Transport Phenomena:

II. Solving transport equations - 1

Take home message:

Fourrier law for heat diffusion:

$$\mathbf{J_D} = -\kappa \nabla T$$

$\mathbf{J_D}$ heat flux [W/m^2] and κ thermal conductivity [W.m^{-1}.K^{-1}]

Heat transport equation:

$$\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \alpha \Delta T + r/\rho C_p$$

$\alpha = \kappa/\rho C_p$ thermal diffusivity [m^2/s],
r production or loss of heat in the volume [W.m^{-3}].

Fick law for mass:

$$\mathbf{J_i} = -D_i \nabla c_i$$

J_i mass or molar flux, c_i mass or molar concentration, D_i diffusion coefficient [m^2/s].

Mass transport equation (for non compressible fluid):

$$\frac{\partial c_i}{\partial t} + \mathbf{u}.\nabla c = \alpha \Delta c + r_i$$

D_i mass diffusion coefficient [m^2/s], r_i source or sink of of species i.

Rheology law for a Newtonian fluid:

$$\boldsymbol{\sigma}_v = \eta(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^{\mathrm{T}})$$

$\boldsymbol{\sigma}_v$ viscous stress, $\mathbf{u}$ velocity, η dynamic viscosity [Pa.s].

Flow equation (for non compressible fluid):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \nu\, \Delta \mathbf{u} - \frac{1}{\rho}\boldsymbol{\nabla} p + \mathbf{g}$$

$\nu = \eta/\rho$ kinematic viscosity [m^2/s].

Boundary conditions at interfaces:

- *continuity* of *temperature* and *concentration* fields
- *continuity* of heat and mass *fluxes* (except when involving a phase transition with *latent heat*)
- *continuity* of *velocities* and *stresses* in flows

1 From mass to heat and viscous stress diffusion

In the previous session, we derived the equation dictating the transport of mass in the case of a dilute solution. Obviously, a possible source (or sink) term should be added to be more complete. The same concept can be applied for temperature, where the equivalent of Fick's law is Fourrier's law:

$$\mathbf{J_D} = -\kappa \nabla T$$

where J_D is the heat flux (in W/m^2) and κ the thermal conductivity.
What is the unit of κ? How would we write the convective flux of heat? By analogy to the transport of mass, recover the heat transport equation. What is the analog of the diffusion coefficient? Although we will not consider radiative fluxes for the moment, do not forget that they can be significant.
Another familiar quantity follows the same type of equation. Any idea? Try to remember Navier & Stokes equation we derived last year.
To make things clear, derive the respective concentration, temperature and velocity profiles in the exemples illustrated in Fig. 1.
What are the corresponding fluxes of mass, temperature and momentum?

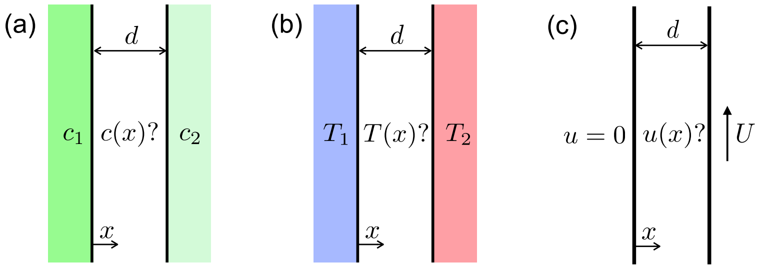

Figure 1: **a.** Hydrogel membrane separating solutions of fixed concentrations of some species **b.** Typical insulating panel separating regions of different temperatures. **c.** Relative motion of two facing surfaces separated by a viscous liquid.

2 Solving transport equations

Transports equations are generally complex to solve has they are coupled to flow equations. However, some basic configurations involving no flow or a very simple advection flow constitute interesting exemples that follow similar solutions.

2.1 Homogenization of a cocktail

Let's go back to the cocktail we used as an illustration of diffusion transport (Fig. 2). To simplify, we consider that the initial conditions correspond to two semi-infinite liquids of concentrations c_0 and $c = 0$, respectively.
How does the typical thickness δ_c of the blurry region evolves in time (in terms of scaling law)? What about the flux of sugar molecules at $z = 0$ (the initial border between the two regions)?
What PDE (Partial Differential Equation) should be solved? Solve this PDE using the reduced

non-dimensional variable $\xi = \frac{y}{2(Dt)^{1/2}}$. What is the expression for the flux of molecules at $z = 0$? Compare with the result from scaling laws.

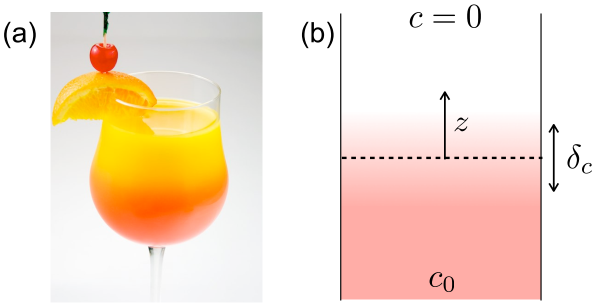

Figure 2: **a.** "Sunrise" cocktail. **b.** Idealized version configuration where two semi-infinite regions of initial concentrations c_0 and $c = 0$ are brought into contact.

2.2 Measuring a diffusion coefficient

One experiment from TPs consists in measuring the diffusion coefficient of fluorescein molecules in water with a microfluidic device (Fig. 3). The chip is designed as a Y channel where two flows converge. A solution of fluorescein is injected in one arm while pure water is injected in the opposite arm. Both flow rates are similar and correspond to an average velocity U. When both streams meet, fluorescein molecules tend to diffuse in the pure water side. The evolution of the concentration profile is monitored by fluorescence imaging in steady conditions.
Derive in terms of simple scaling laws the evolution of the concentration profile along the channel. Under which condition may the diffusive term in the longitudinal direction x direction be neglected? What is the relevant non-dimensional number that characterizes this condition?
In this steady configuration, what is PDE should be solved?
Using the result from the scaling law and the previous result, what is the evolution of the concentration profile along the channel? Doesn't it remind you remind you a situation already explored in the course on Fluid Mechanics?

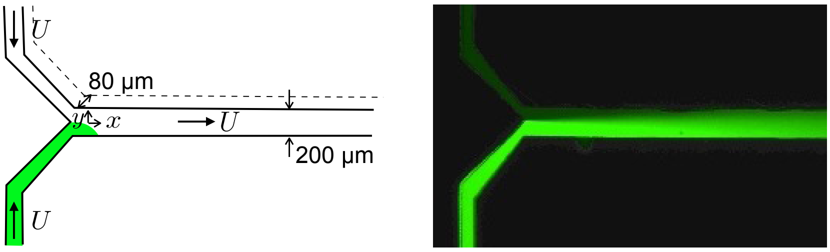

Figure 3: Converging flow in a Y-shaped microfluidic channel (setup used in TPs). A solution of fluorescein in injected in one arm while and pure water in the opposing one.

2.3 Depth of penetration

Ground undergoes cyclic temperature variations during the day or even during the year. As a first approximation, we assume that the temperature of the ground follows a simple law of the form $T(z=0,t) = T_0 + T_1 \cos(\omega t)$ (Fig. 4). How does the temperature evolves as a function of the depth? We can first do the estimate in scaling laws and then derive the exact solution.

Give a numerical estimate of the depth of penetration for a daily or annual cycle for a ground of thermal diffusivity $\alpha \simeq 5.10^{-7}\mathrm{m}^2/\mathrm{s}$.

Caves or basements are reputed to have steady temperatures independently of the external conditions. Does it make sense?

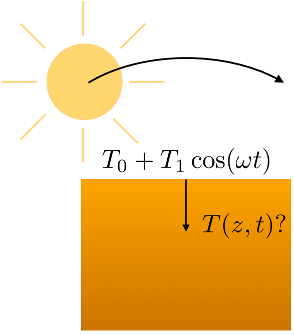

Figure 4: Ground undergoing cyclic variations of surface temperature. What does the temperature evolve as a function of depth?