad g_{\theta\theta}=\frac{\partial y_\alpha}{\partial\theta}\frac{\partial y_\alpha}{\partial\theta} \qquad (4.11c)$$

$$H_r = \frac{\partial y_\alpha}{\partial r}\nabla^2 y_\alpha \qquad H_\theta = \frac{\partial y_\alpha}{\partial \theta}\nabla^2 y_\alpha \tag{4.11d}$$

$$I = g_{rr} + \frac{1}{r^2} g_{\theta\theta} \tag{4.11e}$$

The use of summation convention of summing over repeated indices is used unless specified otherwise. Using (4.4b), (4.10b) and (4.8), the boundary conditions (4.7) can be written as:

$$\begin{aligned} &\left[2\Phi'(I)\frac{\partial y_1}{\partial \theta} + rp\frac{\partial y_2}{\partial r}\right]_{\theta=\pm\pi} = 0 \\ &\left[2\Phi'(I)\frac{\partial y_2}{\partial \theta} - rp\frac{\partial y_1}{\partial r}\right]_{\theta=\pm\pi} = 0 \end{aligned} \tag{4.12a,b}$$

It can be shown using (4.11c) and (4.9) that (4.12a,b) is equivalent to (e.g. by multiplying (4.12a) by $\left.\frac{\partial y_2}{\partial \theta}\right|_{\theta=\pm\pi}$ and (4.12b) by $\left.\frac{\partial y_1}{\partial \theta}\right|_{\theta=\pm\pi}$ and subtracting the resulting expressions leads to 4.13b),

$$p\big|_{\theta=\pm\pi} = \left.\frac{2\Phi'(I)}{g_{rr}}\right|_{\theta=\pm\pi} = \left.\frac{2\Phi'(I) g_{\theta\theta}}{r^2}\right|_{\theta=\pm\pi}, \qquad g_{r\theta}(r, \theta = \pm\pi) = 0 \tag{4.13a,b}$$

The three PDE's ((4.9) and (4.11a,b)) and the boundary conditions (4.13a,b) allows one to solve for the unknown pressure and the deformed positions y_1, y_2.

The only thing left is to specify the work or energy density function $W = \Phi(I)$. Since one is interested in asymptotic behavior where the local fields are unbounded, Stephenson assumes that the behavior of the energy density function for large *I* has the form:

$$\Phi(I \to \infty) = AI^n + BI^{n-1} + o\left(I^{n-1}\right) \tag{4.14}$$

where *n* is a positive real number and can be interpreted as a hardening parameter[1]. The constants *A* and *B* has units of stress, *A* can be interpreted as a reference modulus. For example, for a Mooney-Rivlin material, the strain energy density in plane strain deformation is the same as a neo-Hookean solid which is given by

$$\Phi(I) = \frac{\mu}{2}(I - 2) \tag{4.15}$$

For this special case (4.4) is valid for all levels of deformation with $n = 1, A = \mu/2, B = -\mu$. Note for Mooney-Rivlin solid, $\Phi''(I) = 0$, leading to considerable simplification of (4.11a,b).

---

[1] For a stable material, n > ½ , see Stephenson's paper (*) for a detailed discussion.

Key assumption:

The key assumption is that the dominant solution of (4.9) and (4.11a,b) as $r \to 0$ is *separable*, that is:

$$y_1 = r^{m_1} U_1(\theta), \quad y_2 = r^{m_2} U_2(\theta), \quad p = r^l P(\theta) \qquad \text{as } r \to 0 \qquad (4.16a,b,c)$$

where $m_\alpha, l,$ are unknown constants. The angular functions U_α, P are also unknown. To allow for singular fields with bounded displacements, the unknown constants m_α are restricted to satisfy:

$$1 > min\{m_1, m_2\} = m > 0. \qquad (4.17)$$

Key observation

An important observation by Stephenson [*] is the following: Suppose there exist a solution $\{\vec{y}, \tilde{\sigma}, p\}$ that satisfies the field equations (4.2), (4.3) and (4.6) as well as the traction free boundary conditions imposed on the crack faces (4.7), then $\{\tilde{Q}\vec{y}, \tilde{Q}\tilde{\sigma}, p\}$ is also a solution of the crack problem, where $\tilde{Q}$ is *any* constant proper orthogonal tensor (that is, det$\tilde{Q}$ = 1). With respect to an orthornormal basis, the matrix of $\tilde{Q}$ must have the form:

$$\tilde{Q} \leftrightarrow \begin{bmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{bmatrix} \qquad \varphi \in [0, 2\pi) \qquad (4.18)$$

To see this, note that the equilibrium equation (4.3) is satisfied by $\tilde{Q}\tilde{\sigma}$, since $\tilde{Q}$ is a constant tensor. The deformation gradient due to the application of $\tilde{Q}$ to $\vec{y}$ is $\tilde{Q}\tilde{F}$. Since the work function depends only on the invariants of the Cauchy-Green tensor $\tilde{F}^T\tilde{F}$, it is unaffected by a rigid body rotation, that is

$$\left(\tilde{Q}\tilde{F}\right)^T\left(\tilde{Q}\tilde{F}\right) = \left(\tilde{F}^T\tilde{Q}^T\right)\left(\tilde{Q}\tilde{F}\right) = \tilde{F}^T\tilde{F} \ .$$

This means that

$$\tilde{\sigma} = 2\Phi'(I)\tilde{F} - p\tilde{F}^{-T} \Leftrightarrow \tilde{Q}\tilde{\sigma} = 2\Phi'(I)\tilde{Q}\tilde{F} - p\left(\tilde{Q}\tilde{F}\right)^{-T} \qquad (4.19)$$

So $\tilde{Q}\tilde{\sigma}$ satisfies the constitutive model. Finally, the traction caused by $\tilde{Q}\tilde{\sigma}$ on the crack faces is

$$\left[\tilde{Q}\tilde{\sigma}\bullet\vec{e}_2\right]_{\theta=\pm\pi} = \begin{bmatrix} cos\varphi & -sin\phi \\ sin\phi & cos\phi \end{bmatrix} \underbrace{\begin{pmatrix} \sigma_{12}(r, \theta = \pm\pi) \\ \sigma_{22}(r, \theta = \pm\pi) \end{pmatrix}}_{\vec{0}} = \vec{0} \qquad \text{QED}$$

Let us apply $\tilde{Q}$ to the displacement field specified by (4.16a,b), we have

$$\vec{y}^* \equiv \tilde{Q}\vec{y} = \tilde{Q}\left(y_\alpha \vec{e}_\alpha\right) \Rightarrow \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = r^{m_1} U_1(\theta) \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + r^{m_2} U_2(\theta) \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} + \text{higher order terms}$$

Hence it is always possible to choose a suitable rotation, i.e., φ so that

$$y_\alpha^* = r^m U_\alpha^*(\theta) + o\left(r^m\right) \qquad r \to 0 \tag{4.20}$$

This means that there is no loss in generality to assume that both deformed coordinates has the same power exponent m, i.e.,

$$y_\alpha = r^m U_\alpha(\theta) + o\left(r^m\right) \qquad r \to 0 \tag{4.21}$$

Note that the solution given by (4.21) in general will not satisfied the symmetric conditions imposed by a *Mode I* crack, that is,

$$y_1(x_1, x_2) = y_1(x_1, -x_2) \;\; \& \;\; y_2(x_1, -x_2) = -y_1(x_1, -x_2) \tag{4.22}$$

Therefore, in order to obtain local Mode I fields, it is necessary to consider higher order terms of the expansion of y_α since $U_1(\theta)$ may have to be identical zero in order to satisfy (4.22). In particular, there is no guarantee that such symmetric solution should exist. Indeed, Stephenson showed that there is no solution that satisfies the Mode *II* or *anti-symmetry* condition:

$$y_1(x_1, x_2) = -y_1(x_1, -x_2) \;\; \& \;\; y_2(x_1, -x_2) = y_1(x_1, -x_2) \tag{4.23}$$

In short, there is no Mode *II* in the nonlinear theory!

In summary, to consider the local crack tip field in a plane strain incompressible hyperelastic solid, one assumes a local solution of the form:

$$y_\alpha = r^m U_\alpha(\theta) + r^{m'} V_\alpha(\theta) + \text{higher order terms} \qquad m' > m \tag{4.24}$$

The general procedure consist of substituting (4.24) into the nonlinear governing equations and match terms of equal order, as well as requiring these solutions to satisfy the traction free boundary conditions. In theory, the procedure is straightforward. However, in practice, these calculations can be extremely messy and tricky and will not be presented here (many details can be found in Stephenson [*]). In this lecture I summarize the main relevant results which is for n = 1 and for large n ($n \geq 7/2$)

n = 1

This case corresponds to the Mooney-Rivlin solid. Stephenson's solution ((4.25) in his paper [*]) is:

$$y_1 = a_1 r^{1/2} \sin(\theta/2) + \frac{r}{a^2}\left[a_1 b_1 \cos\theta - a_2 b_2 \sin^2(\theta/2)\right] + O\left(r^{3/2}\right)$$
$$y_2 = a_2 r^{1/2} \sin(\theta/2) + \frac{r}{a^2}\left[a_1 b_2 \sin^2(\theta/2) + a_2 b_1 \cos\theta\right] + O\left(r^{3/2}\right) \qquad (4.25a,b,c)$$
$$p = -2\frac{\mu b_2}{a^2} r^{1/2} \cos(\theta/2) + O(r)$$

where a_α, b_α are arbitrary constants that cannot be determined by local analysis (it depends on the specimen geometry and the loading conditions). Also, a in (4.25c) is defined by

$$a = \sqrt{a_1^2 + a_2^2} > 0 \ . \qquad (4.25d)$$

Although Stephenson does not mentioned this in his paper, the result given by (4.25a,b) gives the following result if $a_\alpha \neq 0$. On the crack faces, $\theta = \pm\pi$, we have, as $r \to 0$,

$$y_1(\theta = \pm\pi) = \pm a_1 r^{1/2},\ y_2(\theta = \pm\pi) = \pm a_2 r^{1/2} \Rightarrow y_2 / y_1 = a_2 / a_1 \qquad (4.26)$$

Let us suppose $a_1 < 0$, then $y_1 > 0$ for the lower
crack face so it must undergo very large rotation - the crack faces open into a straight line locally.

Note that (4.25a,b) cannot satisfy the Mode I symmetry conditions (4.22) unless

$$a_1 = 0, a = |a_2|, b_1 = 0 \qquad (4.27a)$$

Thus, for Mode I, we have

$$y_1 = -\frac{r}{a}\left[b_2 \sin^2(\theta/2)\right] + O\left(r^{3/2}\right)$$
$$y_2 = a_2 r^{1/2} \sin(\theta/2) + O\left(r^{3/2}\right) \qquad (4.27b,c)$$

There is but one problem with the above solution, eqn. (4.27b) predicts that y_1 = 0 directly ahead of the crack tip. While we expect $y_2 = 0$ directly ahead of the crack tip, y_1 should not. In other words, the mapping between the deform and undeform configuration given by (4.27b,c) is not one-one. To remedy this situation, a higher order expansion for y_1 is needed. This was done by Stephenson (see (4.37) in his work). His result is

$$y_1 = -\frac{b_2}{a} r \sin^2(\theta/2) + \frac{1}{a} r^{3/2}\left[\frac{4}{3}\cos^3(\theta/2) + 4\sin^2(\theta/2)\cos(\theta/2)\right] + O\left(r^2\right)$$
$$y_2 = a r^{1/2} \sin(\theta/2) + \frac{1}{a} r^{3/2}\left[c_1 \sin(3\theta/2) - \frac{b_2^2}{2a^2}\sin(\theta/2)\right] + O\left(r^2\right), \qquad |a_2| = a \qquad (4.27d)$$

Thus,

$$y_1\left(r\to 0,\theta=0\right)=\frac{4}{3a}r^{3/2}+O\left(r^2\right) \tag{4.27e}$$

As expected, in Mode I the crack opens up symmetrically, that is, at $\theta=\pm\pi/2$, $-y_1=cr$, where we assume $c=\dfrac{b_2}{a}>0$. In this case, we have $y_2=\pm a_2 r^{1/2}$ for the upper and lower crack faces, thus,

$$y_2=\pm a_2 c^{1/2}\left|y_1\right|^{1/2} \tag{4.27f}$$

Thus, the crack opening displacement of a *Mode I crack* for a Mooney-Rivlin or Neo-hookean solid behaves in the same way as a linear elastic material. However, there is no Mode II !

The case of n $\geq 7/2$

A well-known difficulty with the Mooney-Rivlin model is that it under-predicts strain hardening. This will affect the asymptotic behavior of the crack tip fields. Therefore, it is reasonable to consider the limit of very large *n* in the model, it is for this reason that I have not included the results for $1<n<7/2$. For these cases Stephenson only gives the dominant solution for y_α in his paper (see (3.104) in [*]), where

$$\begin{aligned} y_1 &= -\frac{1}{a}r^{2-m}H(\theta)+o\left(r^{2-m}\right), \qquad m=1-\frac{1}{2n} \\ y_2 &= -ar^m U(\theta)+\frac{1}{a}r^{2-m}\chi(\theta)+o\left(r^{2-m}\right) \end{aligned} \tag{4.28a,b}$$

Here

$$\begin{aligned} U(\theta) &= \sin(\theta/2)\sqrt{1-\frac{2\kappa^2\cos^2(\theta/2)}{1+\omega}}\left[\omega+\kappa\cos\theta\right]^{\kappa/2}, \\ \omega &= \sqrt{1-\kappa^2\sin^2(\theta/2)}, \qquad \kappa=\frac{n-1}{n} \end{aligned} \tag{4.28c,d,e}$$

$$H(\theta)=-\frac{n^{5/2}}{m^2}\left[\omega+\kappa\cos\theta\right]^{2-m}\left[\frac{m}{2-m}F\left(1/2-1/m,1/2;3/2-1/m;\cos^2\xi_0\right)-\kappa\sin\xi_0\right]$$

where *F* stands for the Hypergeometric function (see page 556 in Handbook of Mathematical Functions, by Abromowitz, M and Stegan I, A., for definition and properties of these functions), $m=1-\dfrac{1}{2n}$ and

$$\cos\xi_0\equiv\frac{1}{n\sqrt{2}}\frac{\sqrt{1+\kappa\sin^2\theta-\omega\cos\theta}}{\omega+\kappa\cos\theta}, \quad \theta\in\left[0,\pi\right] \tag{4.28f}$$

Finally,

$$\chi(\theta)=c_2\left[\omega+\kappa\cos\theta\right]^{2-m}\left[\frac{1+\kappa\sin^2\theta-\omega\cos\theta}{n^2\left[\omega+\kappa\cos\theta\right]^2}-1\right] \qquad (4.28g)$$

Note, since $\omega(\theta=0)=1$ (see 4.28d), $\cos\xi_0=0$ at $\theta=0$. In addition, by virtue of (4.28f), *H* is an even function of θ , so y_1 is an even function of x_2. Also, *U* is an odd function of x_2. Hence if we take the constant $c_2=0$ in (4.28g), we will have the dominant terms for a Mode I crack:

$$\begin{aligned} y_1 &= -\frac{1}{a}r^{2-m}H(\theta)+o\left(r^{2-m}\right), \qquad m=1-\frac{1}{2n} \\ y_2 &= ar^mU(\theta)+o\left(r^{2-m}\right) \end{aligned} \qquad (4.28)$$

For *Mode I* loading, according to (4.28a,b), the crack faces are carried into curves in the (y_1,y_2) space given by

$$\begin{aligned} y_1 &= -\frac{1}{a}r^{2-m}H(\pm\pi)\Rightarrow\left[\frac{a}{H(\pm\pi)}(-y_1)\right]^{1/(2-m)}=r \qquad m=1-\frac{1}{2n} \\ y_2 &= ar^mU(\pm\pi)\Rightarrow y_2=\pm k(n)a^{\frac{2}{2-m}}\left|y_1\right|^{\frac{2n-1}{2n+1}}, \qquad y_1<0, \end{aligned} \qquad (4.29)$$

Stephenson has shown that $H(\pm\pi)$ > 0, therefore, $k(n)$ > 0. Note that for very large *n*, Eq. 4.29 implies that the deformed crack faces are *almost* flat. Thus, strain hardening gives crack opening displacements that are substantially different from prediction of linear elasticity. In fact, Stephenson [*] shown that the crack profile for general loading is still given by (4.29) (see (5.3) in [*]), so the crack will always open, there is no Mode II).

Stresses

The 1st Piola stresses in the reference coordinate (r,θ) can be worked out using (4.4b) and the asymptotic expansion of y_α given above. Likewise, the asymptotic true stress fields can be obtained using

$$\tau_{\alpha\beta}=2\Phi'(I)F_{\alpha\gamma}F_{\beta\gamma}-p\delta_{\alpha\beta} \qquad (4.30a)$$

or

$$\tilde{\tau}=\tilde{\sigma}\tilde{F}^T \qquad (4.30a)$$

Let us consider the simple case of *n = 1*, for Mode I, using the dominant terms in (4.37) leads to

$$\sigma_{11}(r,\theta)=\frac{\mu b_2}{a}$$
$$\sigma_{12}(r,\theta)=0 + \textit{higher order terms}$$
$$\sigma_{21}(r,\theta)=-\frac{\mu a}{2\sqrt{r}}sin(\theta/2) \qquad \textit{Mode I, n =1} \qquad (4.31a\text{-}d)$$
$$\sigma_{22}(r,\theta)=\frac{\mu a}{2\sqrt{r}}cos(\theta/2)$$

(4.31a-d)[2]

The true stresses are:

$$\tau_{11}(r,\theta)=\mu(b_2/a)^2 sin^2(\theta/2)$$
$$\tau_{12}(r,\theta)=\frac{\mu b_2}{2}r^{-1/2} sin(\theta/2) \qquad \textit{Mode I , n =1} \qquad (4.32a\text{-}c)$$
$$\tau_{22}(r,\theta)=\frac{\mu a^2}{4r}$$

- Note that the three components of the stresses at the crack tip have different orders of singularity. In Mode I, the normal stress τ_{22} dominates. Therefore, the stress state at the crack tip is that of uniaxial tension.
- This means that the stress state directly ahead of the crack tip is not *hydrostatic* which is predicted by linear theory. Indeed, notice that (4.32a) states that $\tau_{11}(r,\theta)$ is bounded everywhere, in particular, $\tau_{11}(r\to 0,\theta=0)=0$ whereas in linear elasticity $\tau_{11}(r\to 0,\theta=0)=\tau_{22}(r\to 0,\theta=0)=\frac{K_I}{\sqrt{2\pi r}}$.
- Note that the true stresses, expressed in terms of the *material coordinate system* (r,θ) are *separable*.

Mode I, n > 7/2

For a Mode I crack, the true stresses are (see Stephenson, 5.9 and 5.11)

$$\tau_{11}(r,\theta)=o\left(r^{-1+(1/n)}\right)$$
$$\tau_{12}(r,\theta)=Ak_{12}r^{-1+(1/n)}[\omega+\kappa\cos\theta]^{-1/(2n)}\cos\xi_0, \quad k_{12}\equiv 4b_0a^{2n-1}m^{2n}n^{(3/2)-n} \qquad (4.33a\text{-}c)$$
$$\tau_{22}(r,\theta)=Ak_{22}r^{-1}[\omega+\kappa\cos\theta]^{-1}, \quad k_{22}=2a^{2n}m^{2n}n^{1-n}$$

Recall that *A* (see 4.14) can be regarded as a reference modulus, the constants b_0 and *a* cannot be determined by local analysis as they depend on the applied loading and specimen geometry.

[2] This result given by (4.31a-d) is not in Stephenson. It is derived using his results.

Just as the case of n = 1, the near tip stresses in Mode I is still dominated by the normal component of the stress tensor. However, the differences between different stress components become much smaller as the hardening parameter *n* increases. Therefore, hardening recovers the hydrostatic state of stress near the crack tip. The near tip field is controlled by two parameters b_0 and *a* (instead of a single parameter in LEFM). However, the dominant field (which contributes to the J integral or energy release rate) depends only on the parameter *a*.

Stress in deformed coordinates

In most papers on the subject, the stresses (true or 1st Piola) are expressed in terms of the material coordinates x_α. A more practical way is to express these local fields in the deformed coordinates y_α. For example, one describes the shape of the deformed crack by specifying how y_2 varies with y_1. As I shall demonstrate below, the stresses expressed in terms of the deformed coordinates are not separable, and exhibits interesting behavior that are not readily seen when they are expressed in material coordinates. Take for example, a Mode I crack with *n* = 1, where the true stresses are given by (4.32a-c). Let us look at the component of stress that gives the highest singularity τ_{22}. When expressed in material coordinates,

$$\tau_{22}(r,\theta)=\frac{\mu a^2}{4r} \tag{4.34}$$

which is independent of θ. To express τ_{22} in the deformed coordinates, we need to invert the local mapping between the undeformed and the deformed configurations which is given by (4.27a,b). Note that if we ignore the higher order term $r^{3/2}$ in (4.27a), then the mapping is not invertible. Therefore, we have kept this additional term, that is,

$$\begin{aligned} y_1 &= -\frac{r}{a} b_2 \sin^2(\theta/2) + \frac{1}{a} r^{3/2}\left[\frac{4}{3}\cos^3(\theta/2) + 4\sin^2(\theta/2)\cos(\theta/2)\right] \\ y_2 &= a r^{1/2} \sin(\theta/2) \end{aligned} \tag{4.35a,b}$$

Equation (4.34) requires us to express *r* in terms of y_1, y_2 which I have not been able to do. Such close form expression, if exist, are not separable.

To gain insight, let us first examine *directly* ahead of the crack tip, at $\theta = 0$. From (4.25b), we see that $y_2 = 0$, thus, *directly* ahead of the crack tip,

$$y_1 = \frac{4}{3a} r^{3/2} \qquad \theta = 0 \Rightarrow y_2 = 0\,, \tag{4.36}$$

Substituting (4.36) into (4.34), we obtain

$$\tau_{22}(y_1, y_2 = 0) = \frac{\mu a^2}{4}\left(\frac{4}{3ay_1}\right)^{2/3} = \frac{\mu a^2}{4}\left(\frac{4}{3a\rho}\right)^{2/3} \qquad y_1 > 0 \qquad (4.37a)$$

where I have introduced the polar coordinate system (ρ, ϕ) in the deformed configuration, where

$$y_1 = \rho\cos\phi, y_2 = \rho\sin\phi, \qquad -\pi < \phi \le \pi \qquad (4.37b)$$

Eq. (4.37a) implies that the stress singularity directly ahead of the crack tip is reduced when expressed in the deformed coordinates. Note that this result is valid only at $\theta = 0$. On the other hand, let us approach the crack tip from the upper crack face, that is, $\theta \to \pi$, (4.35a,b) implies that

$$|y_2| >> |y_1| \Rightarrow \rho = y_2, \phi \to \pi/2 \Rightarrow r = (\rho/a)^2 \qquad (4.38)$$

This means that,

$$\tau_{22}(\rho, \phi = \pi/2) = \frac{\mu a^2}{4}\left(\frac{a}{\rho}\right)^2 \qquad (4.39)$$

The singularity in τ_{22} is much higher as we approach the crack tip from the crack faces.

For a Mode-I crack in a neo-Hookean material, the asymptotic solution of crack tip deformation field is given in Stephenson (*). Using eq. (4.27d) above, we have, after normalizing y_1, y_2 and r by a^2:[3]

$$\frac{y_1}{a^2} = -\left(\frac{b_2}{a}\right)\left(\frac{r}{a^2}\right)\sin^2\left(\frac{\theta}{2}\right) + \left(\frac{r}{a^2}\right)^{3/2}\left(4\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \frac{4}{3}\cos^3\left(\frac{\theta}{2}\right)\right), \qquad (4.40a)$$

$$\frac{y_2}{a^2} = \left(\frac{r}{a^2}\right)^{1/2}\sin\left(\frac{\theta}{2}\right) + \left(\frac{r}{a^2}\right)^{3/2}\left(c_1\sin^3\left(\frac{\theta}{2}\right) - \frac{1}{2}\left(\frac{b_2}{a}\right)^2\sin\left(\frac{\theta}{2}\right)\right), \qquad (4.40b)$$

To plot the deformation mapping from the undeformed configuration (r, θ) to the deformed configuration (y_1, y_2), we assume c_1=1 and b_2/a = 1. The mapping is shown below for $0 \le r \le 0.1$ and $0 \le \theta \le \pi$.

[3] C.Y. Hui is grateful to Prof. Rong Long at the University of Alberta, who kindly provide the numerical calculations and graphs below .

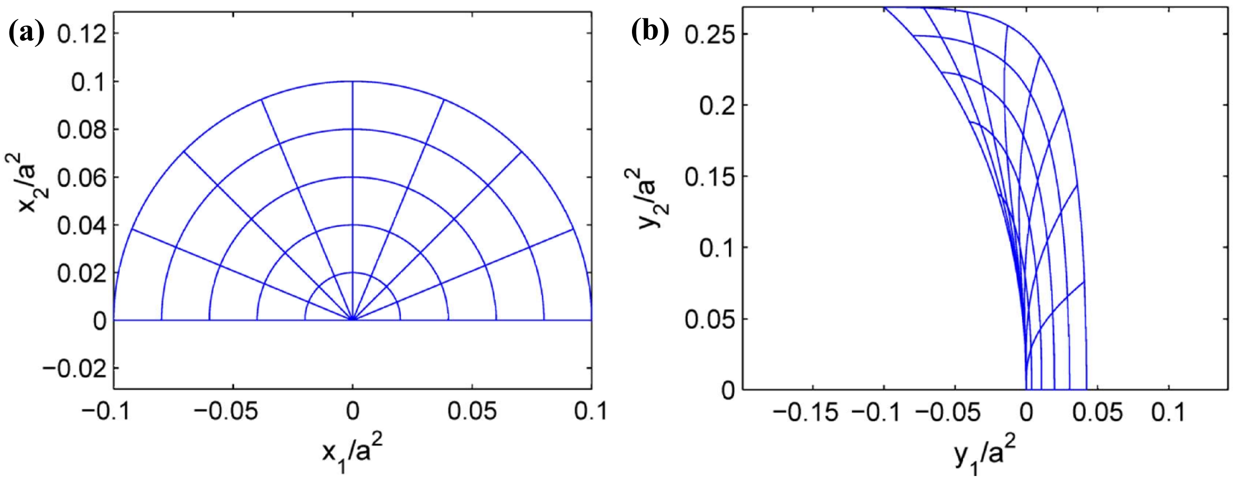

Figure 2.Mapping of the asymptotic crack tip field: (a) a "mesh" in undeformed configuration; the crack surface lies at x_2=0 and x_1<0; (b) deformed "mesh" calculated using (4.40a,b).

2. True stress field

Again using the solution of Stephenson (*), the true stress components expressed in terms of the undeformed coordinates (r,θ) are:

$$\frac{\tau_{11}}{\mu} = \left(\frac{b_2}{a}\right)^2 \sin^2\left(\frac{\theta}{2}\right), \tag{4.41a}$$

$$\frac{\tau_{12}}{\mu} = -\frac{1}{2}\left(\frac{b_2}{a}\right)\left(\frac{r}{a^2}\right)^{-1/2} \sin\left(\frac{\theta}{2}\right) + \sin\theta\cos\theta\,, \tag{4.41b}$$

$$\frac{\tau_{22}}{\mu} = \frac{1}{4}\left(\frac{r}{a^2}\right)^{-1/2} + \left(\frac{3}{2}c_1\cos\theta - \frac{1}{2}\left(\frac{b_2}{a}\right)^2\left(\frac{1}{2} - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right). \tag{4.41c}$$

The color contour of the three stress components in the deformed coordinates, computed using the mapping (4.40a,b) are given below.

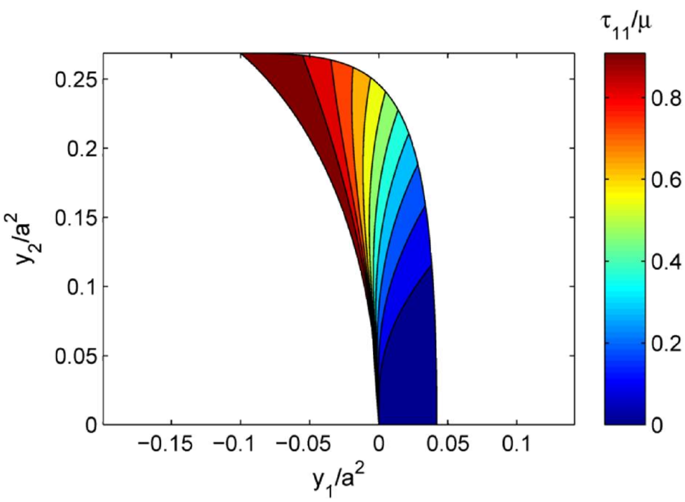

Figure 3 Contour plot of the normalized true stress τ_{11}/μ

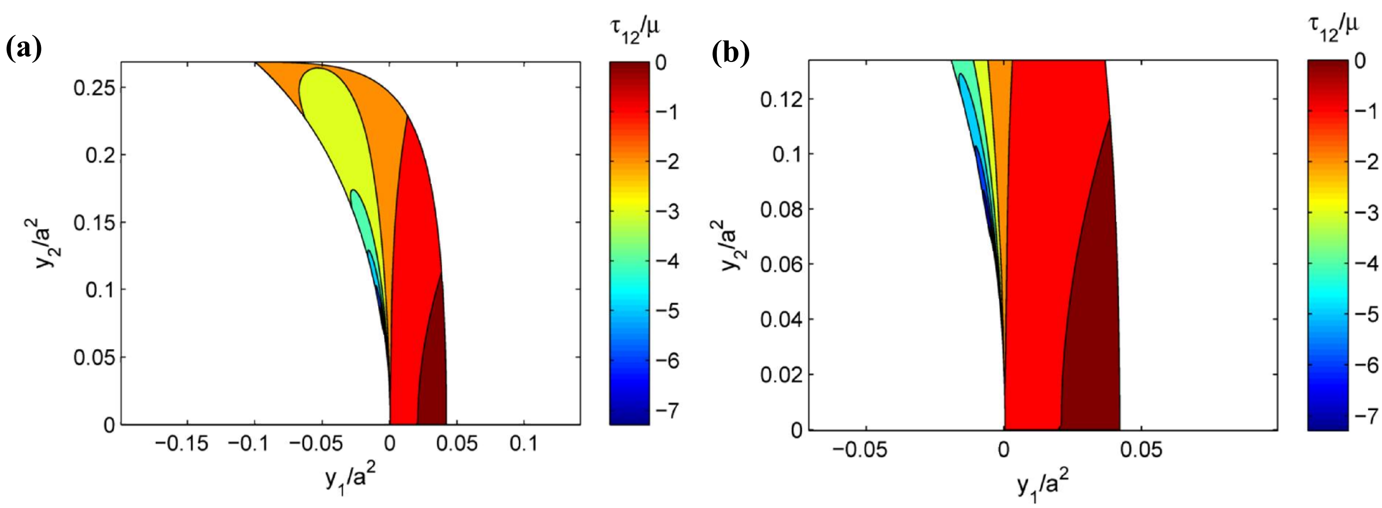

Figure 4 (a) Contour plot of the normalized true stress τ_{12}/μ; (b) zoom-in view of the crack tip.

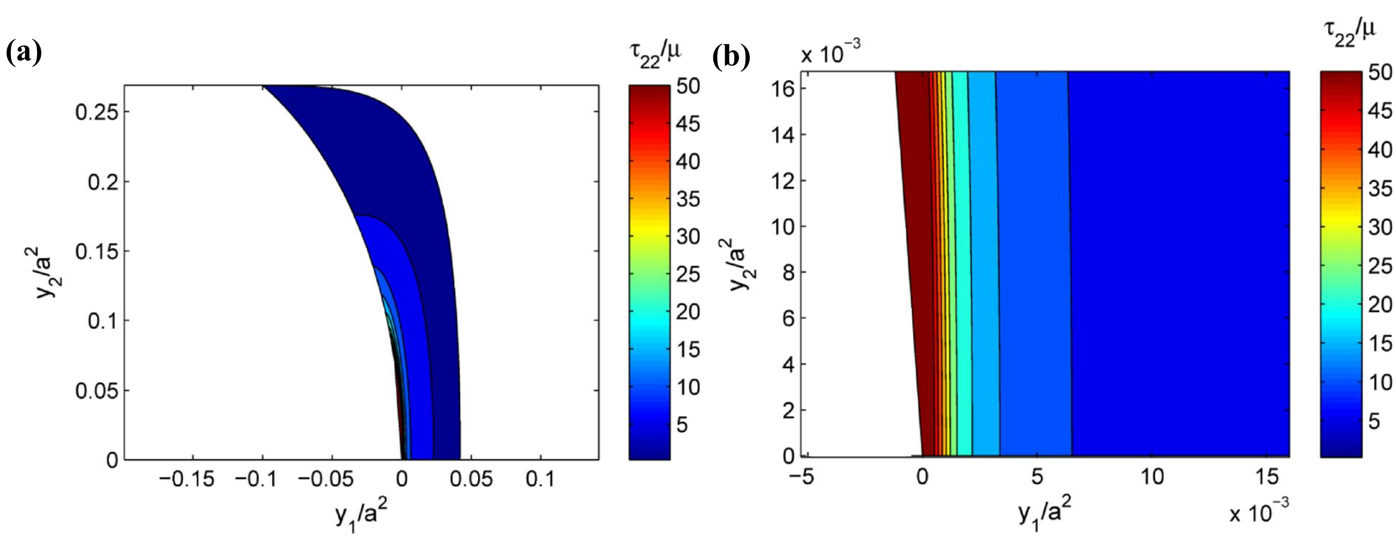

Figure 5 (a) Contour plot of the normalized true stress τ_{22}/μ; (b) zoom-in view of the crack tip.