# CRACK PATHS IN BRITTLE ELASTIC SOLIDS IN 2D AND 3D

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# OUTLINE

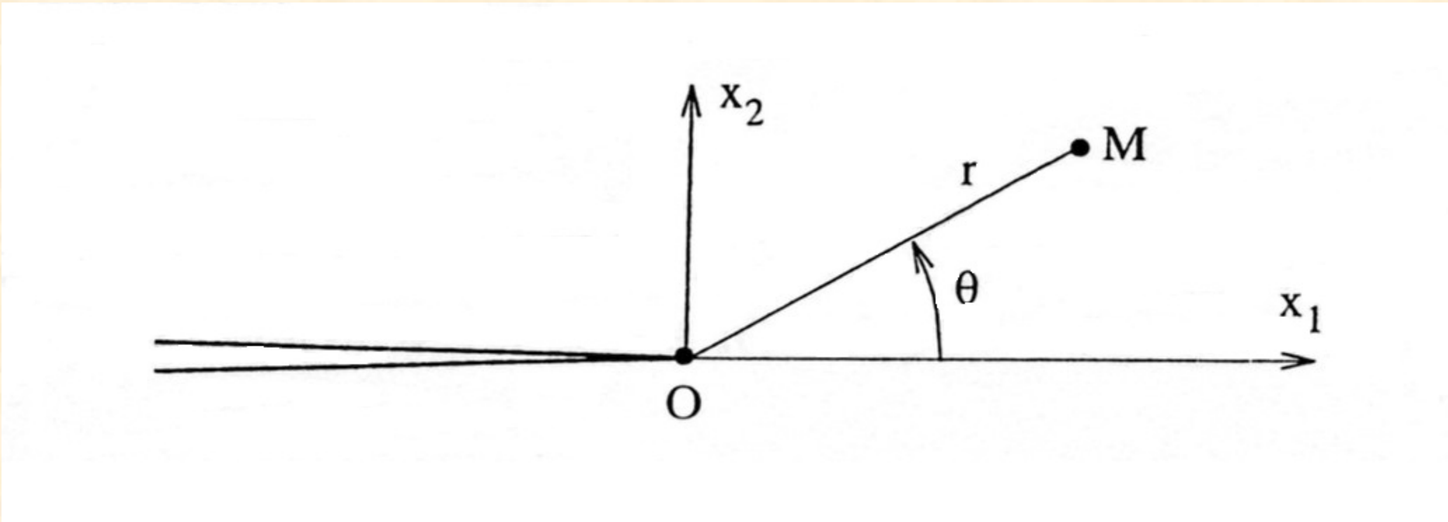
- Preliminaries
- Part A: crack propagation in mixed-mode I+II
- Part B: crack propagation in mixed-mode I+III
- Conclusion

# PRELIMINARIES

- The stress intensity factors
- The energy-release-rate and its relation to the stress intensity factors
- Griffith's and Irwin's crack propagation criteria

# The stress intensity factors

Asymptotic analysis of the stresses near the tip of a crack in a 2D elastic body :



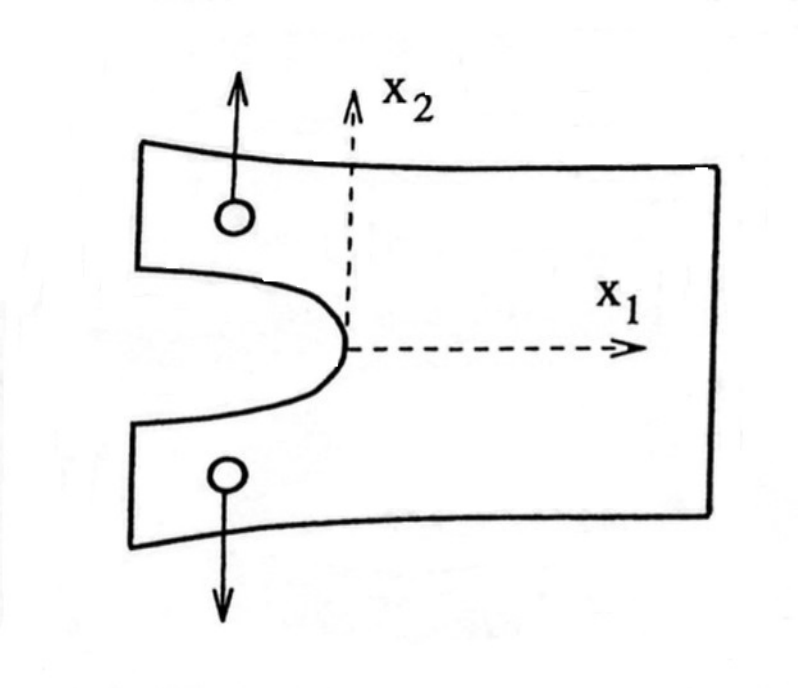
# Plane strain case:

$$\begin{cases} \sigma_{rr} = \dfrac{K_I}{4\sqrt{2\pi r}}\left(5\cos\dfrac{\theta}{2} - \cos\dfrac{3\theta}{2}\right) + \dfrac{K_{II}}{4\sqrt{2\pi r}}\left(-5\sin\dfrac{\theta}{2} + 3\sin\dfrac{3\theta}{2}\right) \\ \sigma_{\theta\theta} = \dfrac{K_I}{4\sqrt{2\pi r}}\left(3\cos\dfrac{\theta}{2} + \cos\dfrac{3\theta}{2}\right) + \dfrac{K_{II}}{4\sqrt{2\pi r}}\left(-3\sin\dfrac{\theta}{2} - 3\sin\dfrac{3\theta}{2}\right) \\ \sigma_{r\theta} = \dfrac{K_I}{4\sqrt{2\pi r}}\left(\sin\dfrac{\theta}{2} + \sin\dfrac{3\theta}{2}\right) + \dfrac{K_{II}}{4\sqrt{2\pi r}}\left(\cos\dfrac{\theta}{2} + 3\cos\dfrac{3\theta}{2}\right) \end{cases}$$

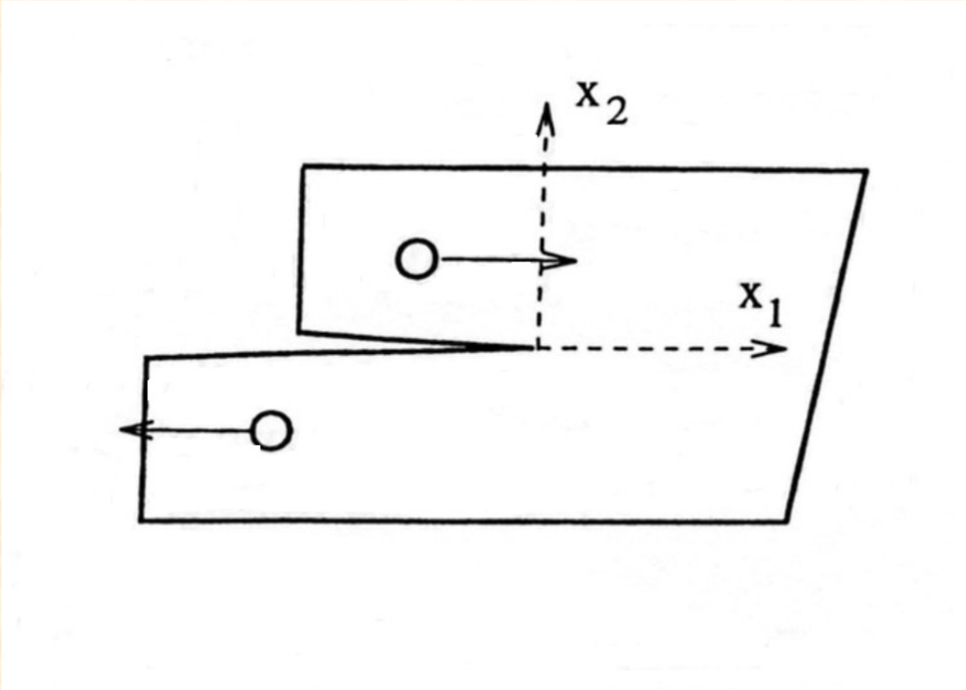
On the crack lips,

$$\begin{cases} u_1 = \pm\dfrac{4(1-\nu^2)}{E} K_{II}\sqrt{\dfrac{r}{2\pi}} \\ u_2 = \pm\dfrac{4(1-\nu^2)}{E} K_I\sqrt{\dfrac{r}{2\pi}} \end{cases}$$

$K_I, K_{II}$ : stress intensity factors (SIF) in mode I and II




Mode I
Opening of the crack
$(K_I > 0)$



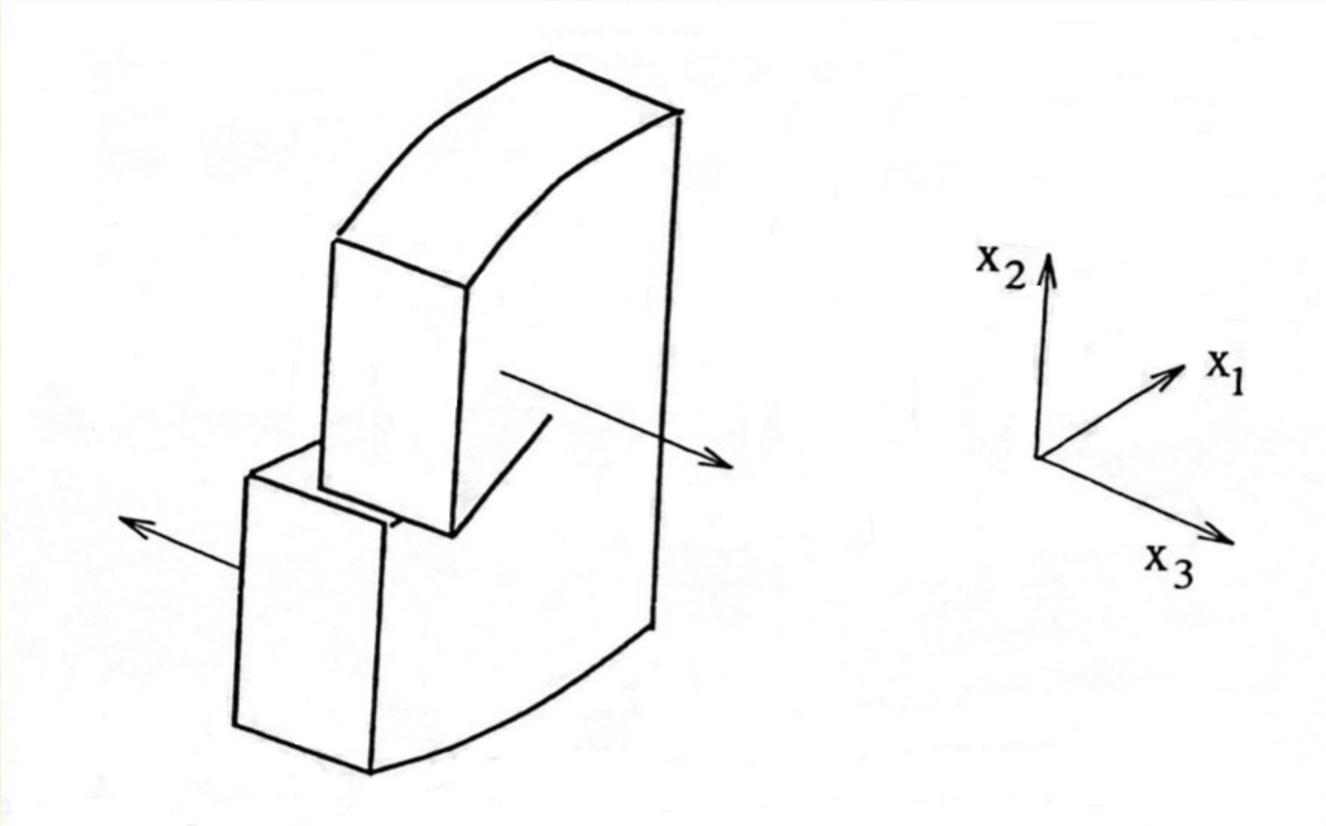

Mode II
Plane shear

# Antiplane case:

$$\begin{cases} \sigma_{13} = -\dfrac{K_{III}}{\sqrt{2\pi r}} \sin\dfrac{\theta}{2} \\ \sigma_{23} = \dfrac{K_{III}}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2} \end{cases}$$

On the crack lips, $$u_3 = \pm \frac{4(1+\nu)}{E} K_{III} \sqrt{\frac{r}{2\pi}}$$

$K_{III}$ : SIF in mode III




Mode III
Antiplane shear

## The energy-release-rate and its relation to the stress intensity factors

Definition of the energy-release-rate $G$ :

$$\boxed{G \equiv -\frac{\partial}{\partial l}\left(U_R - W\right)}$$

$\partial / \partial \mathrm{l}$ : partial derivative with respect to the crack length under constant loading

$U_R$ : elastic potential energy

$-W$ : potential energy of prescribed external forces

## Irwin's formula:

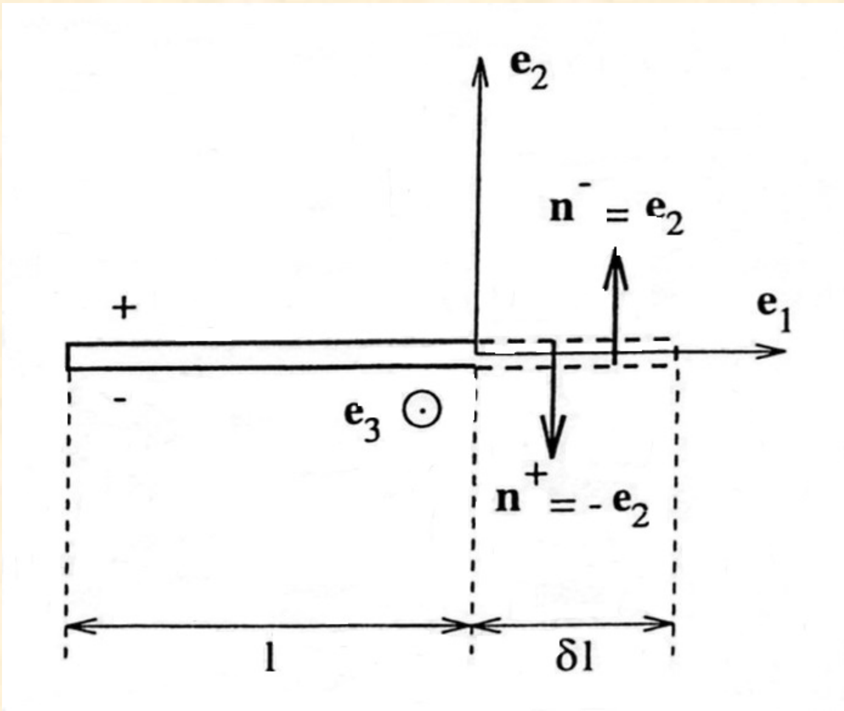
$$\boxed{G = \frac{1-\nu^2}{E}\left(K_I^2 + K_{II}^2\right) + \frac{1+\nu}{E} K_{III}^2}$$

This formula assumes that the crack is extended along its original direction (no kink!)

Sketch of the proof:

Reasoning in two steps :

1) Application of Betti's theorem to two successive stages of crack propagation



$$\longrightarrow \qquad G = -\lim_{\delta l \to 0^+} \frac{1}{2\delta l} \int_\Gamma \mathbf{T}^{(1)} . \mathbf{u}^{(2)} ds$$

$\mathbf{T}^{(1)}$ : traction vector, state (1): length of the crack $l$

$\mathbf{u}^{(2)}$ : displacement, state (2): length of the crack $l + \delta l$

2) Calculation of this integral at the lowest order in $\delta l$, tractions and displacements being replaced by their asymptotic expressions

# Griffith's and Irwin's crack propagation criteria

Irwin:

$$\begin{cases} K_I < K_{Ic} & \Rightarrow \quad \text{no propagation} \\ K_I = K_{Ic} & \Rightarrow \quad \text{possible propagation} \end{cases}$$

Griffith:

$$\begin{cases} G < R & \Rightarrow \quad \text{no propagation} \\ G = R & \Rightarrow \quad \text{possible propagation} \end{cases}$$

These criteria are equivalent with

$$\boxed{R = \frac{1-\nu^2}{E} K_{Ic}^2}$$

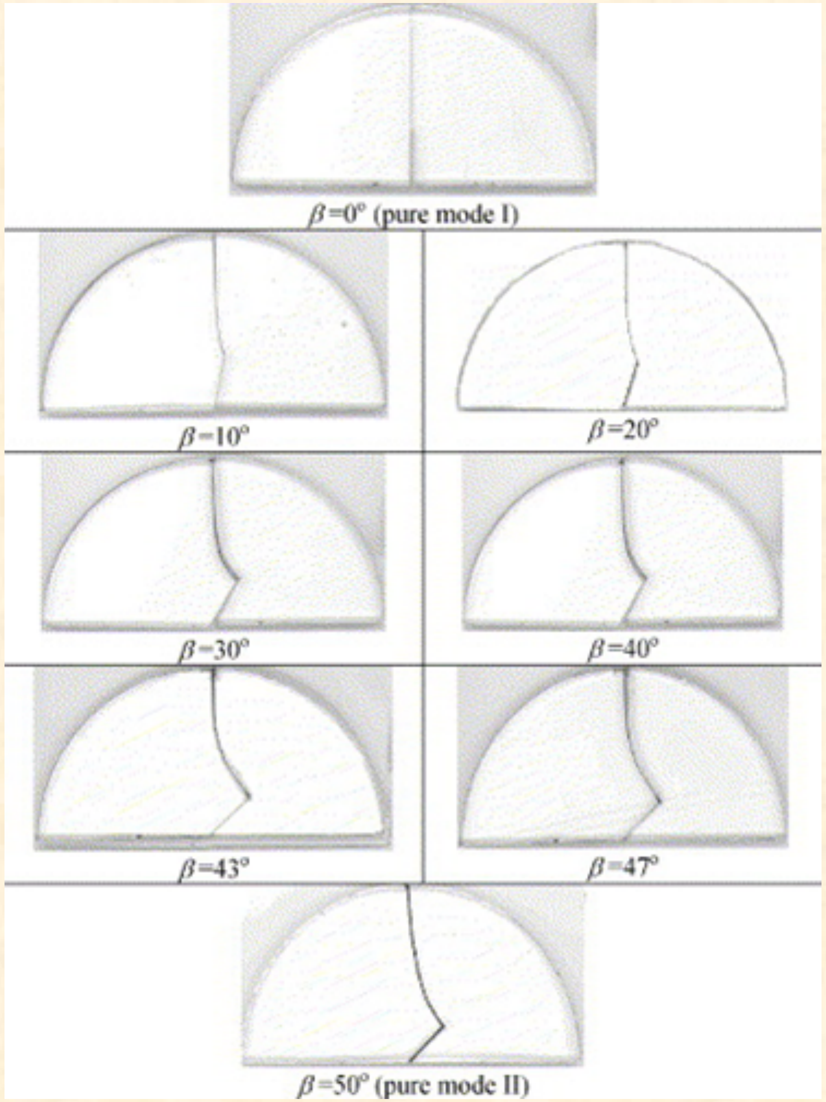
# PART A: CRACK PROPAGATION IN MIXED MODE I+II

- Two experimental examples
- The stress intensity factors just after the kink
- Irwin's extended formula
- Propagation criteria in mode I+II

# Two experimental examples

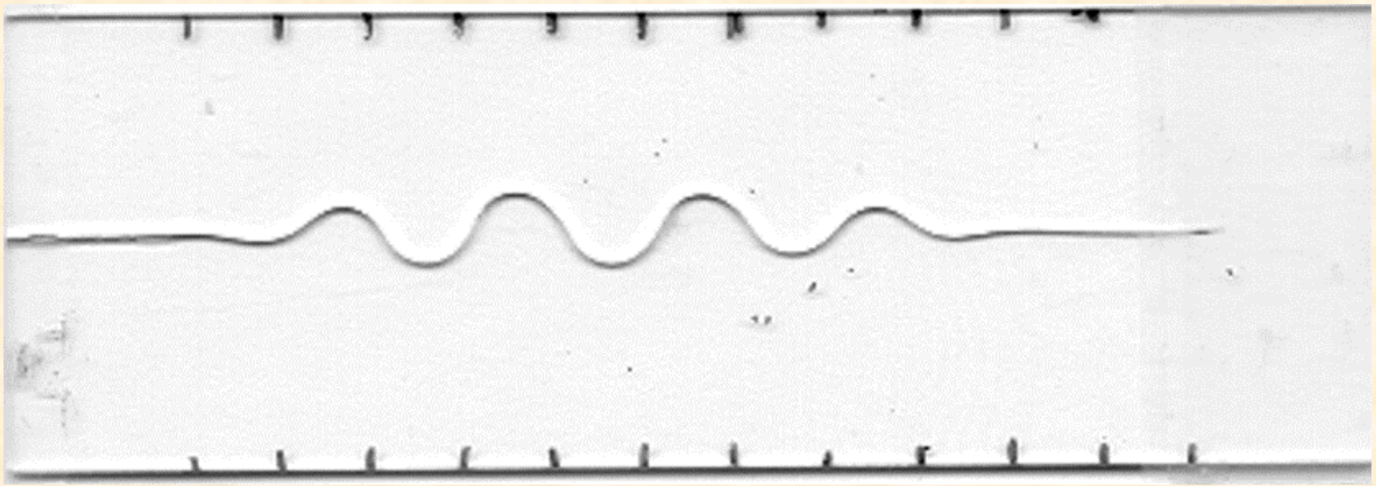
Example 1 (Ayatollahi):

Inclined crack in 3-point bending



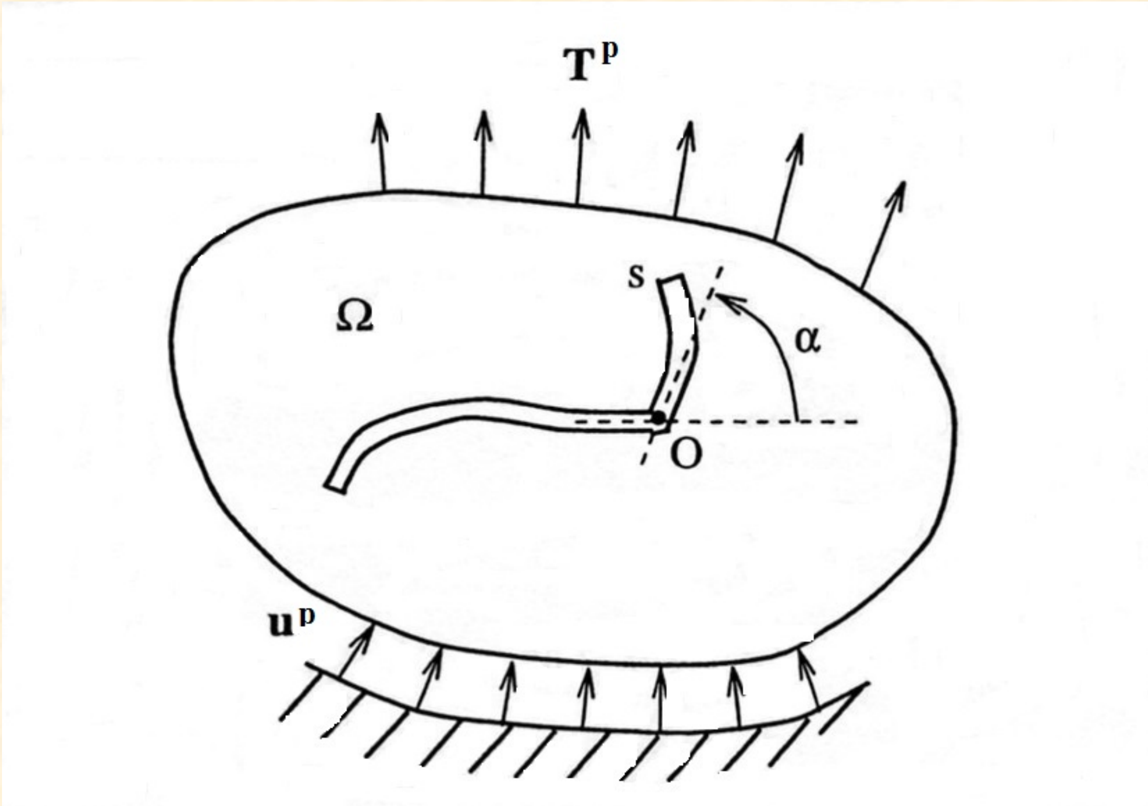
Example 2 (Yang and Ravi-Chandar):

Oscillations of the crack path in a heated strip



## The stress intensity factors just after the kink

Geometric situation:



$\mathbf{K} \equiv \left(K_I, K_{II}\right)$ : SIF just before the kink.

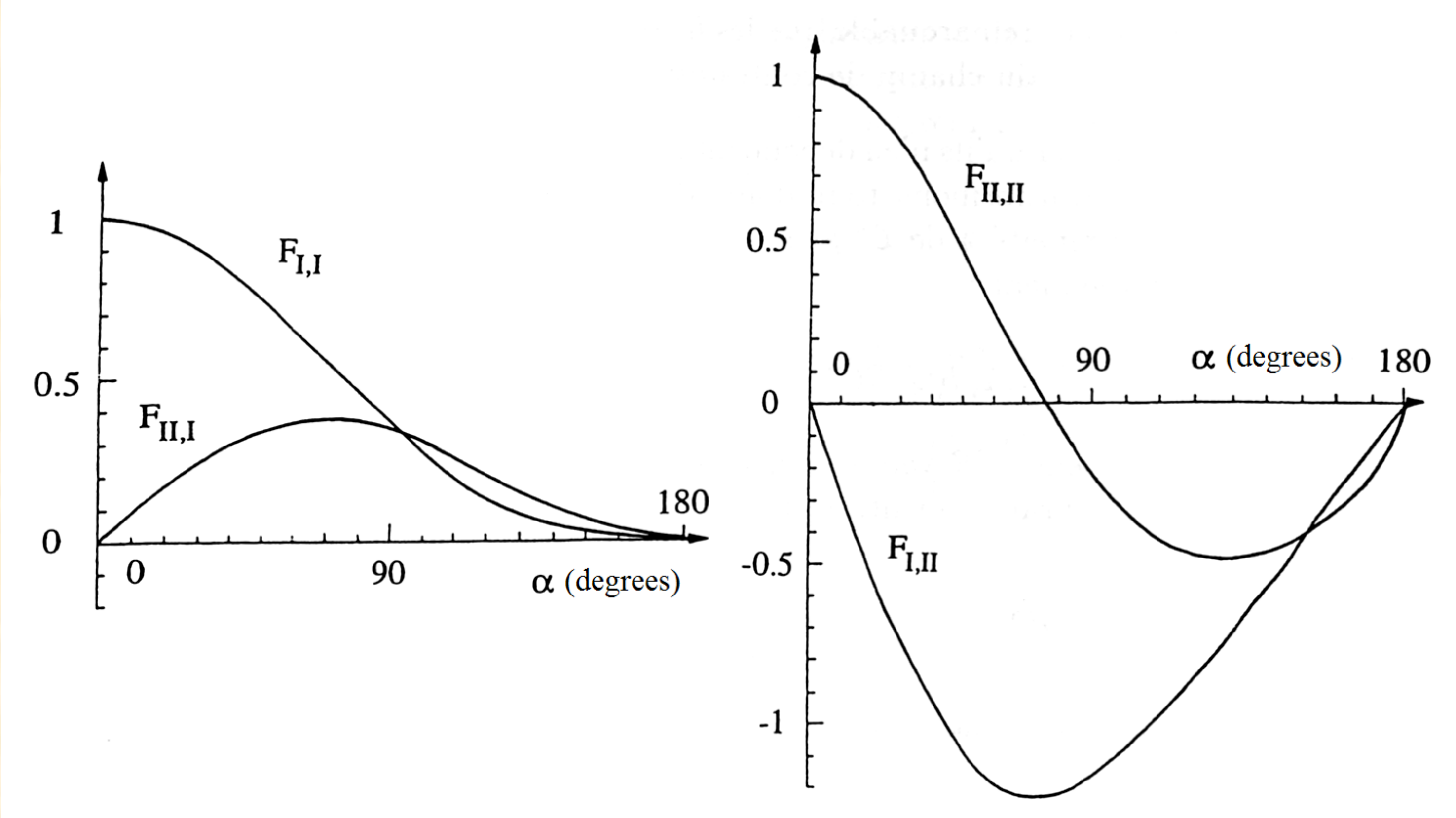
$\mathbf{K}^* \equiv \left(K_I^*, K_{II}^*\right)$ : SIF just after the kink.

Arguments based on scale changes (Karihaloo, Sumi, Leblond, Leguillon) $\longrightarrow$

$$\boxed{\mathbf{K}^* = \mathbf{F}(\alpha).\mathbf{K} \quad \Leftrightarrow \quad \begin{pmatrix} K_I^* \\ K_{II}^* \end{pmatrix} = \begin{bmatrix} F_{I,I}(\alpha) & F_{I,II}(\alpha) \\ F_{II,I}(\alpha) & F_{II,II}(\alpha) \end{bmatrix} . \begin{pmatrix} K_I \\ K_{II} \end{pmatrix}}$$

The SIF just after the kink depend on the geometry and the loading only through the SIF just before the kink and the kink angle.

Calculation of the functions $F_{p,q}(\alpha)$ based on Muskhelishvili's formalism (Wu, Karihaloo, Sumi, Amestoy and Leblond) →




(Amestoy's thesis)

# Irwin's extended formula

Reasoning based on the continuity of the total potential energy $U_R - W$ with respect to the crack extension length $s$ at $s = 0^+$, and on its differentiability for $s > 0$ (Ichikawa and Tanaka)

$$\boxed{G = \frac{1-\nu^2}{E}\left(K_I^{*\,2} + K_{II}^{*\,2}\right)}$$

For $\alpha = 0$, $K_I^* = K_I$, $K_{II}^* = K_{II}$ since $\mathbf{F}(\alpha = 0) = \mathbf{1}$, so that one recovers Irwin's usual formula.

# Propagation criteria in mode I+II

One must now predict

- the intensity of the loading promoting propagation of the crack (like in pure mode I);
- the kink angle.

## Prediction of the kink angle

Experimental observation : a kink occurs as soon as $K_{II} \neq 0$.

Coupled to the hypothesis that the propagation path is regular after the initial kink, this observation suffices to fix the value of the kink angle:

- After the initial kink ($s > 0$), the path is smooth (no kink) so that $K_{II}(s) = 0$.

- Taking the limit $s \to 0^+$, one concludes that

$$\boxed{K_{II}^* = F_{II,I}(\alpha)K_I + F_{II,II}(\alpha)K_{II} = 0}$$

(Goldstein and Salganik's principle of local symmetry)

Comparison with « Griffith's criterion » $G \max/\alpha$ :

If the criteria $K_{II}^* = 0$ and $G \max/\alpha$ coincided, the following identity would hold :

$$\frac{F_{I,I}'(\alpha)}{F_{I,II}'(\alpha)} = \frac{F_{II,I}(\alpha)}{F_{II,II}(\alpha)} \qquad \text{for every } \alpha$$

The calculation of the 6-th order expansion of the functions (Amestoy and Leblond) shows that this identity does not hold. Therefore the two criteria are distinct, although numerically very close.

<u>Prediction of the intensity of the loading promoting propagation</u>

- After the kink, pure mode I $\longrightarrow$ $\boxed{K_I(s) = K_{Ic}}$
- In the limit $s \to 0^+$: $\boxed{K_I^* = K_{Ic}}$

# PART B: CRACK PROPAGATION IN MIXED-MODE I+III

- Basic 3D fracture mechanics
- Some experimental examples
- Pons and Karma’s numerical simulations
- « Continuous propagation »: helical instability of the crack front

## Basic 3D fracture mechanics

- The main term of the stress expansion is obtained by simply adding the plane strain and antiplane solutions (Leblond and Torlai)

$\longrightarrow$ Local SIF $K_I(s), K_{II}(s), K_{III}(s)$
( $s$ : curvilinear length along the crack front)

- Under constant loading, the variation of the total potential energy $U_R - W$ induced by a small crack advance $\delta a(s)$ is given, to first order, by

$$\boxed{\delta\left(U_R - W\right) = -\int_{Crack\ \ front} G(s)\delta a(s)ds}$$

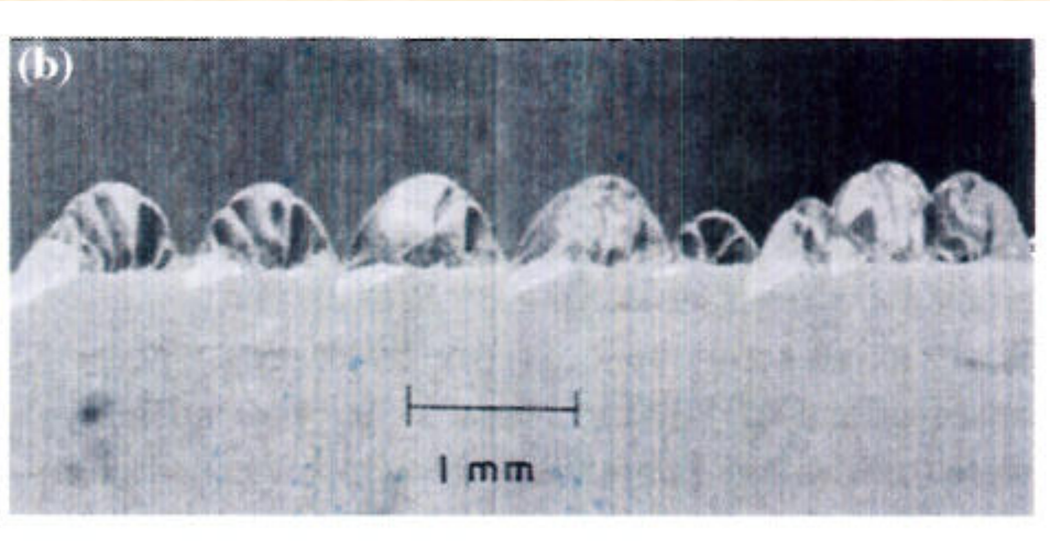
where $G(s)$ is given, for a geometrically smooth propagation of the crack, by a local Irwin formula :

$$\boxed{G(s) = \frac{1-\nu^2}{E}\left(K_I^2(s) + K_{II}^2(s)\right) + \frac{1+\nu}{E}K_{III}^2(s)}$$

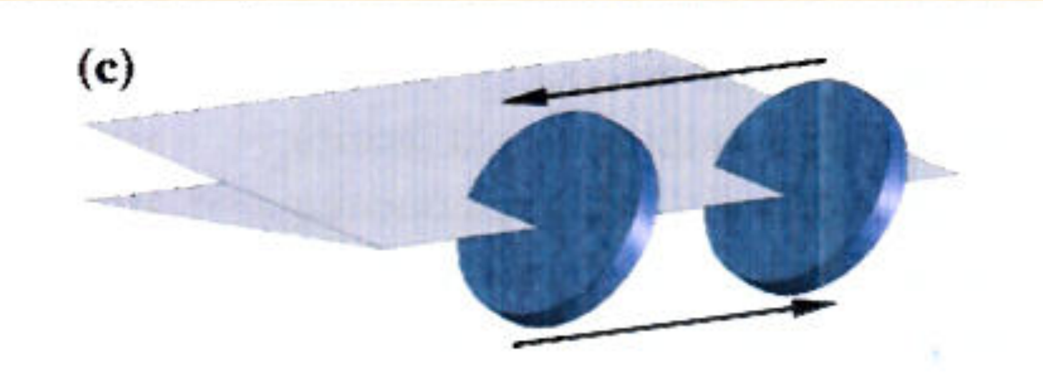
## Some experimental examples

- In all cases the crack evolves toward a situation of pure mode I by rotating about its direction of propagation.
- Propagation of the crack front occurs either at all points of this front (« continuous propagation ») or at discrete locations only (« discontinuous propagation »).

# Palaniswamy and Knauss (pure mode III):




Formation of « platelets » starting from behind the front



# Sommer (mode I+III, combined tension+torsion of glass tubes):

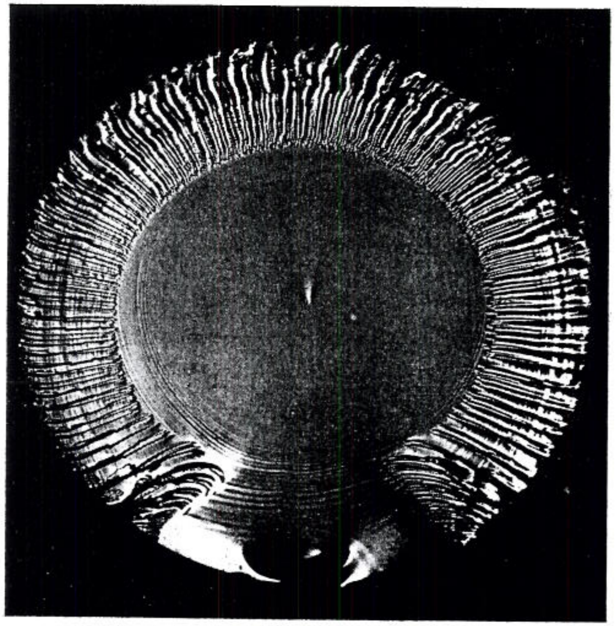


Fig. 2. Fracture surface of a cylindrical bar of glass surrounded with lances.*

pressure $p$ = 860 kg/cm$^2$
shear stress $\tau b$ = 50 kg/cm$^2$
diameter $\phi$ = 0·97 cm.

*We thank Dr. Sommer for his permission for use of this photo as our cover photo to date.

*The Editors*

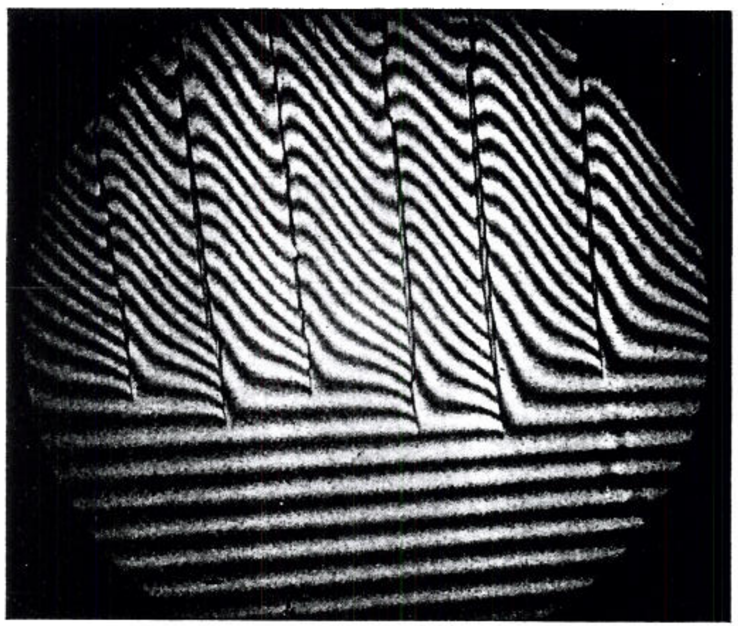
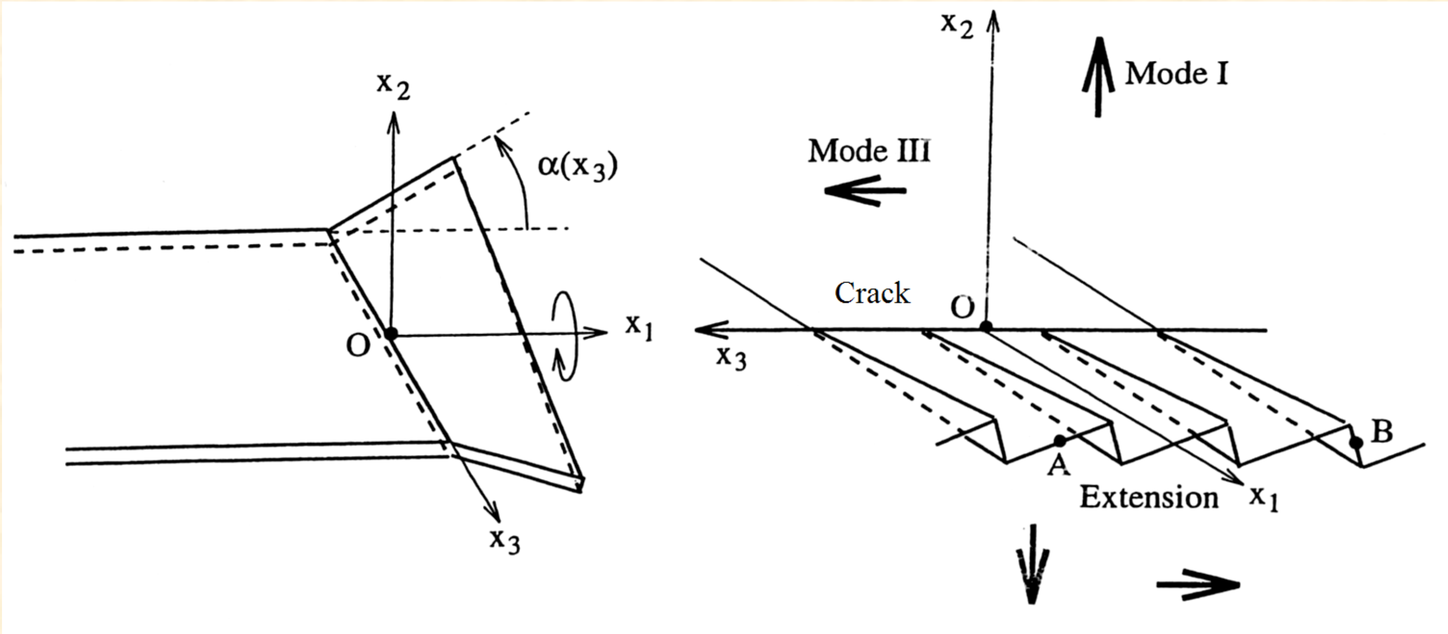


Fig. 1. Lance development in a fracture surface of glass. The picture is taken in an interference microscope ($\lambda = 0{\cdot}54 \times 10^{-3}$ mm thallium light) arrow marks crack growth direction, 250 ×.

Formation of « lances » or « facets » collectively building a « factory roof »

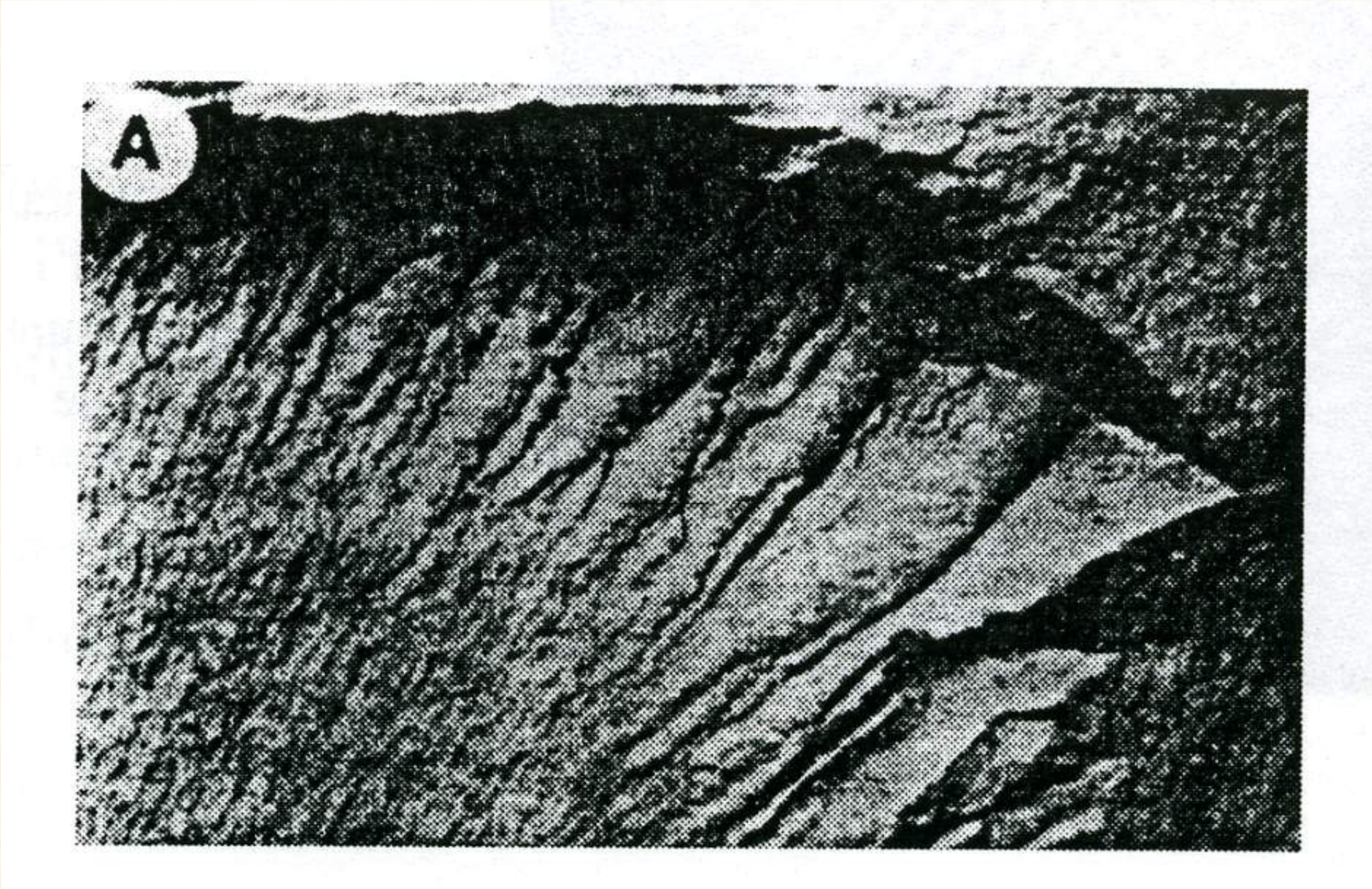
# Schematic geometry of factory roof:




- Existence of « type A » (favored) and « type B » (unfavored) facets

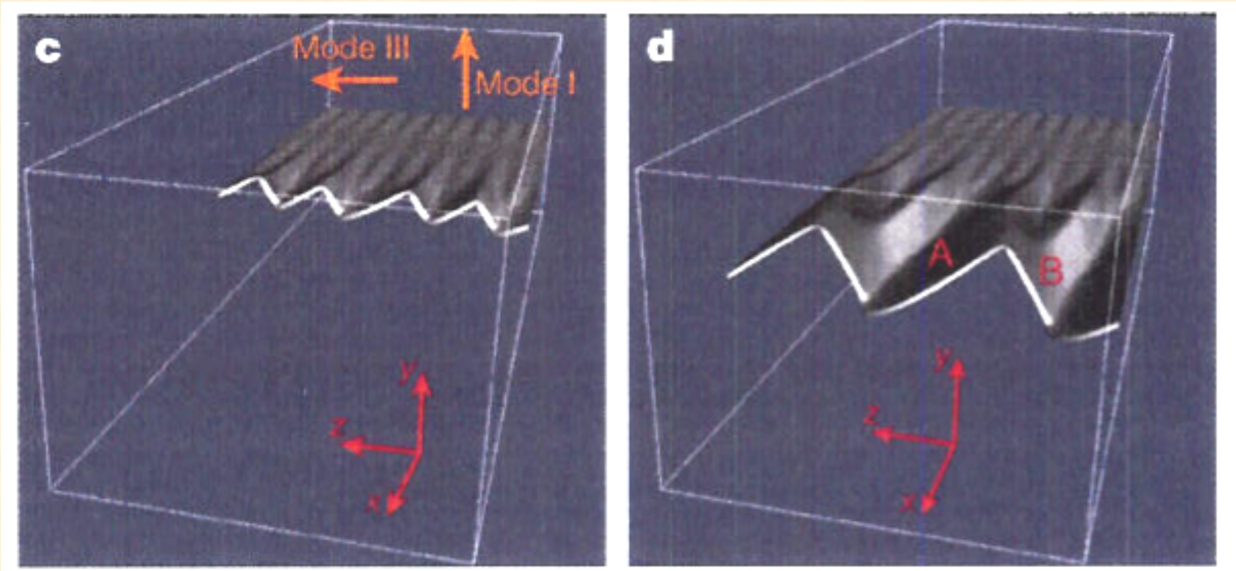
- Type B facets are shorter than type A ones, and often even totally absent (discontinuous propagation)

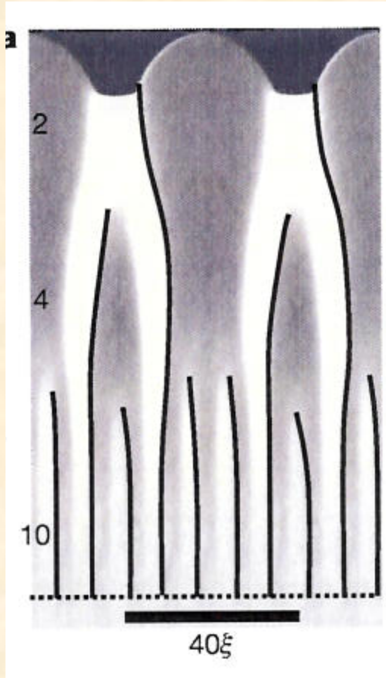
# Pollard and Aydin (geological material):



# Pons and Karma’s numerical simulations

These simulations, based on a « phase-field » model, reproduce both the instability of coplanar propagation in mode I+III and the gradual « coarsening » of facets.



The model used is approximately equivalent to enforce

- a Griffith-type condition $G(s) = R = Cst.$
- Goldstein and Salganik's principle of local symmetry $K_{II}(s) = 0$

all along the crack front.

The simulations therefore suggest that a theoretical explanation of crack front instability in mode I+III (for continuous propagation) might be found by using this double criterion.

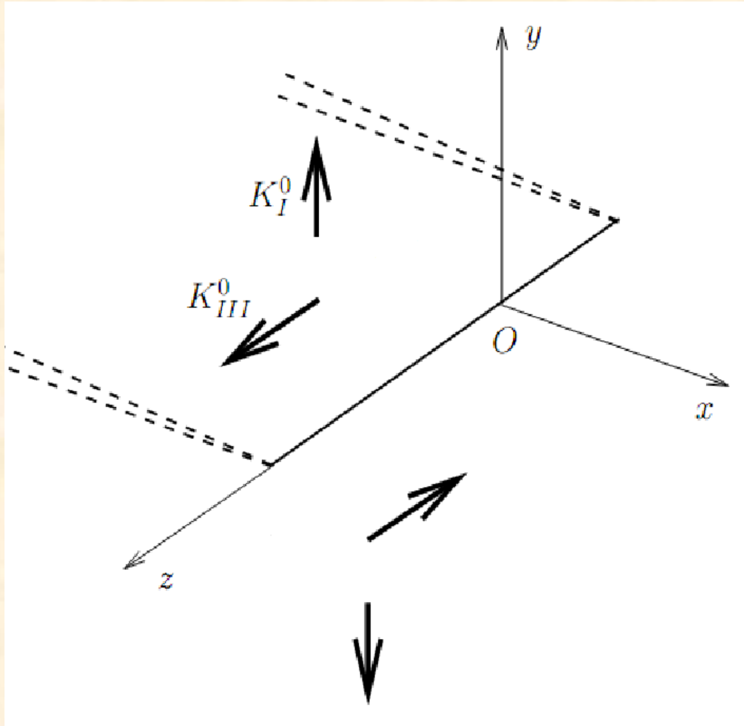
# « Continuous propagation »: helical instability of the crack front

(Leblond, Karma and Lazarus)

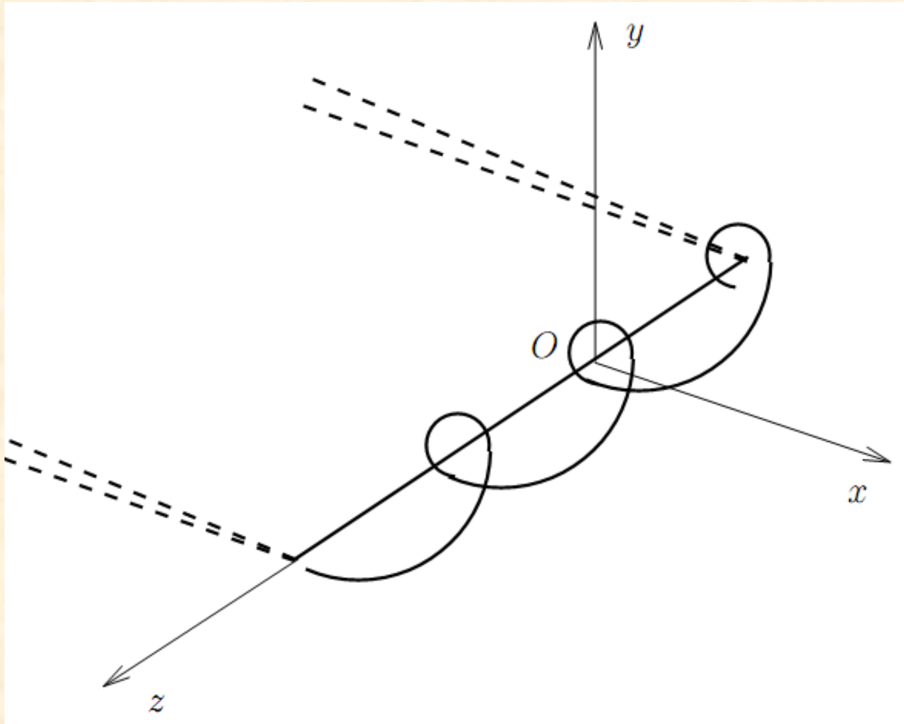
Continuous propagation is assumed. The theoretical analysis is based on:

- formulae (Movchan, Willis, Gao, Rice) providing the exact expression of the stress intensity factors to first order in the in-plane and out-of-plane perturbations of the crack;
- Griffith's criterion and Goldstein and Salganik's principle of local symmetry.

# Geometrical configurations:




Initial situation




Perturbed situation

Perturbed front: elliptic helix with size growing exponentially with the distance of propagation.

## In-plane and out-of-plane perturbations of the crack:

$$\begin{cases} \varepsilon\phi_x(x,z) & = & \varepsilon A_x\, e^{x/a} \cos(kz) \\ \varepsilon\phi_y(x,z) & = & \varepsilon A_y\, e^{x/a} \sin(kz) \end{cases} , \quad a > 0, k > 0$$

## Perturbed stress intensity factors:

$$\begin{cases} \delta K_I(z) & = & -\dfrac{K_I^0}{2}\,\varepsilon A_x k\cos(kz) + K_{III}^0\,\varepsilon A_y k\cos(kz)\left(-2 + \dfrac{1-2\nu}{\sqrt{2}(1-\nu)}F(ka)\right) \\ \delta K_{II}(z) & = & \dfrac{K_I^0}{2}\,\varepsilon\,\dfrac{A_y}{a}\sin(kz)\left(1 + \dfrac{2-3\nu}{2-\nu}ka\right) + \dfrac{2}{2-\nu}K_{III}^0\,\varepsilon A_x k\sin(kz) \\ \delta K_{III}(z) & = & \dfrac{2(1-\nu)^2}{2-\nu}K_I^0\,\varepsilon A_y k\cos(kz) - \dfrac{2+\nu}{2(2-\nu)}K_{III}^0\,\varepsilon A_x k\cos(kz) \end{cases}$$

where $F(u) \equiv \sqrt{\dfrac{u}{1+u}}$

(these expressions apply to the position $x = 0$ of the front and must be multiplied by $e^{x/a}$ if $x \neq 0$ ).

Application of the double criterion:

- Principle of local symmetry $\longrightarrow$

$$\boxed{\frac{A_y}{A_x} = -4\frac{K_{III}^0}{K_I^0}\frac{ka}{2-\nu+(2-3\nu)ka}}$$

This equation defines the elliptic shape of the projection of the crack front onto the plane $Oxy$.

- Griffith's criterion $\longrightarrow$

$$ka = \frac{(2+\nu)\left(\dfrac{K_{III}^{0}}{K_{I}^{0}}\right)^{2} + (1-\nu)(2-\nu)}{\left[3(2-\nu) - 4\sqrt{2}\,(1-2\nu)F(ka)\right]\left(\dfrac{K_{III}^{0}}{K_{I}^{0}}\right)^{2} - (1-\nu)(2-3\nu)}$$

This equation defines the ratio of the characteristic distances $a$ and $\lambda \equiv 2\pi/k$ of the perturbation in the directions $Ox$ and $Oz$.

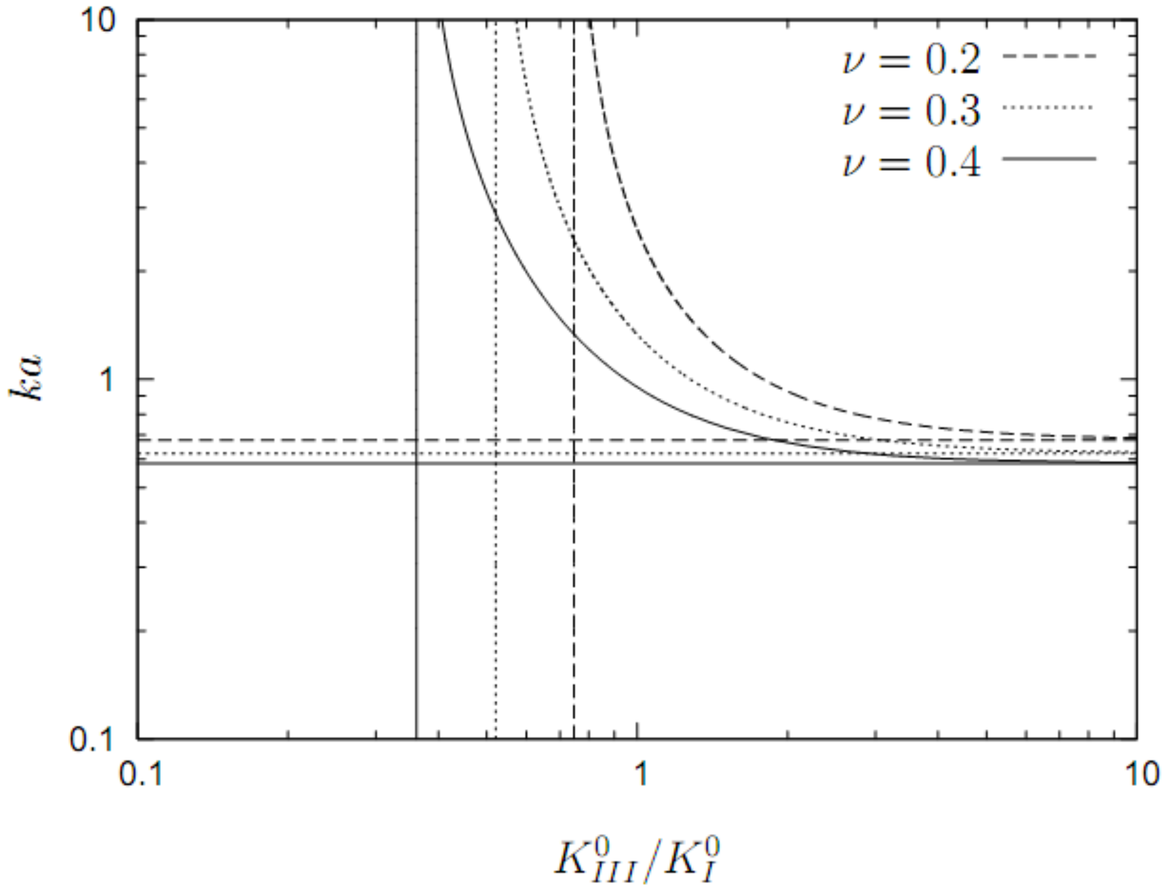
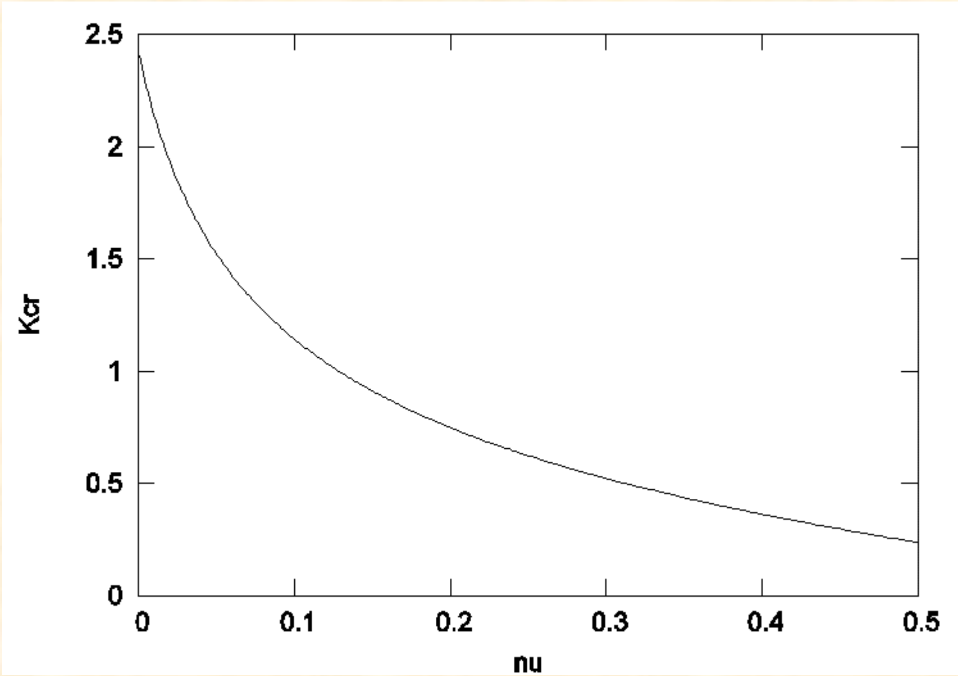



Fig. 3. Plot of $ka$ versus $K_{III}^0/K_I^0$ for $\nu = 0.2$, 0.3 and 0.4.

A non-trivial solution (mode of bifurcation) exists when $K_{III}^0/K_I^0$ is larger than some threshold $(K_{III}^0/K_I^0)_{\mathrm{cr}}$.

# Calculation of the threshold $(K^0_{III}/K^0_{I})_{\rm cr}$:

$$\boxed{\left(\frac{K^0_{III}}{K^0_{I}}\right)_{\rm cr} = \left[\frac{(1-\nu)(2-3\nu)}{3(2-\nu)-4\sqrt{2}\,(1-2\nu)}\right]^{1/2}}$$



This theoretical threshold is considerably higher than that observed experimentally. Possible explanations:

- Large effect of imperfections below the bifurcation threshold
- Deviations from coplanarity of a different type (discontinuous propagation)

# CONCLUSION

- The problem of crack kinking in mode I+II (2D problem) is now well understood and may be considered as settled.
- In contrast much remains to be done on the problem of formation of inclined fracture facets in mode I+III (3D problem).

MANY THANKS FOR YOUR KIND AND PATIENT ATTENTION!