# Dynamic fracture

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*Lecture 3 presented at the*

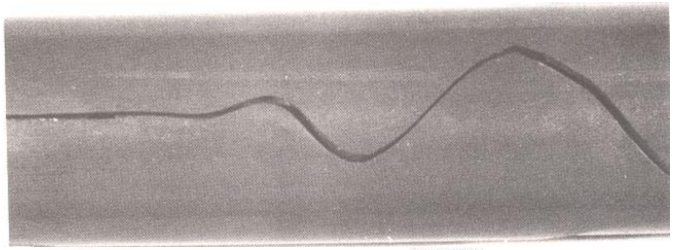
***University of Pierre and Marie Curie***

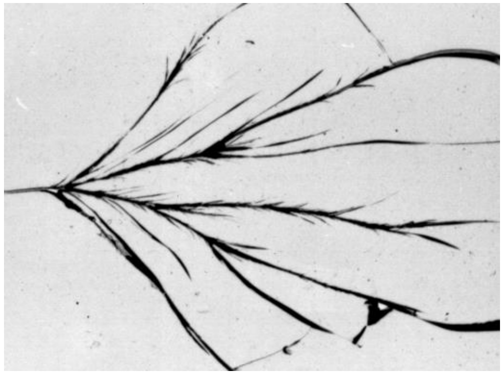
***May 15, 2014***

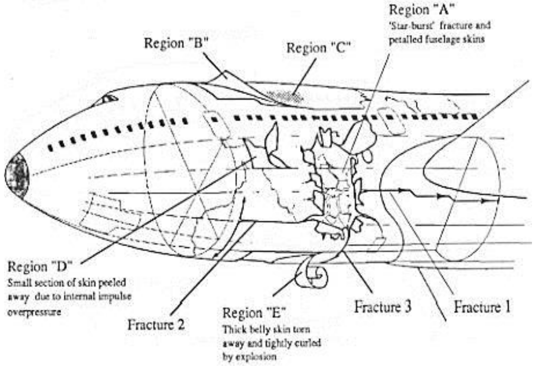
# Motivating problems

- Pipelines and pressure vessels
- Nuclear reactor containment vessels
- Airplanes
- Armor penetration and protection
- Earth’s crust
- Scientific curiosity

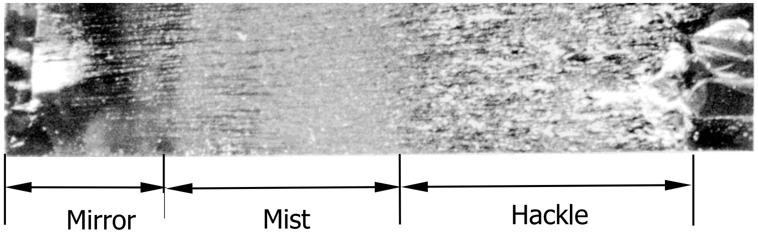
National Transportation Safety Board
http://www.ntsb.gov







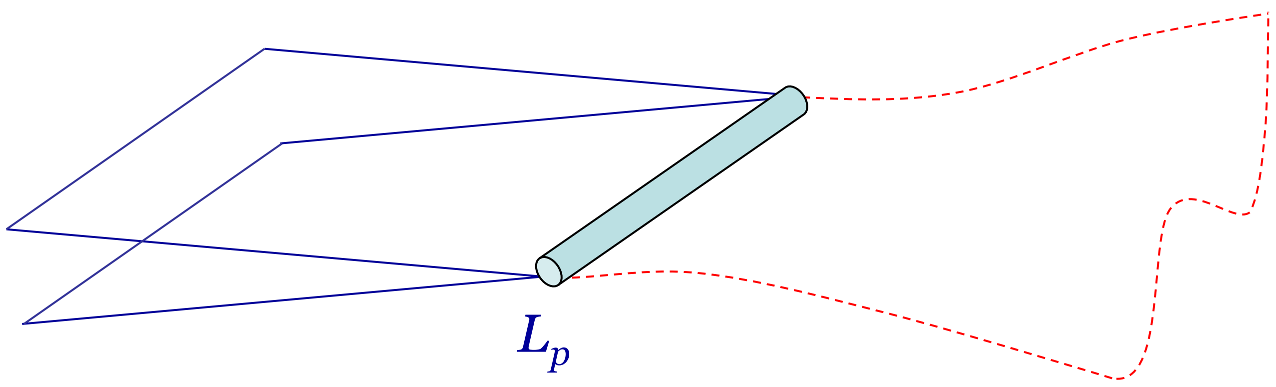

UK Air Accidents Investigation Report , N739PA at Lockerbie, Scotland on 21 December 1988




*Aerial view of the San Andreas fault slicing through the Carrizo Plain in the Temblor Range east of the city of San Luis Obispo. (Photograph by Robert E. Wallace, USGS.)*
http://pubs.usgs.gov/publications/text/San_Andreas.html

# The continuum view of fracture

**Outer Problem**: Knowledge of constitutive laws and balance equations is adequate to calculate the energy release rate, $G$




**Inner Problem**: Are the details of the failure mechanisms and fracture processes important ***only*** in determining the fracture energy, $\Gamma$?

# Review of linear elastodynamics

$$\mathbf{y}(\mathbf{x},t) = \mathbf{x} + \mathbf{u}(\mathbf{x},t)$$

$$\boldsymbol{\varepsilon}(\mathbf{x},t) = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]$$

$$\boldsymbol{\sigma}(\mathbf{x},t) = \lambda\varepsilon_{kk}\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{f} = \rho\ddot{\mathbf{u}}$$

$$\boxed{(\lambda+\mu)\nabla\left(\nabla\cdot\mathbf{u}\right) + \mu\nabla^2\mathbf{u} + \mathbf{f} = \rho\ddot{\mathbf{u}}}$$

***- Navier's equations of motion***

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}^*(\mathbf{x},t) \quad \text{on} \quad \partial_1 R$$

$$\mathbf{s}(\mathbf{x},t) = \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n} = \mathbf{s}^*(\mathbf{x},t) \quad \text{on} \quad \partial_2 R$$

***- boundary conditions***

***- initial conditions***

## Bulk waves

$$\nabla^2\mathbf{u} = \nabla\left(\nabla\cdot\mathbf{u}\right) - \nabla\times\nabla\times\mathbf{u}$$

$$\Rightarrow (\lambda+2\mu)\nabla\left(\nabla\cdot\mathbf{u}\right) - \mu\nabla\times\nabla\times\mathbf{u} = \rho\ddot{\mathbf{u}}$$

If deformation is irrotational: $\nabla\times\mathbf{u} = 0$

$$\boxed{\nabla^2\mathbf{u} = \frac{1}{C_d^2}\ddot{\mathbf{u}} \quad \text{with} \quad C_d = \sqrt{\frac{\lambda+2\mu}{\rho}}}$$

$\Rightarrow$ dilatational or pressure waves

If deformation is equivoluminal: $\nabla\cdot\mathbf{u} = 0$

$$\boxed{\nabla^2\mathbf{u} = \frac{1}{C_s^2}\ddot{\mathbf{u}} \quad \text{with} \quad C_s = \sqrt{\frac{\mu}{\rho}}}$$

$\Rightarrow$ distortional or shear waves

# Wave speeds

| Material | Modulus of elasticity $E$ | Poisson's Ratio $\nu$ | Density $\rho$ | Dilatational Wave Speed $C_d$ | Distortional Wave Speed $C_s$ | Plate Wave speed | Rayleigh Wave Speed $C_R$ |
|---|---|---|---|---|---|---|---|
| | GPa | | Mg/m³ | m/s | m/s | m/s | m/s |
| High strength steel | 200 | 0.3 | 7.8 | 5875 | 3140 | 5308 | 2913 |
| Tungsten and alloys | 406 | 0.3 | 13.4 | 6386 | 3414 | 5770 | 3167 |
| Aluminum alloys | 70 | 0.3 | 2.7 | 5908 | 3158 | 5338 | 2929 |
| Alumina | 390 | 0.22 | 3.9 | 10685 | 6402 | 10251 | 5858 |
| Silicon Nitride | 350 | 0.22 | 3.2 | 11175 | 6695 | 10721 | 6127 |
| Silica glass | 70 | 0.22 | 2.6 | 5544 | 3322 | 5319 | 3040 |
| Homalite-100 | 4.5 | 0.34 | 1.2 | 2402 | 1183 | 2059 | 1104 |
| Plexiglas | 3.4 | 0.34 | 1.2 | 2088 | 1028 | 1790 | 960 |
| Polycarbonate | 2.6 | 0.34 | 1.2 | 1826 | 899 | 1565 | 839 |
| Rubber | 0.1 | 0.499 | 0.85 | 4434 | 198 | 396 | 189 |
| | 0.01 | 0.499 | 0.85 | 1402 | 63 | 125 | 60 |

# Lamé solution

$$\mathbf{u} = \nabla\varphi + \nabla\times\boldsymbol{\psi}$$ Helmholtz resolution

$$\boxed{\nabla^2\varphi = \frac{1}{C_d^2}\ddot{\varphi}} \quad \boxed{\nabla^2\boldsymbol{\psi} = \frac{1}{C_s^2}\ddot{\boldsymbol{\psi}}}$$

Consider a plane wave propagating in the $x_1$ direction

$$u_1 = \varphi'(x_1 - C_d t)$$
$$u_2 = -\psi_3'(x_1 - C_s t)$$
$$u_3 = \psi_2'(x_1 - C_s t)$$

Dilatational waves are longitudinal waves

Distortional waves are transverse waves

Plane strain

$$u_\alpha = u_\alpha(x_1, x_2, t); \quad u_3 \propto x_3$$

$$\begin{cases} \varepsilon_{\alpha\beta}(x_1, x_2, t) = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \\ \varepsilon_{33} = \text{const}; \ \varepsilon_{3\alpha} = 0 \end{cases}$$

$$\begin{cases} \sigma_{\alpha\beta}(x_1, x_2, t) = \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu\varepsilon_{\alpha\beta} \\ \sigma_{33}(x_1, x_2, t) = \nu\sigma_{\gamma\gamma} \\ \sigma_{3\alpha}(x_1, x_2, t) = 0 \end{cases}$$

$$\varphi = \varphi(x_1, x_2, t); \ \boldsymbol{\psi} = \psi(x_1, x_2, t)\mathbf{e}_3$$

$$\boxed{\nabla^2\varphi = \frac{1}{C_d^2}\ddot{\varphi}} \quad \boxed{\nabla^2\psi = \frac{1}{C_s^2}\ddot{\psi}}$$

# Elastodynamics continued

$$u_1 = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}$$

$$u_2 = \frac{\partial \varphi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}$$

$$\sigma_{11} = \lambda \nabla^2 \varphi + 2\mu \left[ \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right]$$

$$\sigma_{22} = \lambda \nabla^2 \varphi + 2\mu \left[ \frac{\partial^2 \varphi}{\partial x_2^2} - \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right]$$

$$\sigma_{12} = \mu \left[ 2 \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi}{\partial x_2^2} - \frac{\partial^2 \psi}{\partial x_1^2} \right]$$

*Note that boundary conditions couple the derivatives of the dilatational and distortional potentials, regardless of whether the displacements or tractions are prescribed.*

# Rayleigh surface waves

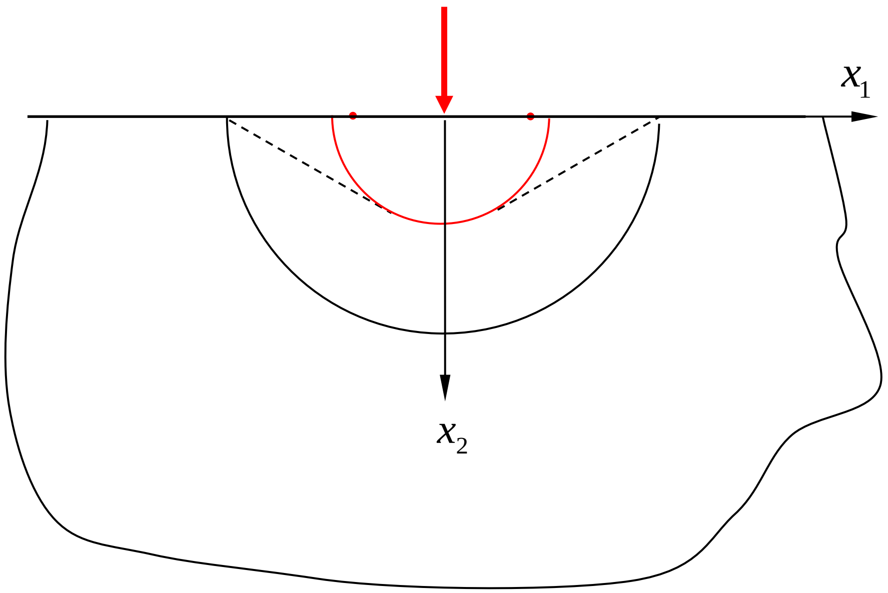
$$\varphi = A\exp(-\alpha_d \kappa x_2)\exp\{i\kappa(x_1 - ct)\}$$

$$\psi = B\exp(-\alpha_s \kappa x_2)\exp\{i\kappa(x_1 - ct)\}$$

$$\alpha_d = \sqrt{1-\frac{c^2}{C_d^2}} \quad \text{and} \quad \alpha_s = \sqrt{1-\frac{c^2}{C_s^2}}$$

$$\sigma_{22}(x_1, 0^{\pm}, t) = 0$$

$$\sigma_{12}(x_1, 0^{\pm}, t) = 0$$




## *- boundary conditions*

$$(1+\alpha_s^2)A + i2\alpha_s B = 0$$

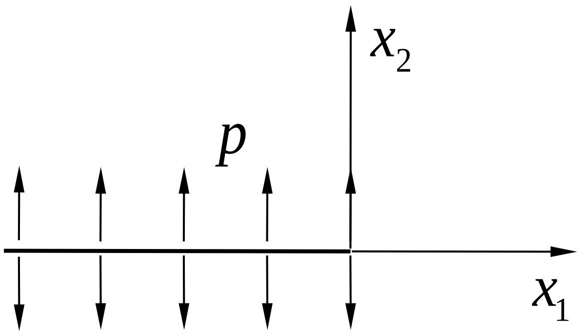
$$-i2\alpha_d A + (1+\alpha_s^2)B = 0$$

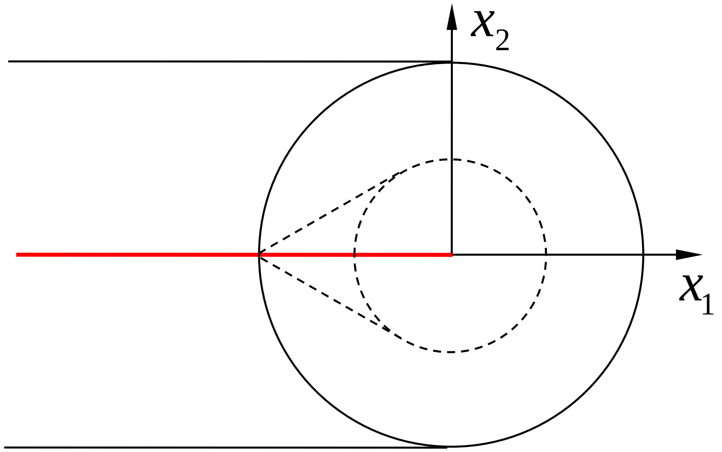
$$\Rightarrow R(c) = 4\alpha_d \alpha_s - (1+\alpha_s^2)^2 = 0$$

$$4\sqrt{1-k_R^2/k^2}\sqrt{1-k_R^2} - \left(2-k_R^2\right)^2 = 0$$

$$\boxed{k_R = \frac{C_R}{C_s} = \frac{0.862+1.14\nu}{1+\nu} < 1}$$

# Stationary crack under dynamic loading




$K_I(t) \cdots$ dynamic stress intensity factor

$$\sigma_{\alpha\beta}(r,\theta) = \frac{K_I(t)}{\sqrt{2\pi r}} f^{Is}_{\alpha\beta}(\theta) + \ldots \text{ as } r \to 0$$

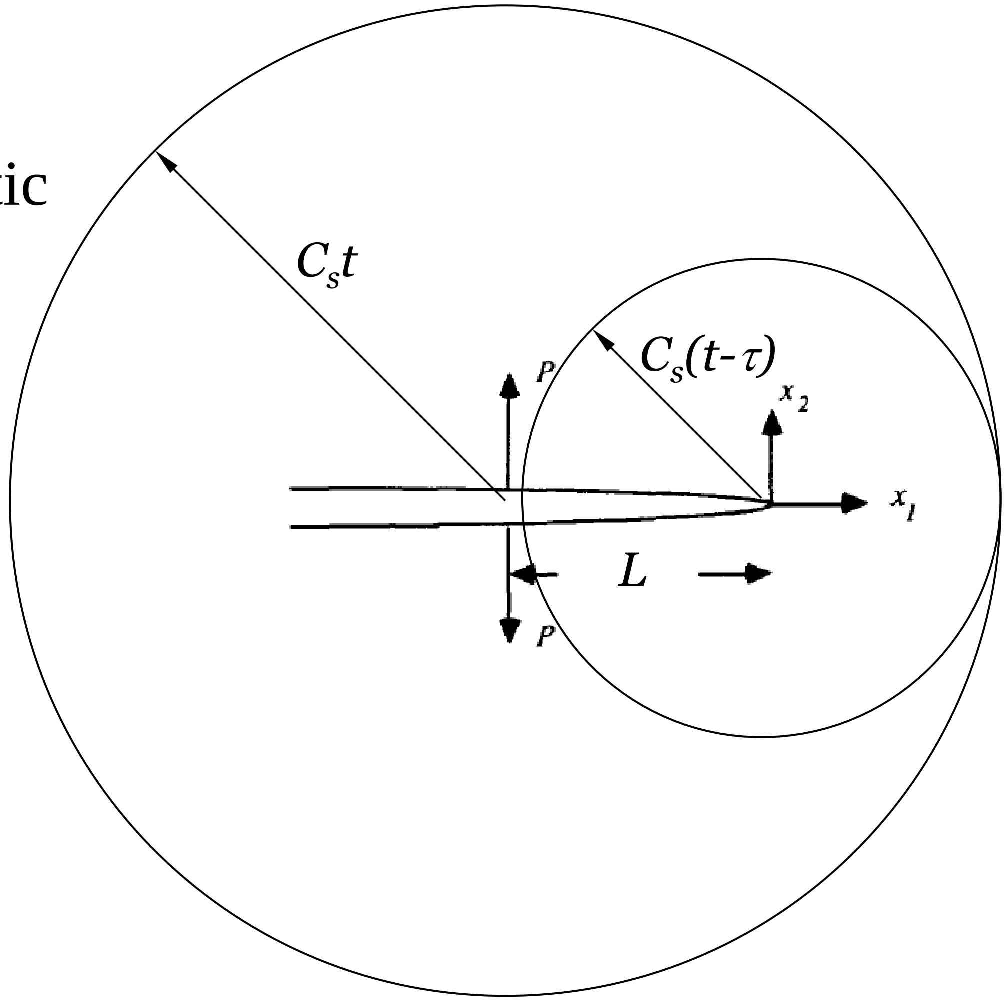
$$f^{Is}_{11}(\theta) = \cos\tfrac{1}{2}\theta\left[1 - \sin\tfrac{1}{2}\theta \sin\tfrac{3}{2}\theta\right]$$
$$f^{Is}_{22}(\theta) = \cos\tfrac{1}{2}\theta\left[1 + \sin\tfrac{1}{2}\theta \sin\tfrac{3}{2}\theta\right]$$
$$f^{Is}_{12}(\theta) = \cos\tfrac{1}{2}\theta \sin\tfrac{1}{2}\theta \cos\tfrac{3}{2}\theta$$

# Example - 1

$$K_I = P\sqrt{\frac{2}{\pi L}} \cdots \text{quasi-static}$$

*- after passage of the shear wave scattered from the crack tip*

$$K_I(t) = P\sqrt{\frac{2}{\pi L}}$$

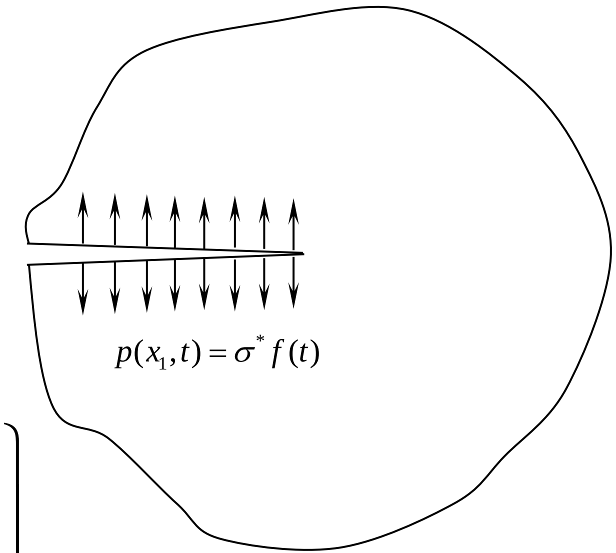


# Example - 2

$$\sigma_{22}(x_1,0,t) = -\sigma^* H(t) \qquad -\infty < x_1 < 0$$

$$\sigma_{12}(x_1,0,t) = 0 \qquad -\infty < x_1 < \infty$$

$$u_2(x_1,0,t) = 0 \qquad 0 < x_1 < \infty$$



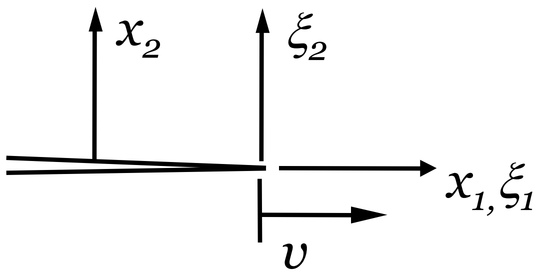

$$\left.\begin{aligned} &\sigma_{22}\left(x_1,0^+,t\right) = \sigma_+\left(x_1,t\right) - \sigma^* H\left(t\right) H\left(-x_1\right) \\ &\sigma_{12}\left(x_1,0^+,t\right) = 0 \\ &u_2\left(x_1,0^+,t\right) = u_-\left(x_1,t\right) \end{aligned}\right\} \text{for } -\infty < x < \infty$$

$$\boxed{K_I\left(t\right) = \frac{2\sigma^* \sqrt{C_d t(1-2v)/\pi}}{\left(1-v\right)}}$$

Reflects the fact that there is no characteristic time or length in the problem.

# Steadily moving crack

Introduce transformation: $\xi_1 = x_1 - v t,\ \xi_2 = x_2$




Rescale coordinates: $\begin{cases} \zeta_d = r_d e^{i\theta_d} = \xi_1 + i\alpha_d \xi_2 \\ \zeta_s = r_s e^{i\theta_s} = \xi_1 + i\alpha_s \xi_2 \end{cases}$

$$\alpha_d = \sqrt{1 - \frac{v^2}{C_d^2}} \qquad \boxed{\alpha_s = \sqrt{1 - \frac{v^2}{C_s^2}}} \qquad \boxed{\Rightarrow v < C_s}$$

$$\nabla^2 \varphi(r_d, \theta_d) = 0$$

$$\nabla^2 \psi(r_s, \theta_s) = 0$$

$$\varphi(r_d, \theta_d) = r_d^{\lambda} f(\theta_d; \lambda)$$

$$\psi(r_s, \theta_s) = r_s^{\lambda} g(\theta_s; \lambda)$$

$$\Rightarrow \begin{cases} \varphi(r_d, \theta_d) = A r_d^{\lambda} \cos \lambda \theta_d \\ \psi(r_s, \theta_s) = B r_s^{\lambda} \sin \lambda \theta_s \end{cases}$$

# Crack tip stress field

Impose traction free boundary conditions on crack faces:

$$(1+\alpha_s^2)A\cos(\lambda-2)\pi+2\alpha_s B\cos(\lambda-2)\pi=0$$

$$2\alpha_d A\sin(\lambda-2)\pi+(1+\alpha_s^2)B\sin(\lambda-2)\pi=0$$

$$\Rightarrow \lambda=\frac{1}{2}n+1, \text{ for } n=1,2,\ldots$$

$$\sigma_{\alpha\beta}(r,\theta)=\frac{K_I}{\sqrt{2\pi r}}f_{\alpha\beta}^{I}(\theta;v)+T\left(\alpha_d^2-\alpha_s^2\right)\delta_{\alpha1}\delta_{\beta1}+\cdots$$

$$u_\alpha(r,\theta)=\frac{K_I\sqrt{r}}{\sqrt{2\pi}}g_\alpha^I(\theta;v)+\cdots$$

# Crack tip stress field

- $K_I \cdots$ to be obtained from solution of boundary-initial value problem.
- Singularity in the stress (and strain field)
- Crack opening appears parabolic
- Similar field for mode II
- Other features of the stress field

$$K_I = \lim_{\xi_1 \to 0} \sqrt{2\pi\xi_1}\,\sigma_{22}\left(r, 0^{\pm}\right)$$

dynamic stress intensity factor

$$\sigma_{\alpha\beta}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f^I_{\alpha\beta}\left(\theta; v\right) + T\left(\alpha_d^2 - \alpha_s^2\right)\delta_{\alpha 1}\delta_{\beta 1} + \cdots$$

$$u_\alpha(r,\theta) = \frac{K_I\sqrt{r}}{\sqrt{2\pi}} g^I_\alpha\left(\theta; v\right) + \cdots$$

# Crack tip stress field

$$f_{11}^{I}(\theta;v) = \frac{1}{R(v)}\left\{(1+\alpha_s^2)(1+2\alpha_d^2-\alpha_s^2)\frac{\cos\frac{1}{2}\theta_d}{\gamma_d^{1/2}} - 4\alpha_d\alpha_s\frac{1}{\gamma_s^{1/2}}\cos\tfrac{1}{2}\theta_s\right\}$$

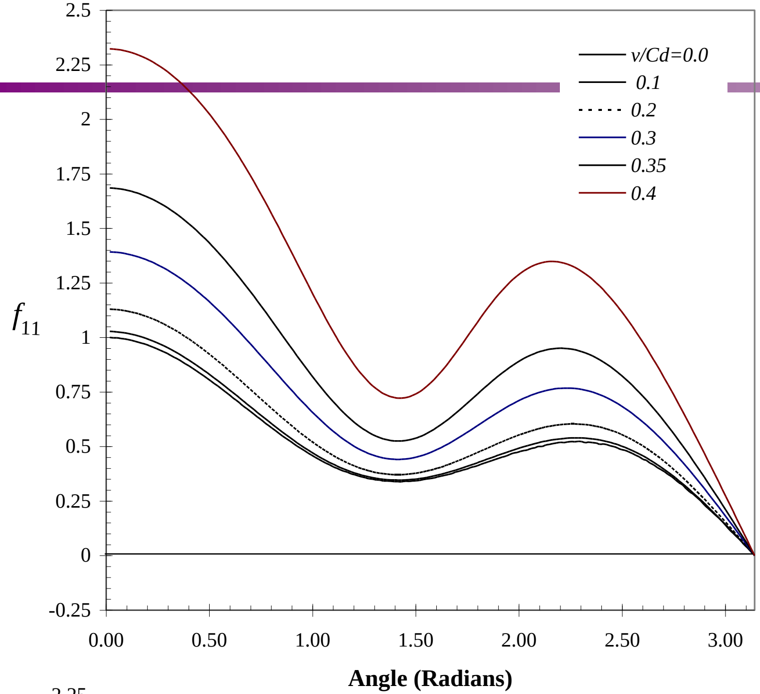
$$f_{22}^{I}(\theta;v) = \frac{1}{R(v)}\left\{-(1+\alpha_s^2)^2\frac{\cos\frac{1}{2}\theta_d}{\gamma_d^{1/2}} + 4\alpha_d\alpha_s\frac{1}{\gamma_s^{1/2}}\cos\tfrac{1}{2}\theta_s\right\}$$

$$f_{12}^{I}(\theta;v) = \frac{2\alpha_d(1+\alpha_s^2)}{R(v)}\left\{\frac{\sin\frac{1}{2}\theta_d}{\gamma_d^{1/2}} - \frac{\sin\frac{1}{2}\theta_s}{\gamma_s^{1/2}}\right\}$$

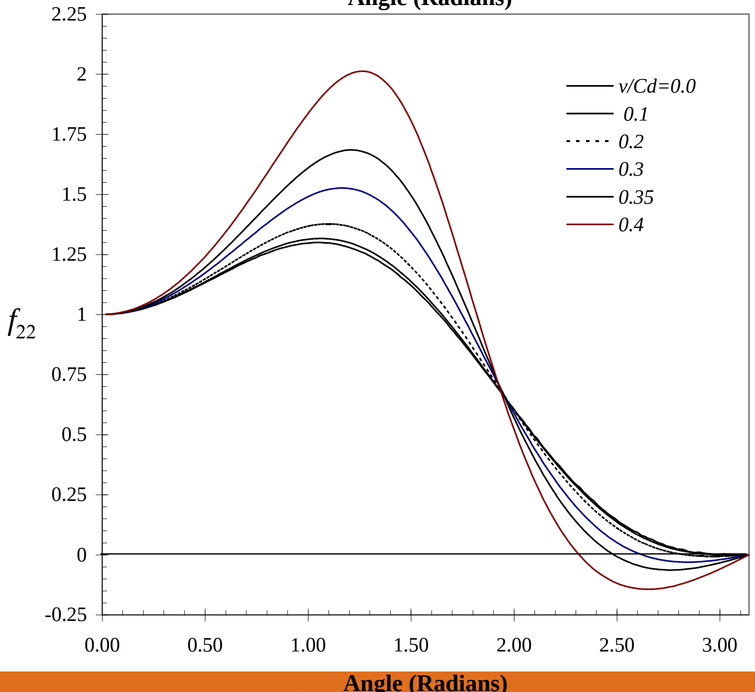
$$\gamma_d = \sqrt{1-(\mathrm{v}\sin\theta/C_d)^2} \text{ and } \gamma_s = \sqrt{1-(\mathrm{v}\sin\theta/C_s)^2}$$

$$\tan\theta_d = \alpha_d\tan\theta \text{ and } \tan\theta_s = \alpha_s\tan\theta$$

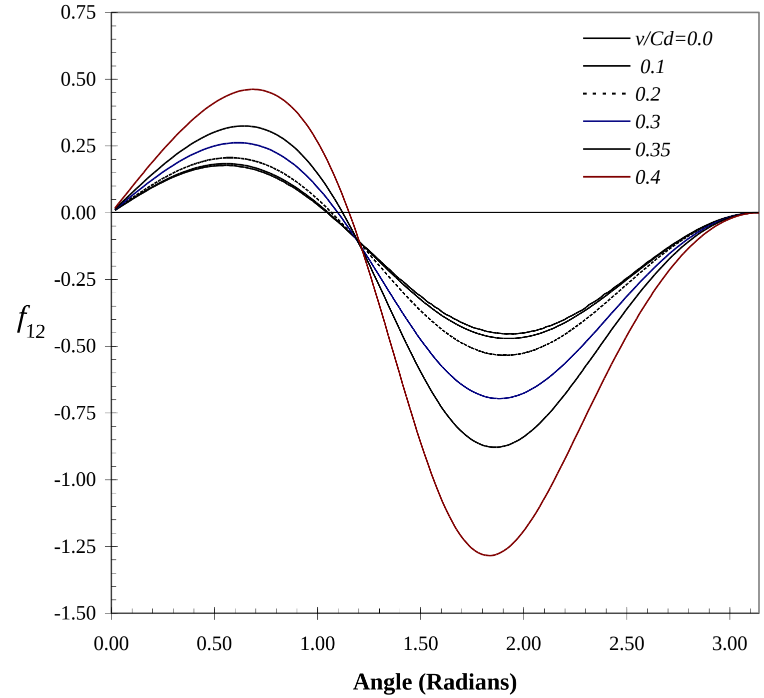
$$R(v) = 4\alpha_d\alpha_s - (1+\alpha_s^2)^2$$



v/Cd=0.0
0.1
0.2
0.3
0.35
0.4
$f_{11}$
2.5
2.25
2
1.75
1.5
1.25
1
0.75
0.5
0.25
0
-0.25
0.00
0.50
1.00
1.50
2.00
2.50
3.00
Angle (Radians)

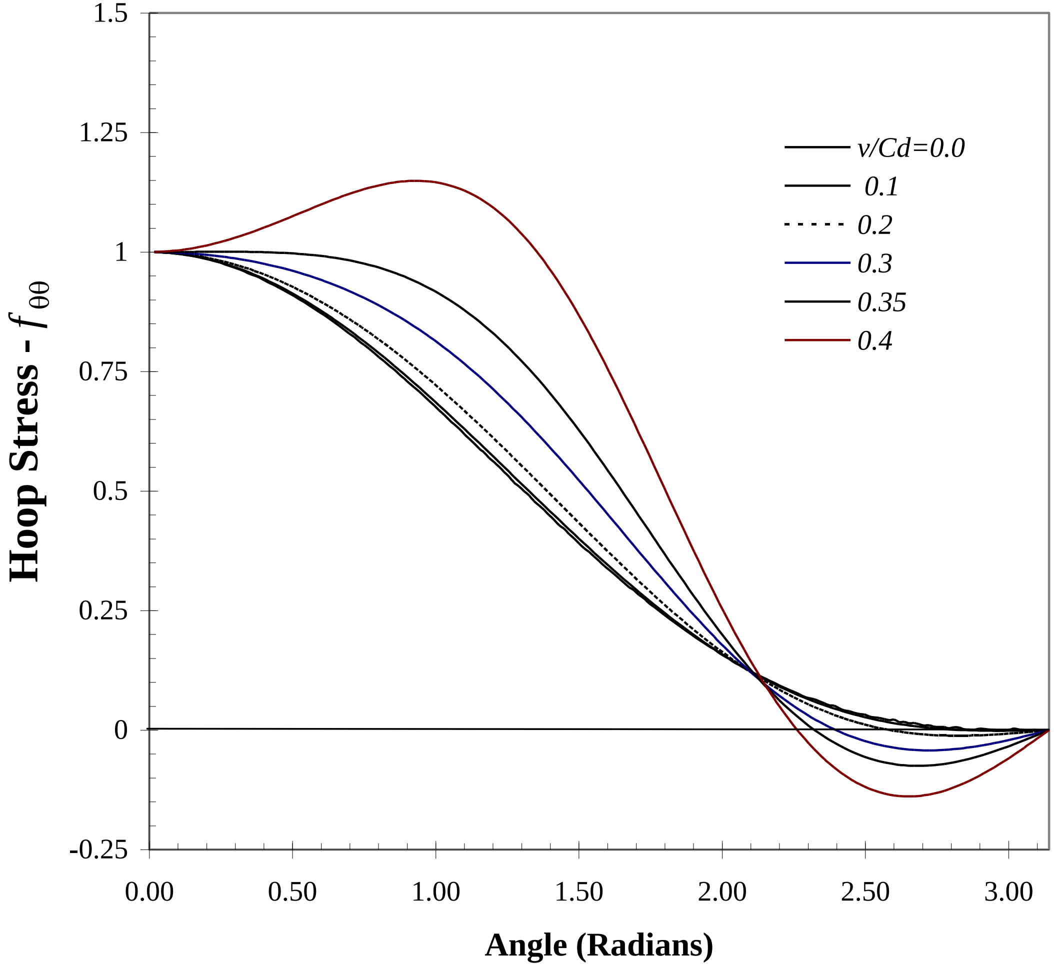



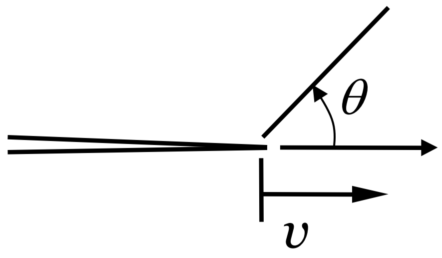
v/Cd=0.0
0.1
0.2
0.3
0.35
0.4
$f_{22}$
2.25
2
1.75
1.5
1.25
1
0.75
0.5
0.25
0
-0.25
0.00
0.50
1.00
1.50
2.00
2.50
3.00
Angle (Radians)

v/Cd=0.0
0.1
0.2
0.3
0.35
0.4
$f_{12}$
0.75
0.50
0.25
0.00
-0.25
-0.50
-0.75
-1.00
-1.25
-1.50
0.00
0.50
1.00
1.50
2.00
2.50
3.00
Angle (Radians)

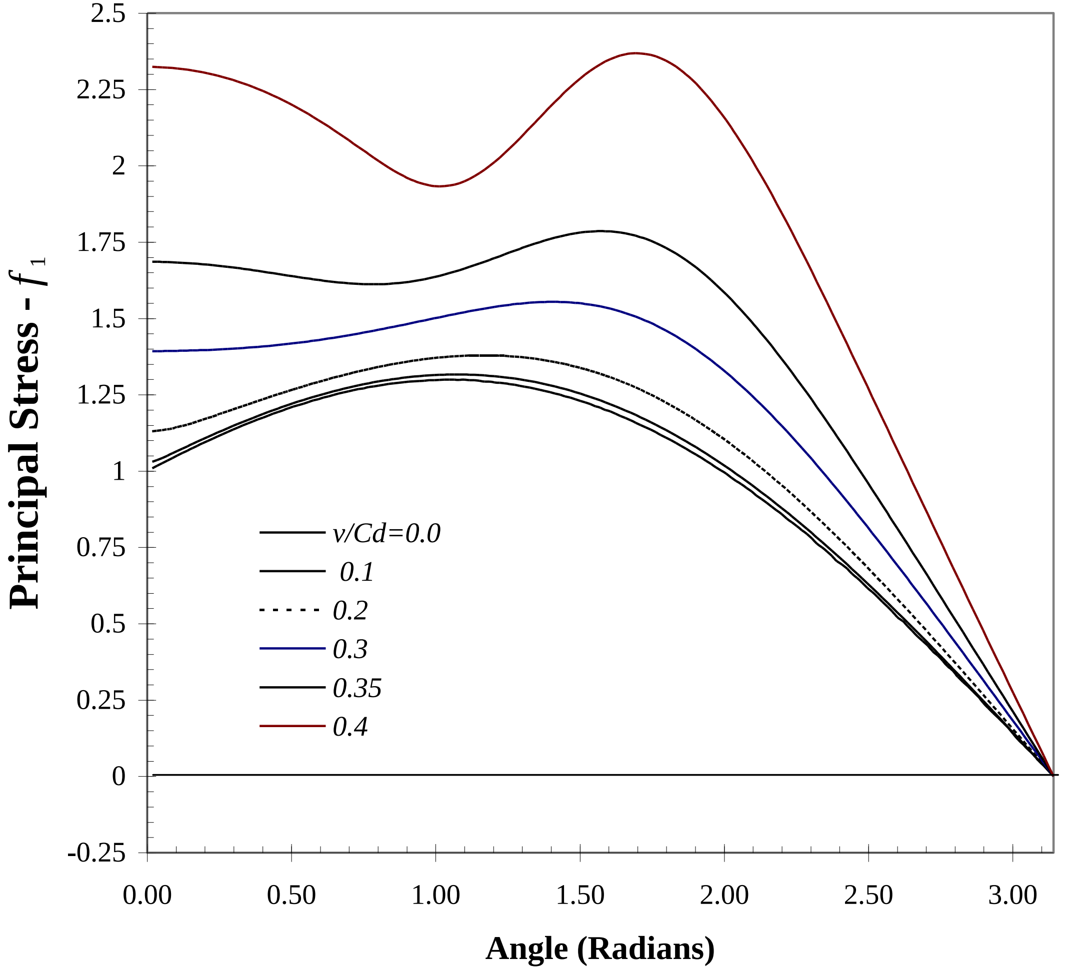
# Yoffe's observation

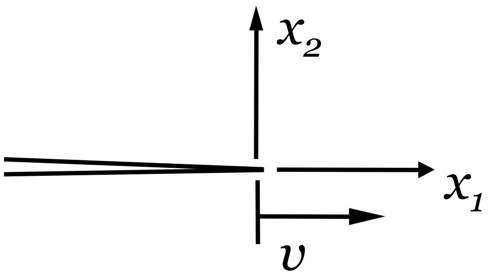



Yoffe (1951) suggested that the shift in the orientation of the maximum hoop stress could be the underlying cause of crack branching.

# Rice's observation



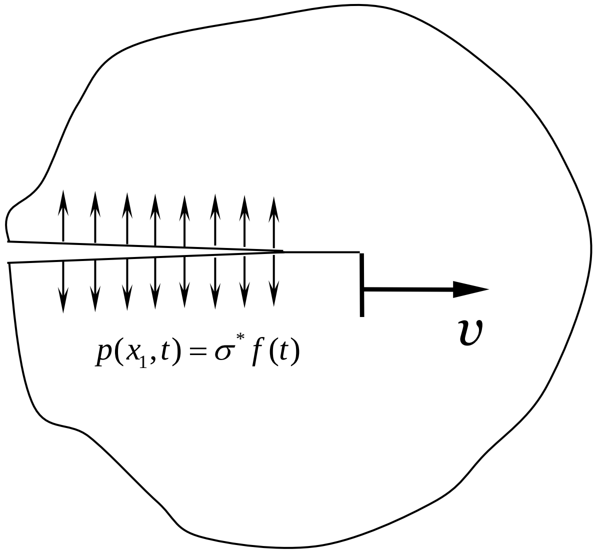

$$f_{11}^{I}(\theta;v) > f_{22}^{I}(\theta;v)$$

# Dynamic SIF for moving cracks

**Freund's theorem:**

$$K_I(t,l,\dot{l}) = k(v)K_I^0(t,l,0)$$

$$k(v) \approx \frac{(1-v/C_R)}{\sqrt{1-v/C_d}}$$




$$K_I = \begin{cases} 2\sigma^* \dfrac{\sqrt{C_d t(1-2v)/\pi}}{(1-v)} & t<\tau \\ 2\sigma^* k(v)\left[\dfrac{\sqrt{C_d t(1-2v)/\pi}}{(1-v)} - \sqrt{\dfrac{2v(t-\tau)}{\pi}}\right] & t>\tau \end{cases}$$

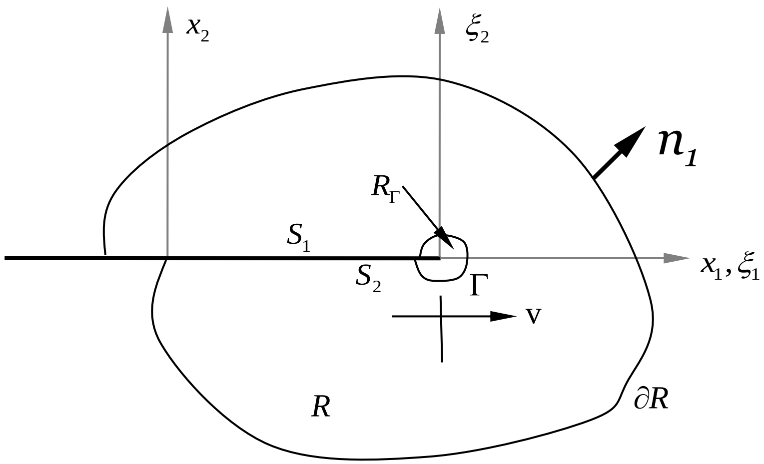
# Energy release rate

$$P-\left(\dot{U}+\dot{T}\right)=-\lim_{\Gamma\to 0}\int_{\Gamma}\left[\sigma_{\alpha\beta}\dot{u}_{\alpha}n_{\beta}+\frac{1}{2}\left(\rho\dot{u}_{\alpha}\dot{u}_{\alpha}+\sigma_{\alpha\beta}u_{\alpha,\beta}\right)vn_{1}\right]ds=G(v)/v$$

$$\boxed{G=\frac{1-\nu^{2}}{E}A_{I}(v)K_{I}^{2}\cdots\text{dynamic energy release rate}}$$

$$A_{I}(v)=\frac{v^{2}\alpha_{d}}{(1-\nu)C_{s}^{2}R(v)}$$

- *Note similarity to the quasi-static energy release rate*
- *R(v) has a root at the Rayleigh wave speed – this implies a limit for crack speed*
- *Similar calculations for other modes*
- *G must equal the fracture energy Γ*




$$\boxed{G=\Gamma\Leftrightarrow f\left(K_{I},v\right)=0}$$

***Dynamic fracture criterion***

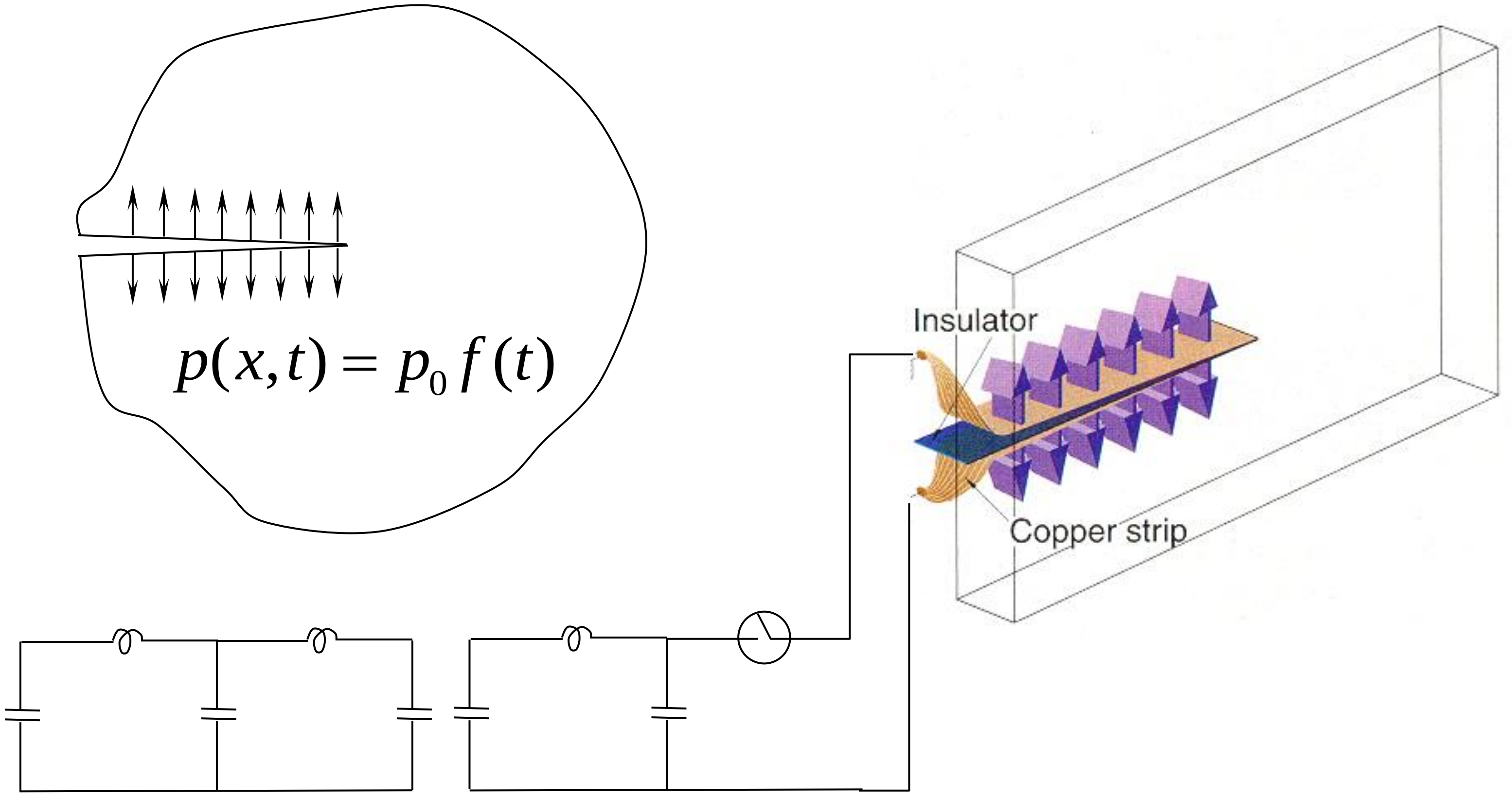
# Fracture criterion - 1

- **Brittle fracture** - little or no inelastic deformation in the process zone; failure is by microcracking (bond breaking, cleavage) – glass? – not really! Perhaps single crystals!?
- **Quasi-brittle** – still no inelastic deformation, but a rather diffuse damage through microcracking (heterogeneity of the microstructure is the main feature) – even glass appears quasi-brittle!
- **Ductile failure** – significant amount of inelastic deformation; failure by void nucleation, growth and coalescence – some polymers, and most metals

# Fracture criterion - 2

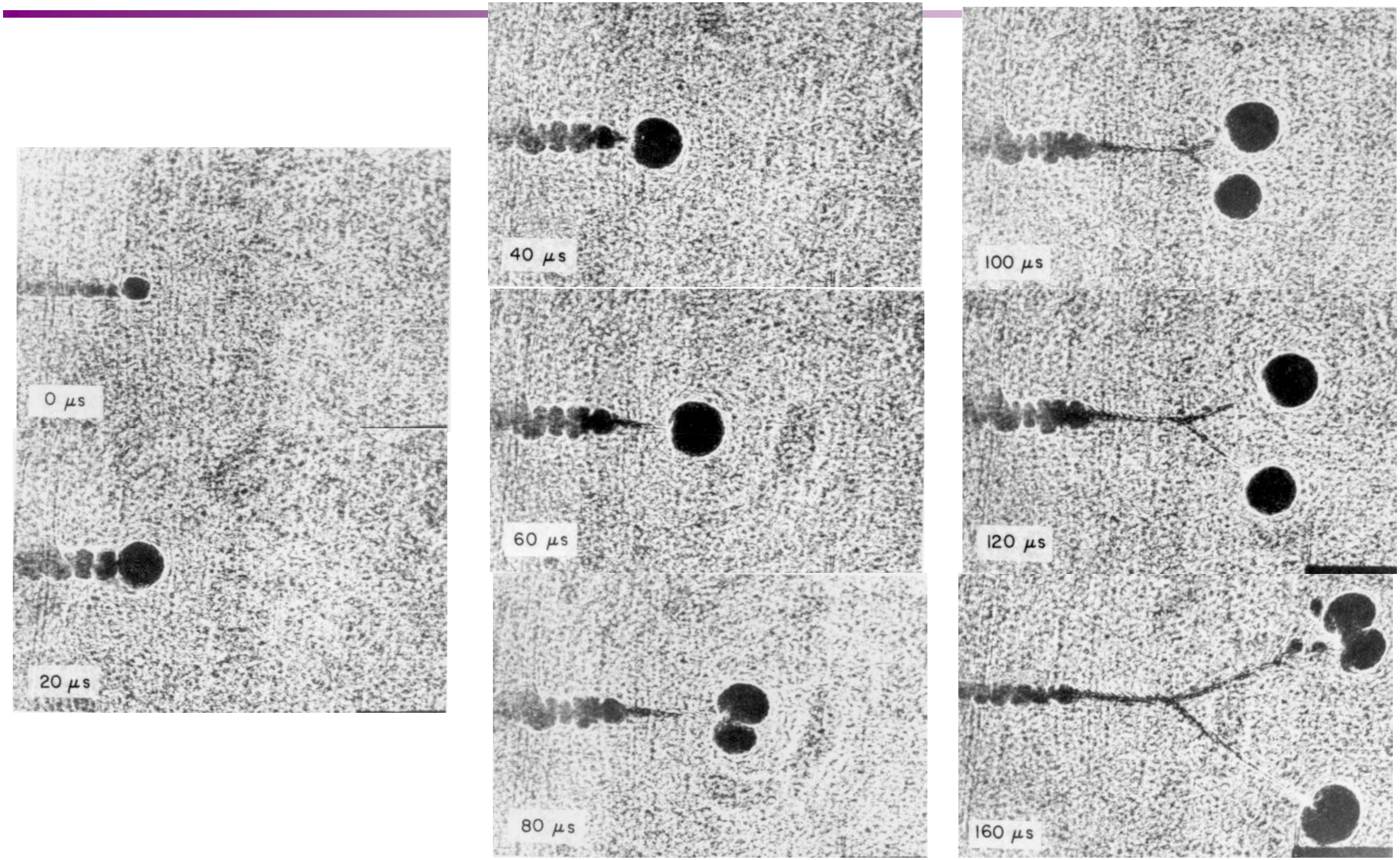
- $\Gamma$ varies over several orders of magnitude
  - True surface energy is ~ O(1) J/m$^2$
  - Glasses and ceramics~ 10 J/m$^2$
  - Polymers ~1 kJ/m$^2$
  - Metals ~ 100 kJ/m$^2$
- Differences arise due to different mechanisms of deformation and failure
- Must be determined through calibration experiments, such as the pioneering work of Obreimoff (1930)

# Dynamic fracture experiments



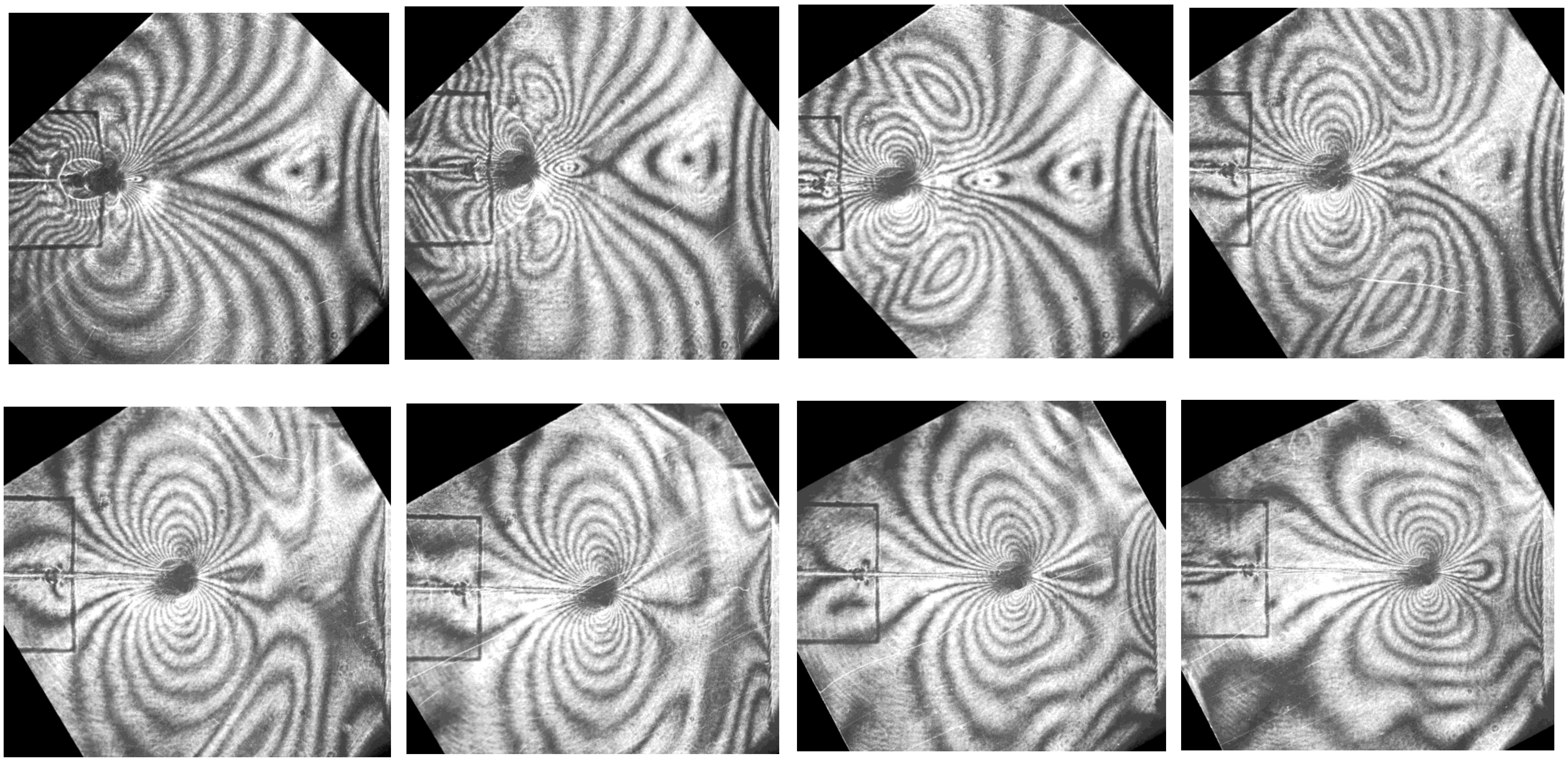

Homalite-100, electromagnetic loading
Ravi-Chandar and Knauss, *Int J Fract*, 1982

# Caustics



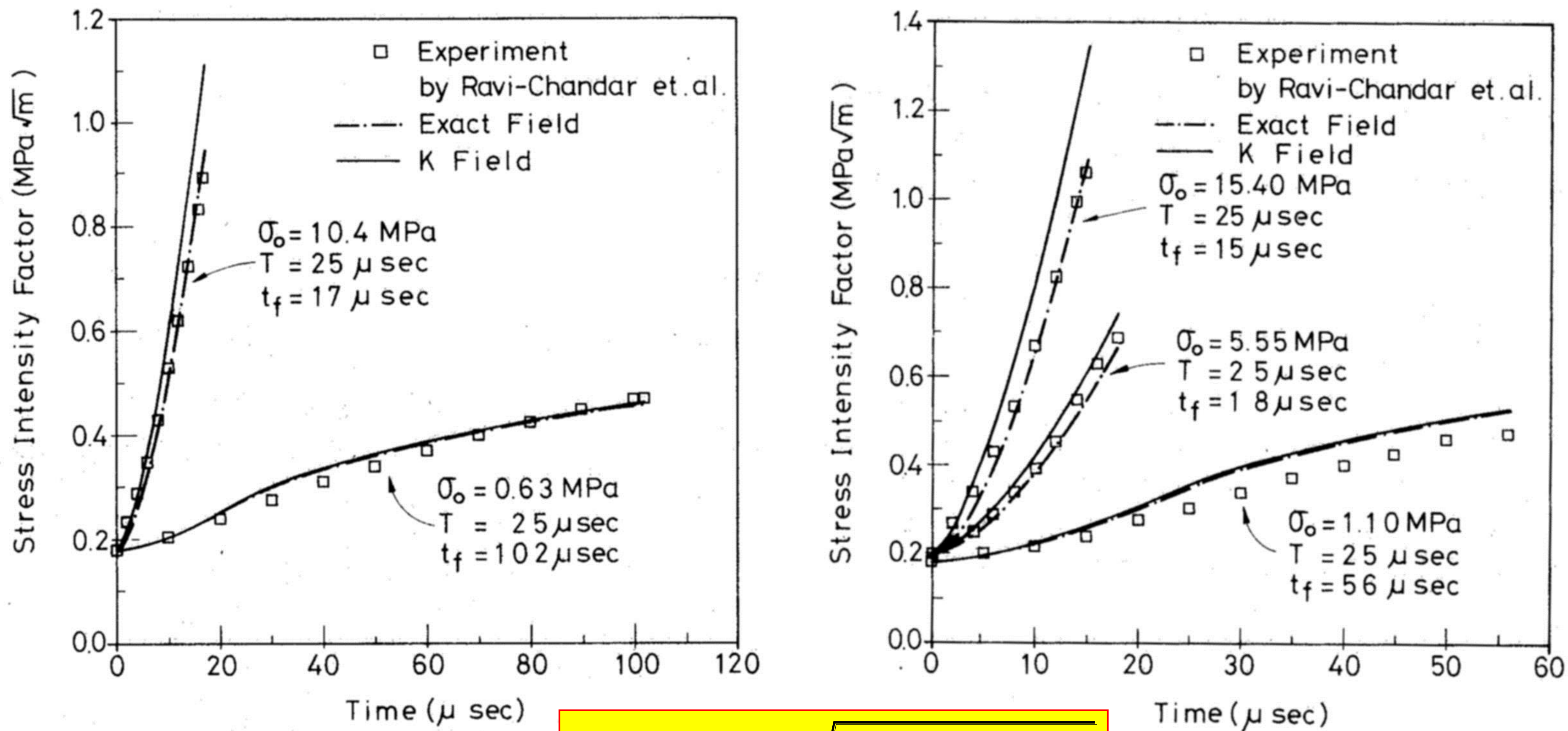

Homalite-100, electromagnetic loading
Ravi-Chandar and Knauss, *Int J Fract*, 1984

# Photoelasticity



Polycarbonate, quasi-static loading
Taudou, Potti and Ravi-Chandar, *Int J Fract*, 1992

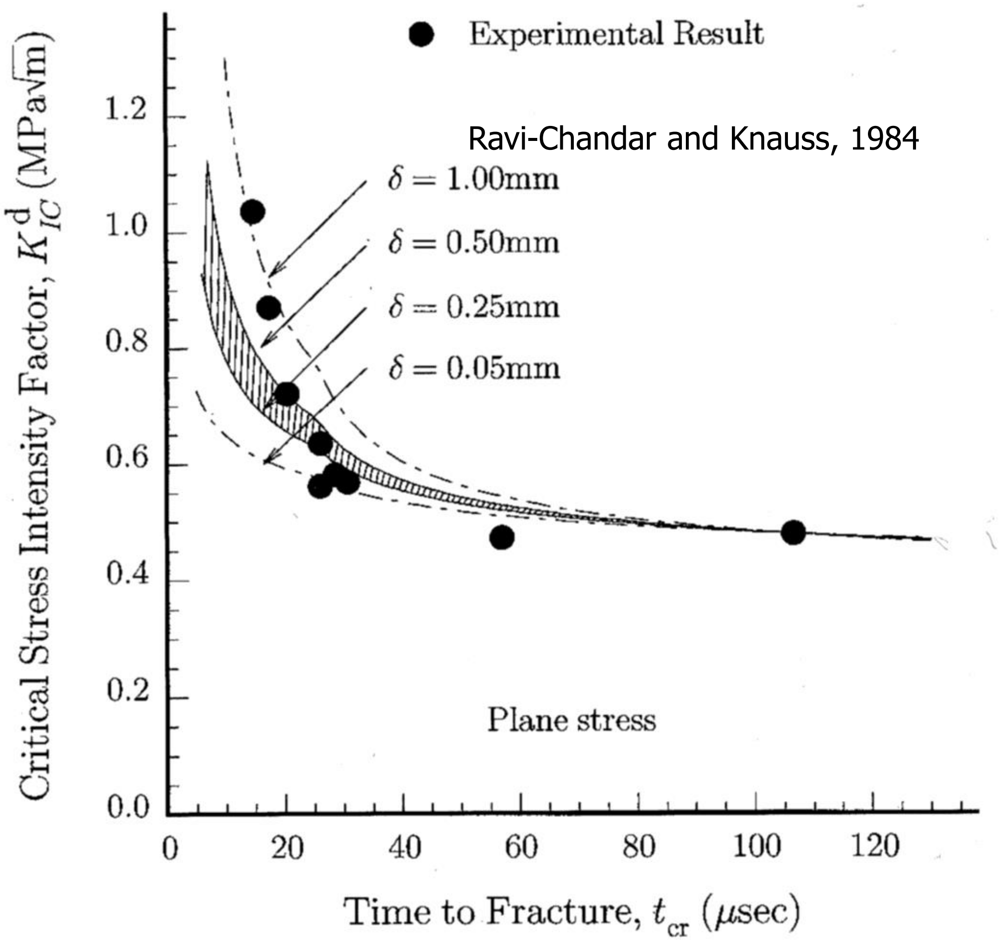
# Stationary crack under dynamic loading

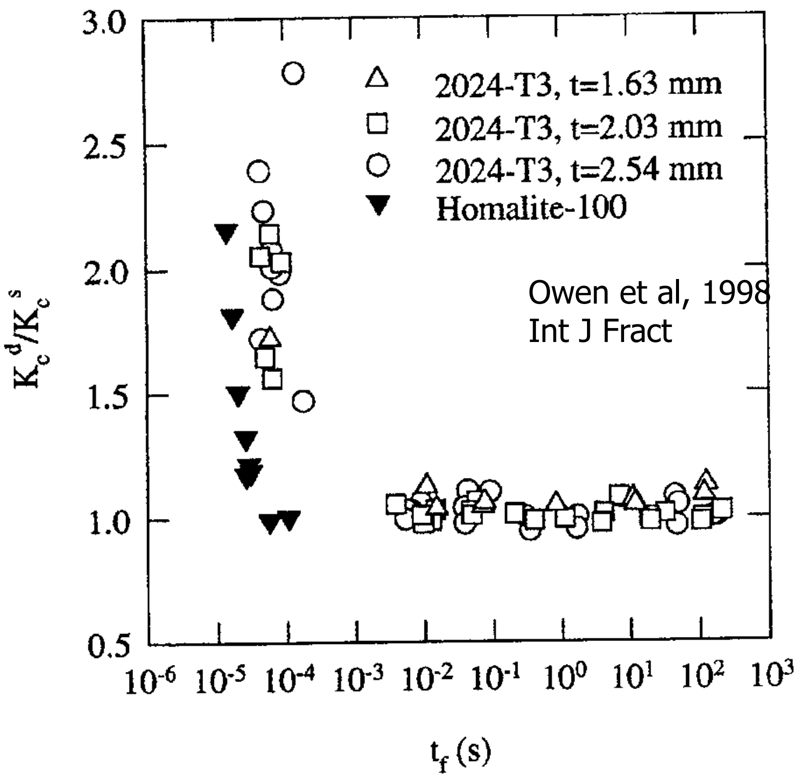



$$K_I\left(t\right)=\frac{2\sigma^*\sqrt{C_d t(1-2v)/\pi}}{\left(1-v\right)}$$

# Practical fracture criterion - 1

- Dynamic crack initiation toughness



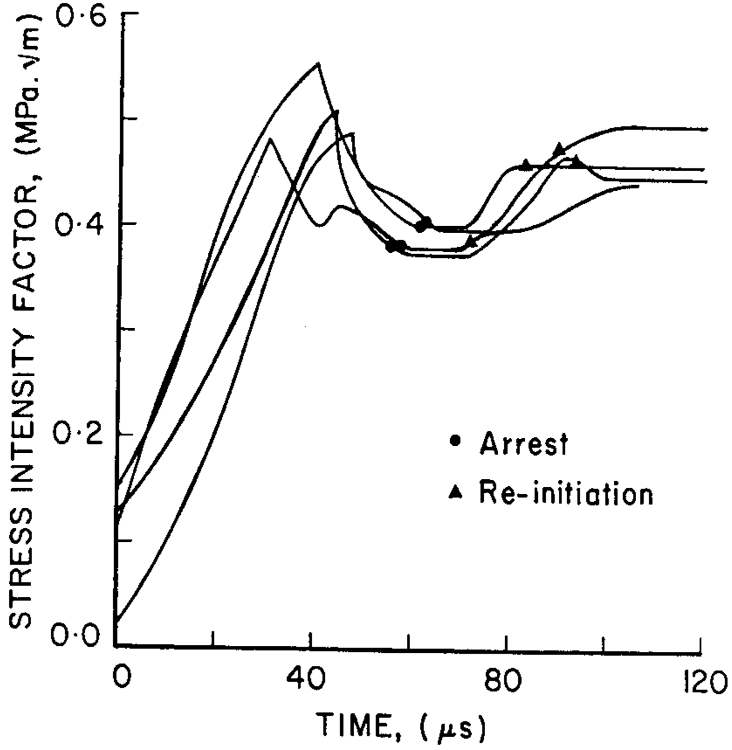

$$G(v) = \frac{K_I^2(t,v)}{E} A_I(v) = \Gamma$$

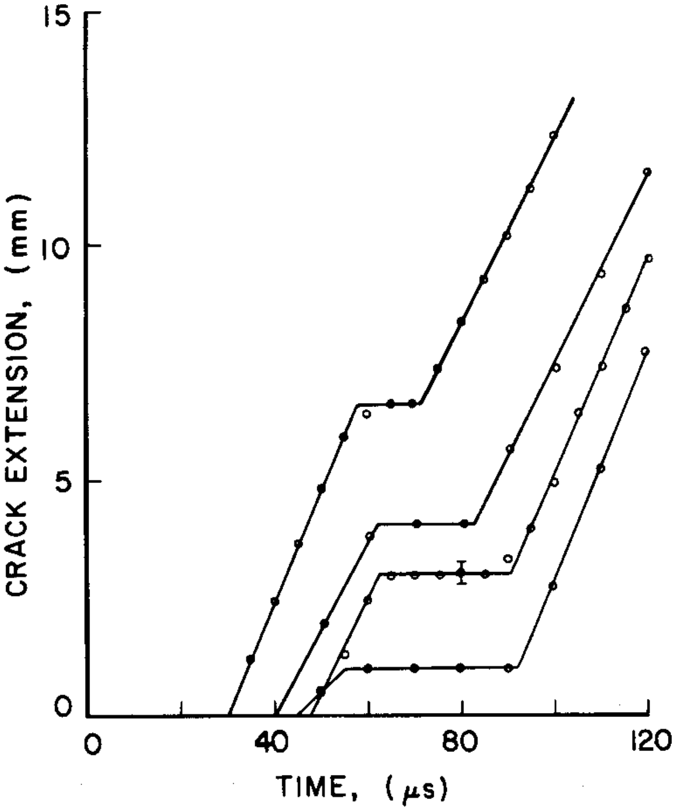
$$\boxed{K_I^{dyn}(t_f) = K_{Id}\left(T, \dot{K}_I^{dyn}\right)}$$

Ravi-Chandar and Knauss, 1984

# Practical fracture criterion - 2

- Dynamic crack arrest toughness:



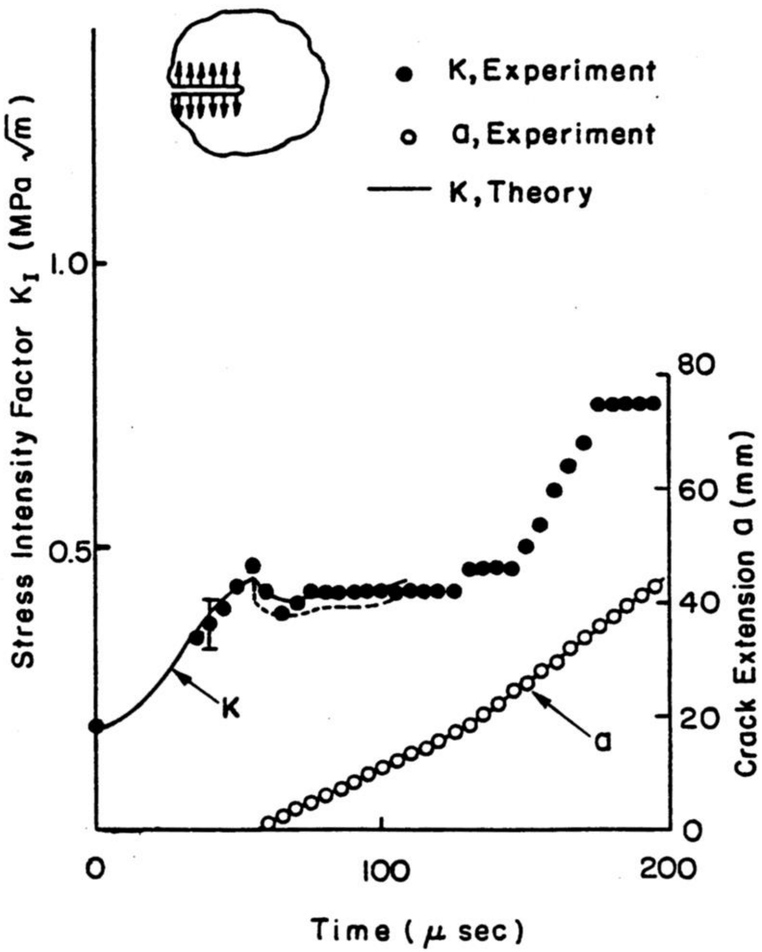



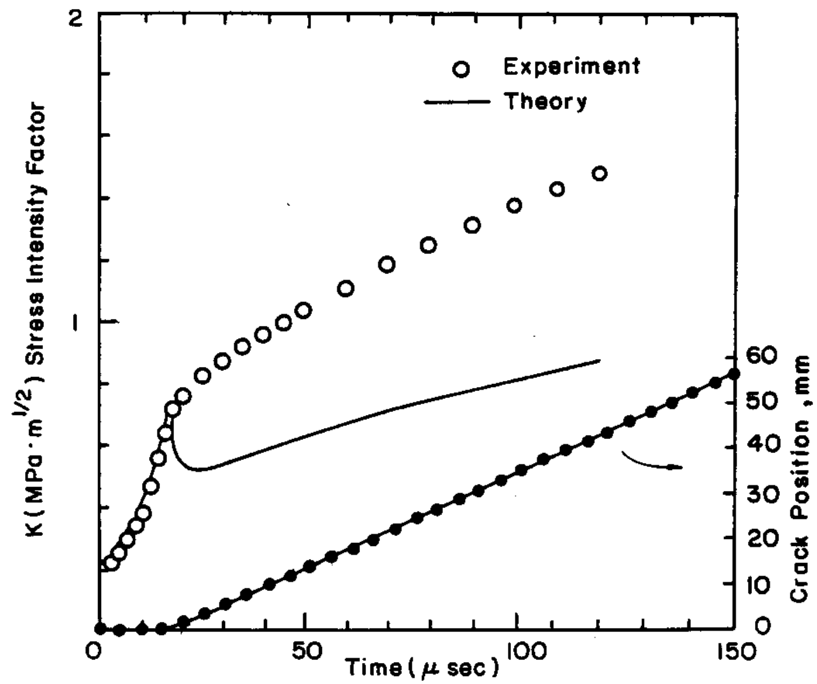

Ravi-Chandar and Knauss, 1984

$$\boxed{K_I^{dyn}(t) \leq K_{Ia}\left(T\right)}$$

# Practical fracture criterion - 3

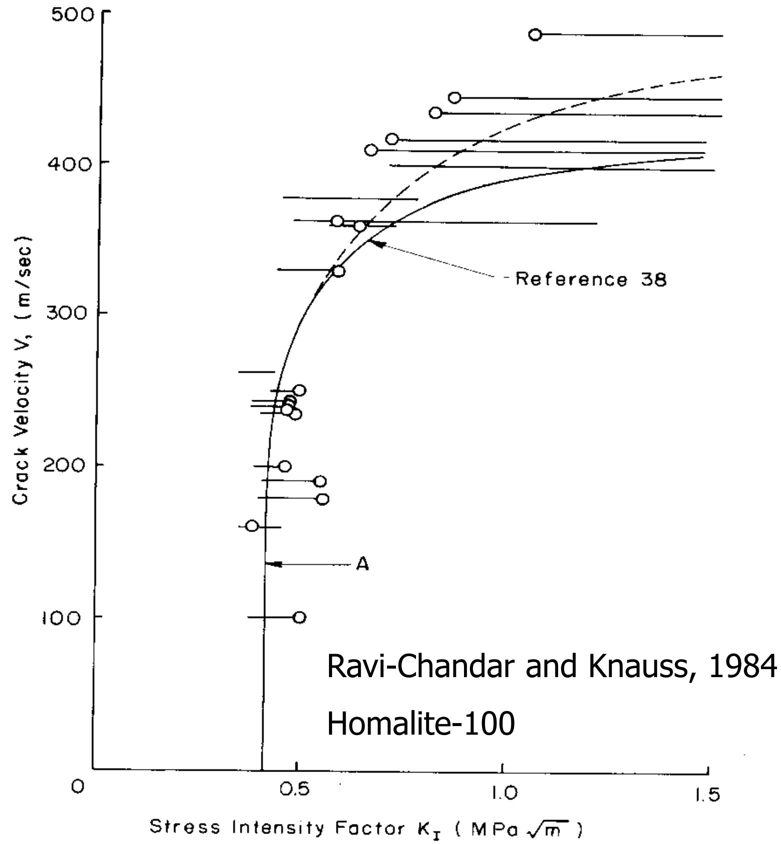
- Dynamic crack propagation toughness

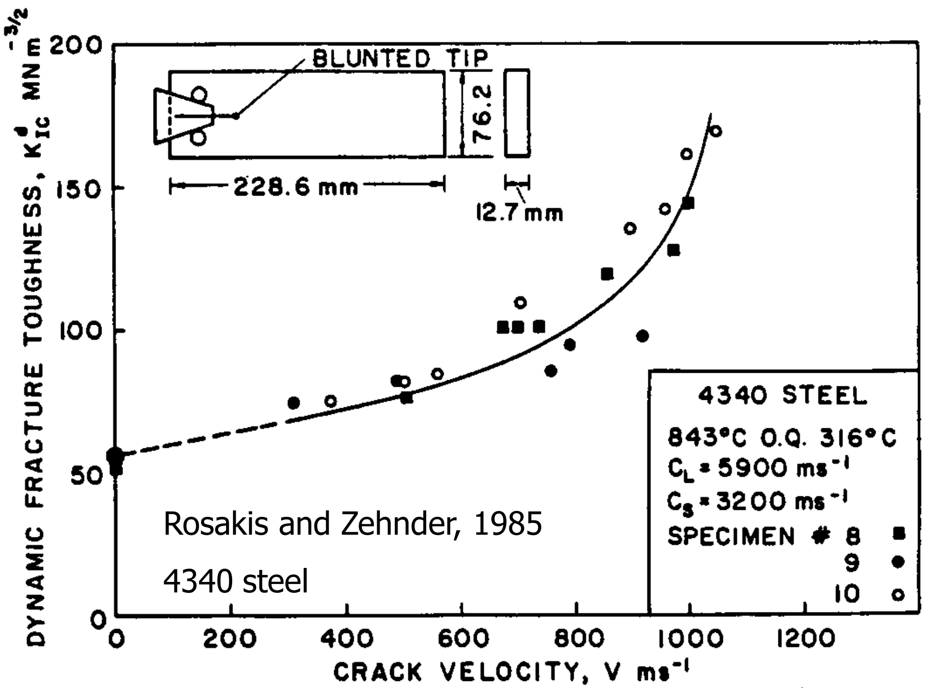


# Dynamic crack propagation toughness

$$\boxed{K_I^{dyn}(t,v) = K_{ID}\left(v;\dot{K}_I^{dyn},T\right)}$$



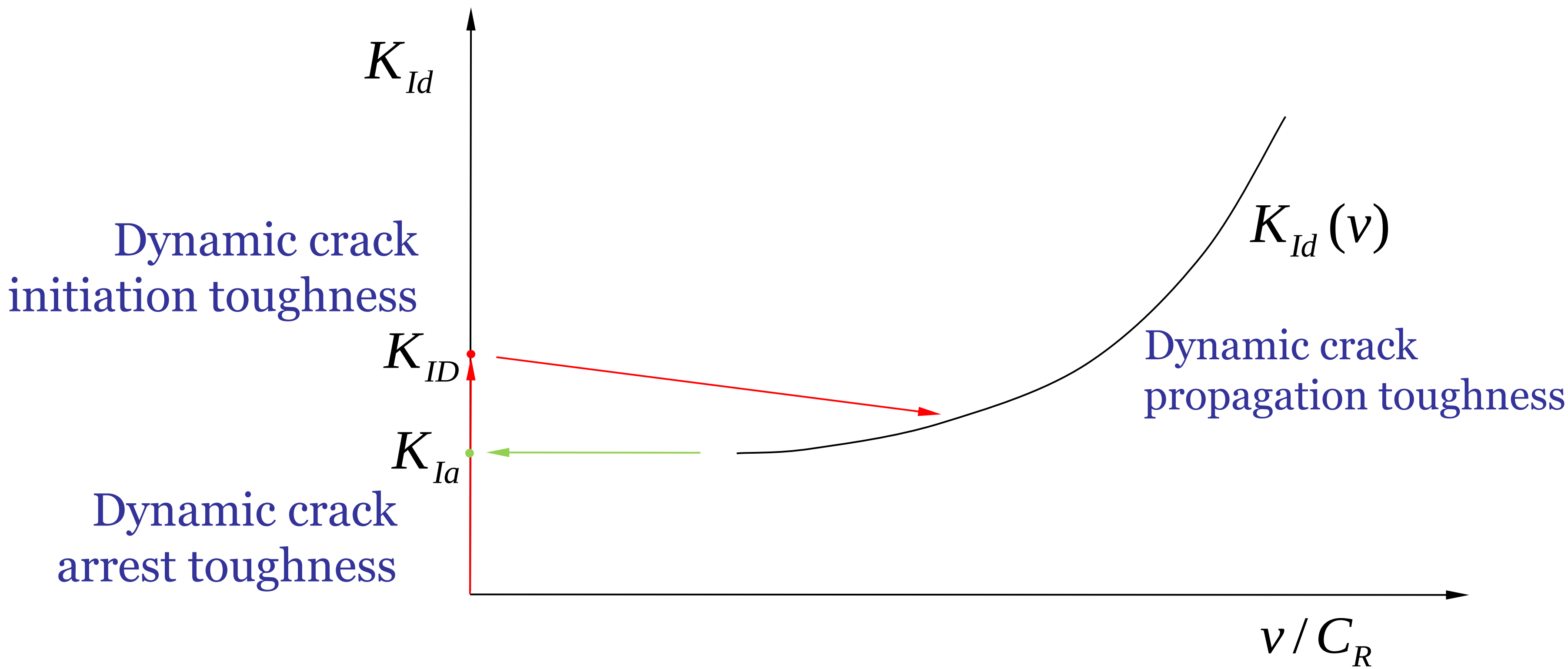
# Dynamic fracture criteria

$$K_I^{dyn}(t_f) = K_{Id}\left(T, \dot{K}_I^{dyn}\right)$$ ...Dynamic initiation toughness

$$K_I^{dyn}(t,v) = K_{ID}\left(v; \dot{K}_I^{dyn}, T\right)$$ ...Dynamic growth toughness

$$K_I^{dyn}(t) \le K_{Ia}\left(T\right)$$ ...Dynamic arrest toughness

# Consequences of fracture *criteria*



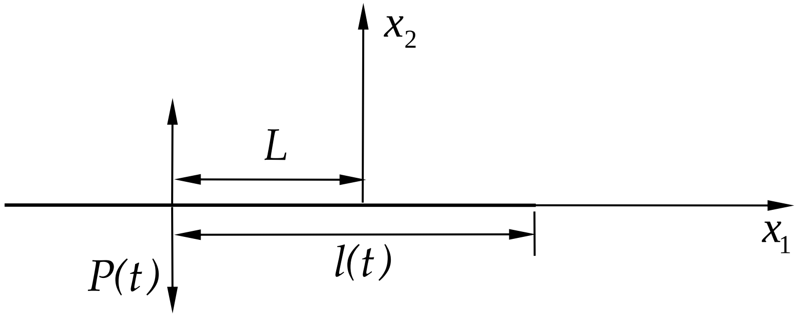

For $v < 0.2\ C_R$, elastodynamic fracture theory works quite well

# Experimental observations

- For $v < 0.2\ C_R$, elastodynamic fracture theory works quite well
- Limiting speed $v \sim 0.5\ C_R$, is not predicted by the theory
- Rapid increase in fracture toughness in nominally brittle materials
- Features of surface roughening and crack branching

# Example – Crack arrest specimen

$$K_I^{dyn}(t,0) = P\sqrt{\frac{2}{\pi L}} \quad \text{for } t \le 0$$




$$K_I^{dyn}(t,\dot{l}) = k(\dot{l})P\sqrt{\frac{2}{\pi(L+l(t))}} \quad \text{for } t \ge 0$$

$$K_I^{dyn}(t,0) = P\sqrt{\frac{2}{\pi(L+l)}}$$

$$P\sqrt{\frac{2}{\pi L}} = K_{Id}\left(T,\dot{K}_I^{dyn}\right)\cdots\text{crack initiation}$$

$$k(\dot{l})P\sqrt{\frac{2}{\pi\left(L+l(t)\right)}} = K_{\mathrm{ID}}\left(v;\dot{K}_I^{dyn},T\right)\cdots\text{crack propagation}$$

$$k(\dot{l})P\sqrt{\frac{2}{\pi\left(L+l(t)\right)}} = K_{\mathrm{Ia}}\left(T\right)\cdots\text{crack arrest}$$