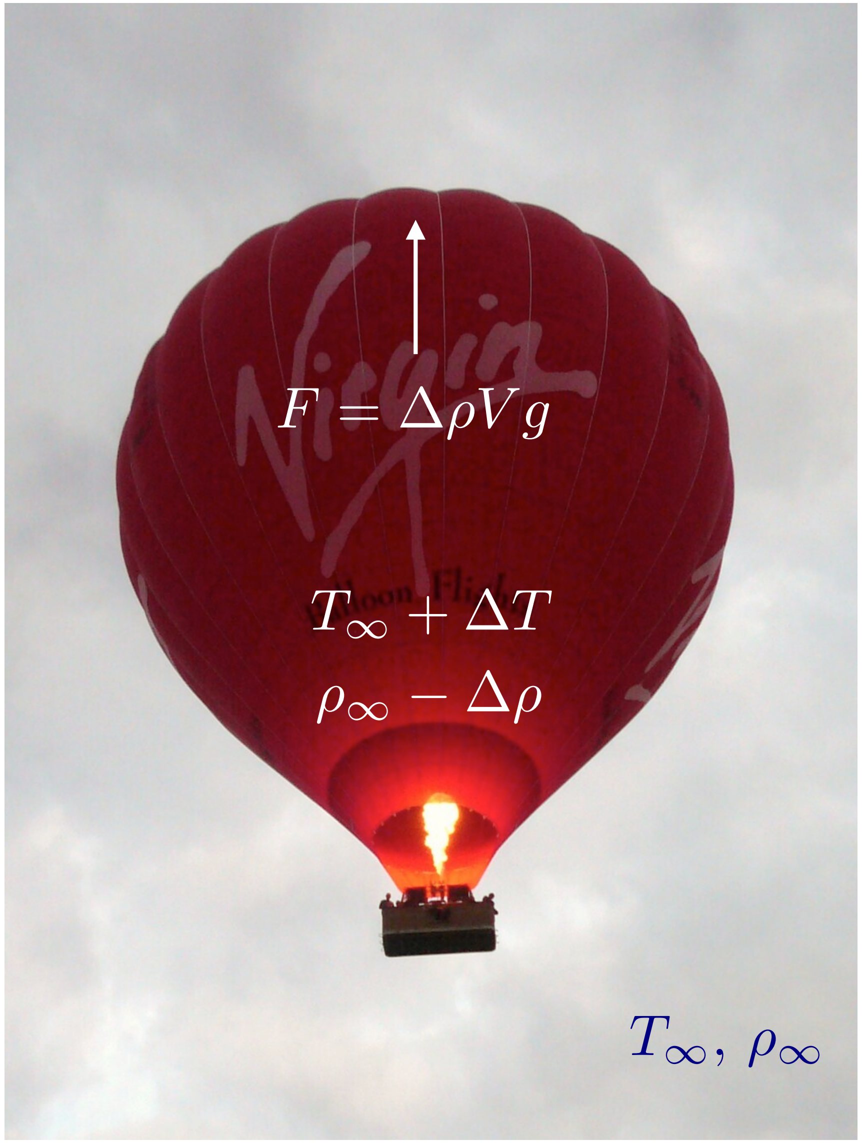
6. Natural convection

6. Natural convection

Principle: hot air balloon (montgolfière)



Thermal expansion coefficient:

$$\rho(T) = \rho_\infty \left(1 - \beta(T - T_\infty)\right)$$

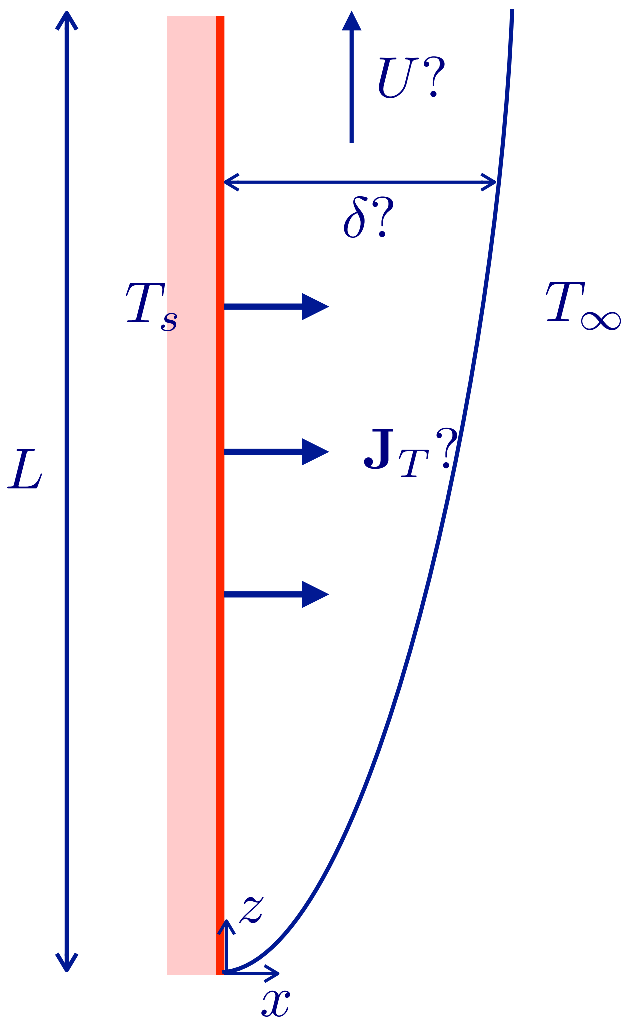
$$\boxed{\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}}$$

hot fluid rises up

⇒ natural convection ↔ gravity

6. Natural convection

Simplified configuration: vertical wall



$$U_\infty = 0$$

Fluid flow induced by temperature contrast
⇒ 1st approximation, single boundary layer

We need to solve 2 coupled equations:

Flow equation:

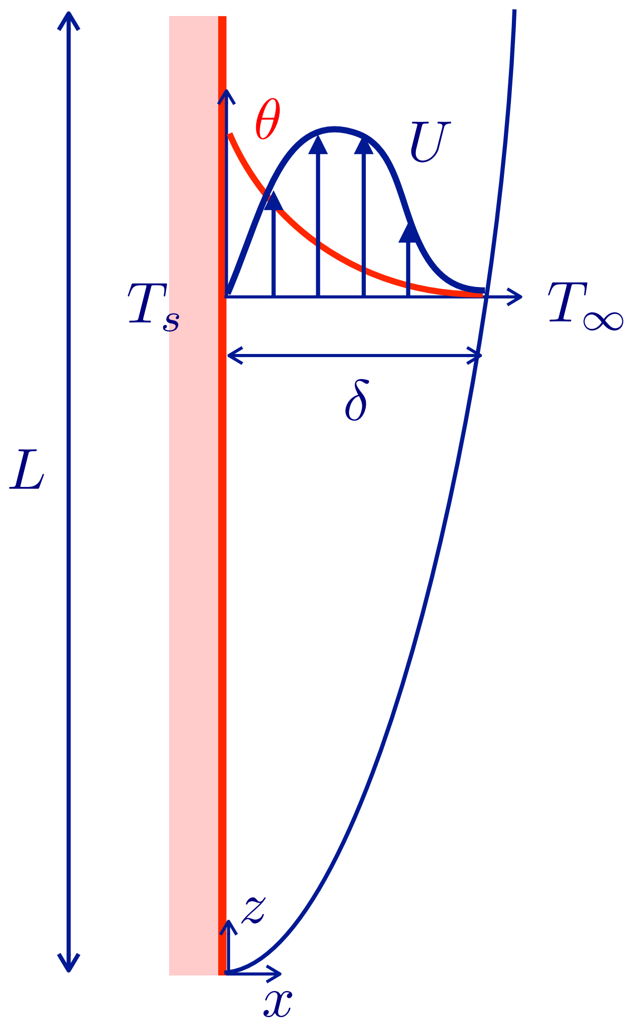
$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} P + \eta \Delta \mathbf{u} + \rho_\infty(1 - \beta(T - T_\infty))\mathbf{g}$$

Heat equation:

$$\frac{\partial T}{\partial t} + (\mathbf{u}.\boldsymbol{\nabla})T = \alpha \Delta T$$

6. Natural convection

Profiles in the boundary layer



$U_\infty = 0$

$\theta = T - T_\infty$

6. Natural convection

Doing our best to simplify !

$$\cancel{\rho\frac{\partial \mathbf{u}}{\partial t}} + \underset{\rho_\infty}{\rho}(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}P + \eta\Delta\mathbf{u} + \rho_\infty(1-\beta(T-T_\infty))\mathbf{g}$$

steady Boussinesq approximation: $\rho \simeq \rho_\infty$
except in $\rho\mathbf{g}$

for a uniform temperature $\mathbf{u}=0$ and $T=T_\infty \Rightarrow -\boldsymbol{\nabla}P + \rho_\infty\mathbf{g} = 0$

We assume to get the same $\boldsymbol{\nabla}P$ in the presence of convection (1st order approximation)

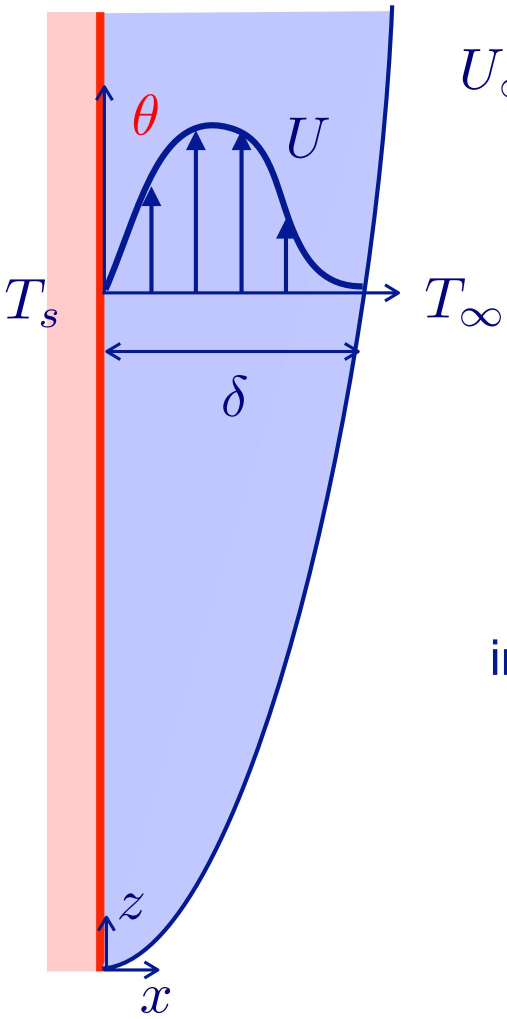
Remains $\rho_\infty(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} \simeq \eta\Delta\mathbf{u} + \rho_\infty\beta\theta\mathbf{g}$ with $\theta = T - T_\infty$

$$\cancel{\frac{\partial \theta}{\partial t}} + (\mathbf{u}.\boldsymbol{\nabla})\theta = \alpha\Delta\theta$$

steady

6. Natural convection

Balance shear force on the wall / buoyancy force



$$U_\infty = 0$$

shear stress $\sigma = \eta \left.\frac{\partial u_z}{\partial x}\right|_{x=0}$

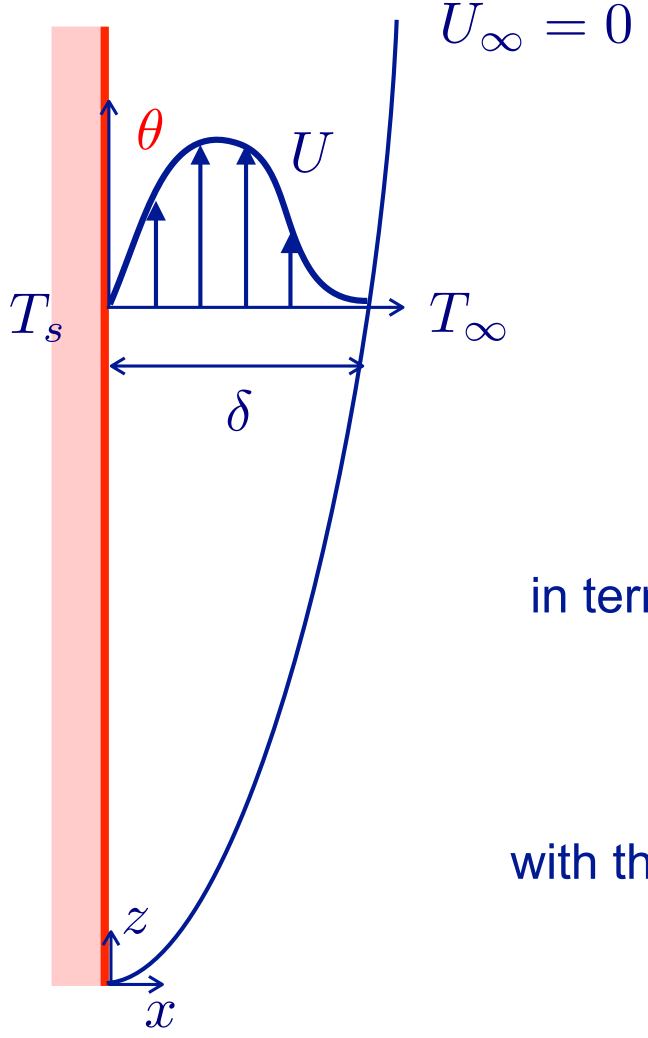
$$W \int_0^z \eta \left.\frac{\partial u_z}{\partial x}\right|_{x=0} dz = W \int_0^z \rho_\infty g \beta \theta \delta(z)\, dz$$

in terms of scaling: $\eta \frac{U}{\delta} \sim \rho_\infty g \beta \theta \delta$

$$U \sim \frac{g\beta\theta\delta^2}{\nu}$$

6. Natural convection

Heat equation



$$\cancel{\frac{\partial \theta}{\partial t}} + (\mathbf{u}.\boldsymbol{\nabla})\theta = \alpha \Delta \theta$$

we consider $u_z \gg u_x$

and $z \gg \delta$

$$u_z \frac{\partial \theta}{\partial z} \simeq \alpha \frac{\partial^2 \theta}{\partial x^2}$$

in terms of scaling:

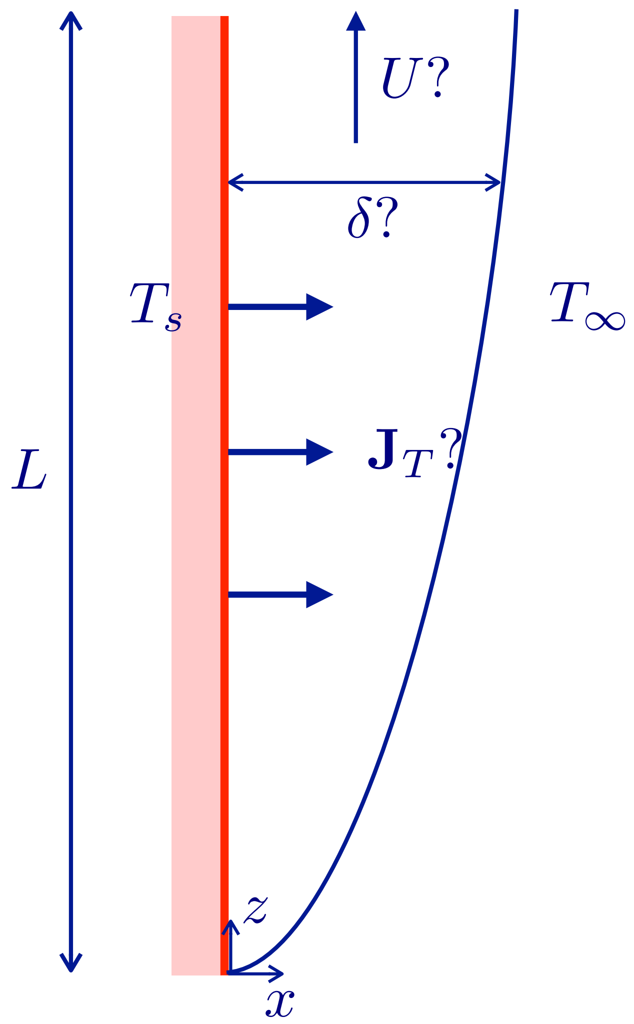
$$\frac{U}{z} \sim \frac{\alpha}{\delta^2}$$

with the previous relation: $\dfrac{g\beta\theta\delta^2}{\nu z} \sim \dfrac{\alpha}{\delta^2}$

$$\boxed{\delta \sim \left(\frac{\nu\alpha}{\beta\theta g} z\right)^{1/4}}$$

6. Natural convection

Getting the heat flux



$$U_\infty = 0 \qquad J_T = \kappa \frac{\partial T}{\partial x} \sim \kappa \frac{\theta}{\delta} \sim \kappa \frac{\theta}{L} \mathrm{Nu}$$

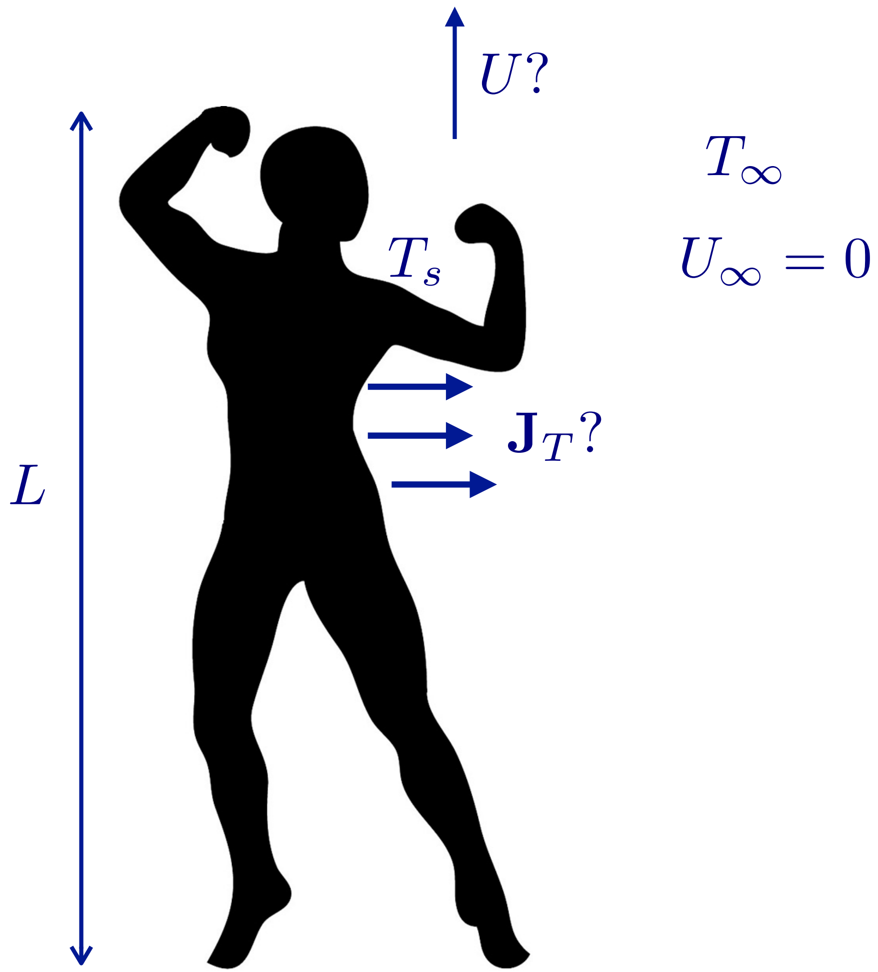
$$\boxed{\mathrm{Nu} \sim \mathrm{Ra}^{1/4}}$$

Rayleigh number: $\boxed{\mathrm{Ra} = \dfrac{\beta g L^3 \theta}{\nu \alpha}}$

convection velocity: $\boxed{U \sim (\beta g \theta z)^{1/2}}$

6.1 Human plume

Getting the heat flux



Naked human: confort for

$$T_\infty \sim 27°\mathrm{C} \;\Rightarrow\; \theta \sim 10°\mathrm{C}$$

$$\mathrm{Ra} \sim 3.10^{10} \;\Rightarrow\; \mathrm{Nu} \sim 400$$

$$h_T = \frac{\kappa}{L}\mathrm{Nu} \sim 6.5\,\mathrm{W.m^{-2}.K^{-1}}$$

$$J_T = h_T(T_s - T_\infty) \sim 65\,\mathrm{W/m^2}$$

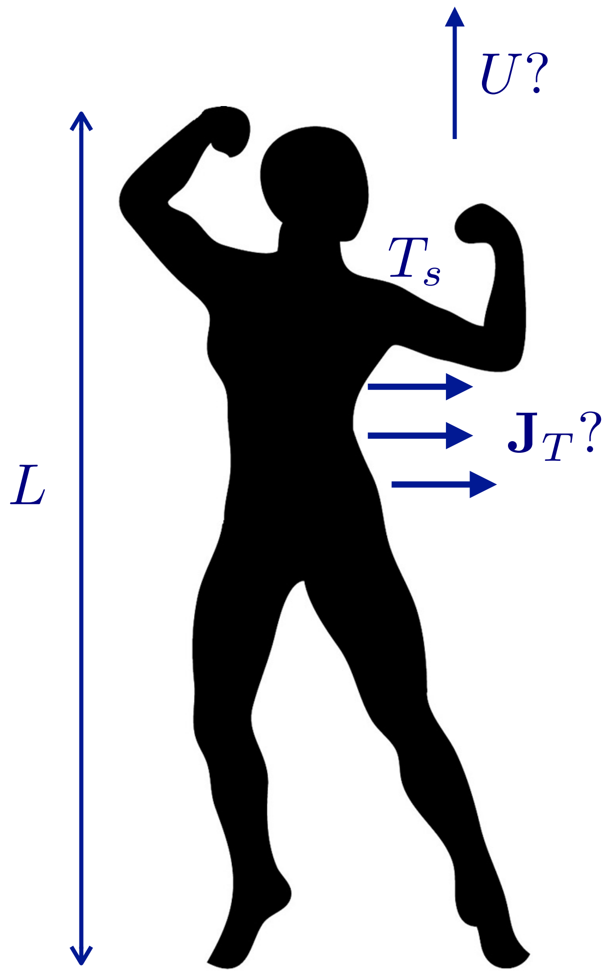
$$S \simeq 1.6\,\mathrm{m}^2$$

$$P = J_T S \simeq 100\,\mathrm{W}$$

good order of magnitude

6.1 Human plume

with a sweater: same $T_s - T_\infty \sim 10°\mathrm{C}$



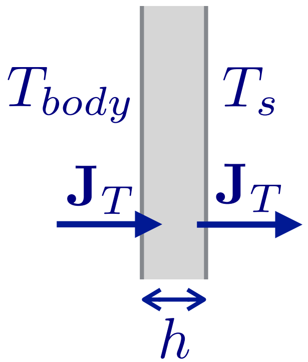
could be $T_\infty \sim 17°\mathrm{C}$ and $T_s \sim 27°\mathrm{C}$

T_∞

$U_\infty = 0$

compatible IR imaging

Thickness of the sweater?



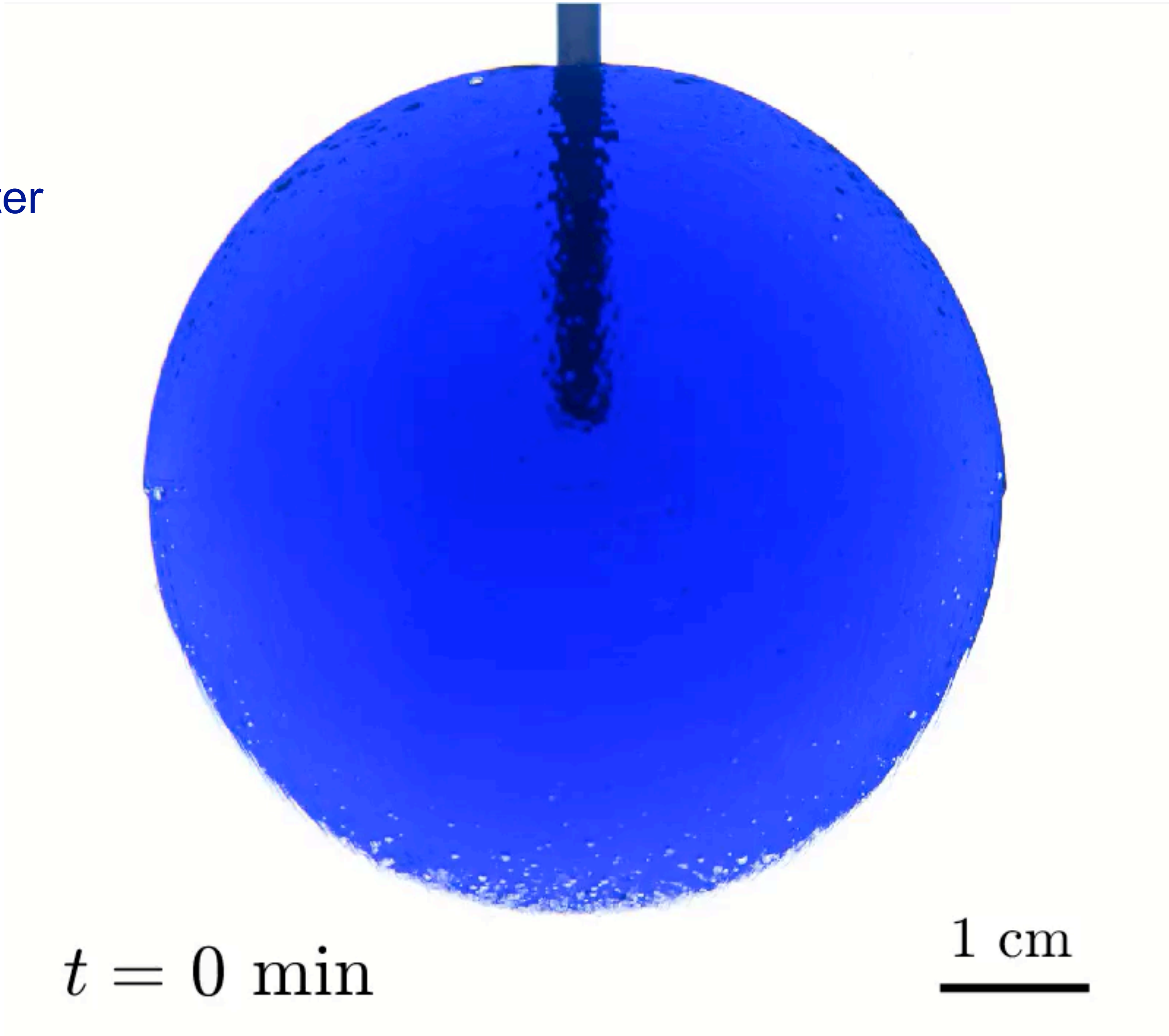
$$J_T = \frac{\kappa_{wool}}{h}(T_{body} - T_s)$$

$$\kappa_{wool} \sim 0.05\,\mathrm{W.m^{-1}.K^{-1}}$$

$$h \sim 8\,\mathrm{mm}$$

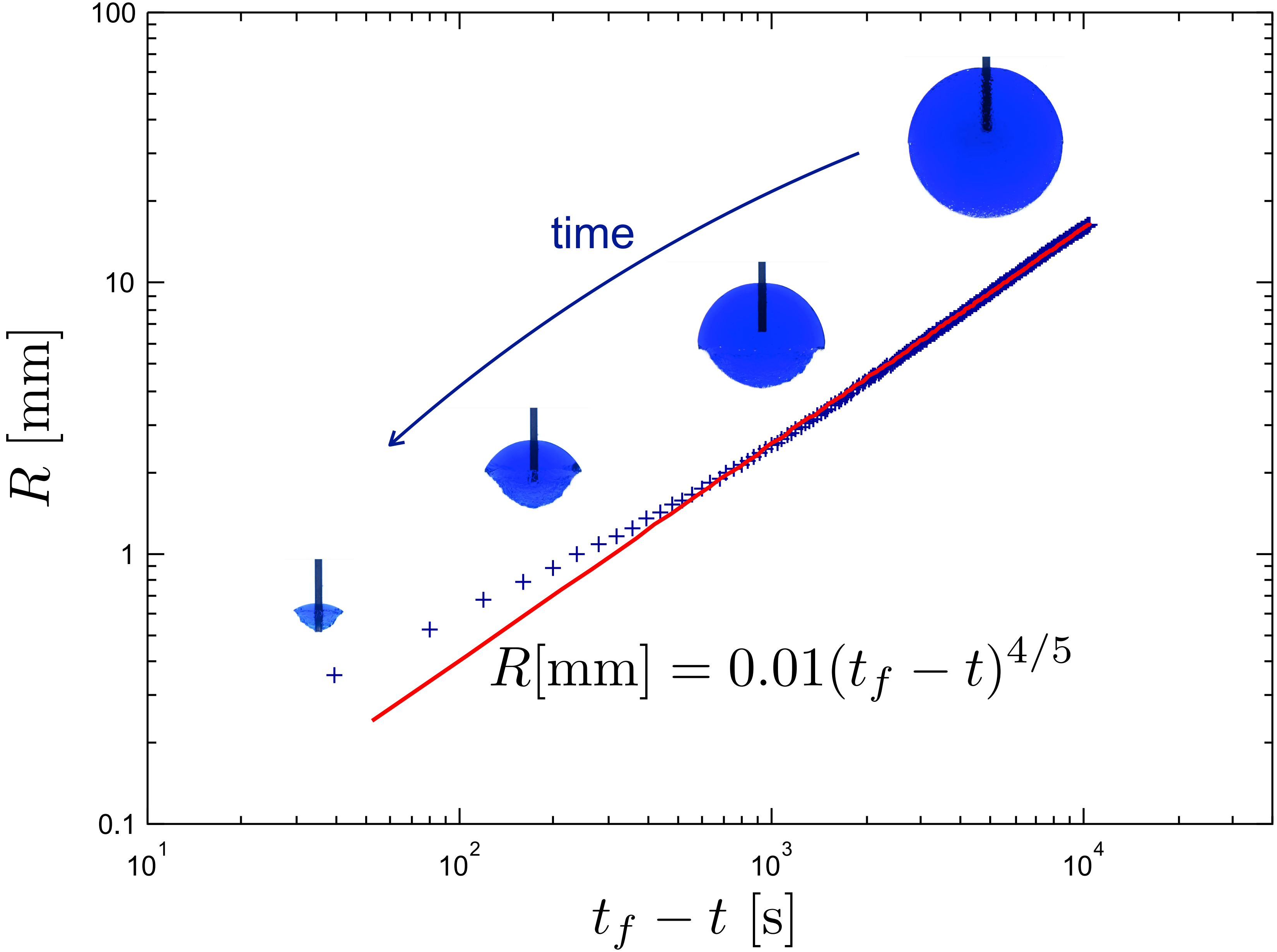
6.2 Dissolution of a lollipop

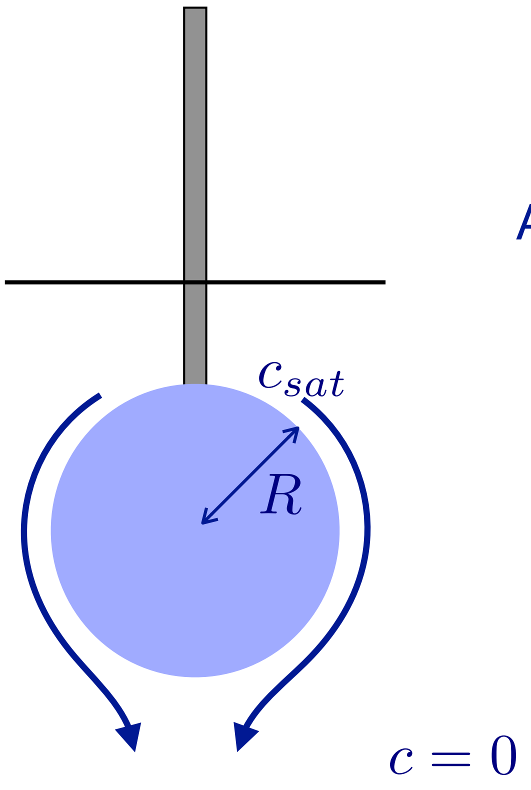
water



M.S.D Wykes et al. Physical Review Fluids (2018) shadowgraph

6.2 Dissolution of a lollipop





At the surface of the lollipop $c = c_{sat}$ (mass concentration)

liquid density: $\rho = \rho_0 + c = \rho_0\,(1 + c/\rho_0)$

$$c/\rho_0 \updownarrow \beta\theta$$

$$\boxed{Ra = \frac{g\, c_{sat}/\rho_0\, R^3}{\nu D}}$$

$$J_m \sim \frac{D}{R}(c_{sat} - 0)\left(\frac{g\, c_{sat}/\rho_0\, R^3}{\nu D}\right)^{1/4}$$

mass conservation: $$\frac{dm}{dt} = -\rho_s 4\pi R^2 \frac{dR}{dt} = 4\pi R^2 J_m$$

$$\frac{dR}{dt} \sim -\frac{D c_{sat}}{\rho_s}\left(\frac{g\, c_{sat}/\rho_0}{\nu D}\right)^{1/4} \frac{1}{R^{1/4}}$$

$$\frac{dR/R_0}{dt} \sim -\frac{Dc_{sat}}{\rho_s R_0}\left(\frac{g\, c_{sat}/\rho_0}{\nu D R_0}\right)^{1/4} \frac{1}{(R/R_0)^{1/4}}$$

$1-(R/R_0)^{5/4} = t/\tau$ with $\boxed{\tau \sim \frac{\rho_s R_0}{Dc_{sat}}\left(\frac{\nu D R_0}{g\, c_{sat}/\rho_0}\right)^{1/4}}$

$$\boxed{R = R_0\,(1-t/\tau)^{4/5}}$$

Comparison with experiment: exponent 4/5 → OK

$$R_0 = 3\,\mathrm{cm} \qquad c_{sat} = 0.3\,\mathrm{g/cm^3} \qquad \rho_s = 1.43\,\mathrm{g/cm^3}$$

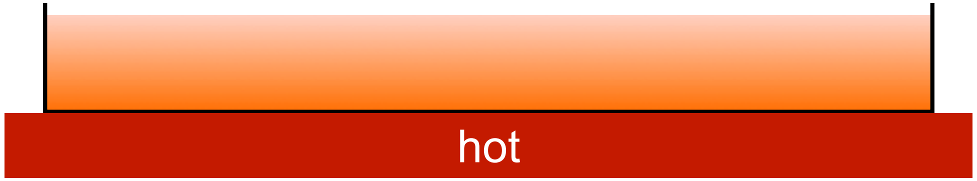
$D \simeq 4.3 \cdot 10^{-10}\,\mathrm{m^2/s}$ (sucrose in pure water)

$\nu = 10^{-6}\,\mathrm{m^2/s}$ (pure water) $\leftrightarrow$ $8 \cdot 10^{-4}\,\mathrm{m^2/s}$ (saturated sucrose)

$\tau \simeq 250\,\mathrm{min}$ with ν_{water}

experiment: $\tau \simeq 175\,\mathrm{min}$

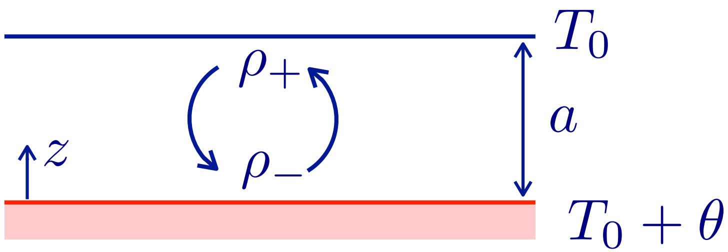
7. Rayleigh-Bénard instability



https://youtu.be/nQUH9nGTZTY

→ course on Instabilities: Laurent Duchemin

Rayleigh-Bénard instability



temperature profile:

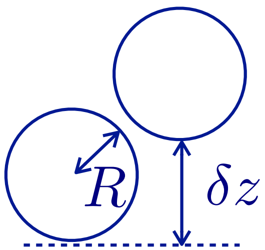
$$T(z) = T_0 + \theta - \theta z/a$$

density profile:

$$\rho(z) = \rho_0 \left(1 - \beta(T(z) - T_0)\right)$$

$$\rho(z) = \rho_0 \left(1 - \beta(\theta - \theta z/a)\right)$$

light at the bottom $\Rightarrow$ unstable?



After a perturbation a blob of size R moves up by δz

mismatch with ambiant density: $\delta\rho = -\rho_0\, \beta \dfrac{\theta}{a} \delta z$

buoyancy force: $\dfrac{4}{3}\pi R^3 \delta\rho g = \dfrac{4}{3}\pi R^3 \rho_0 g \beta \dfrac{\theta}{a} \delta z$

Buoyancy balanced by viscous drag (we assume $Re \ll 1$)

$$\frac{4}{3}\pi R^3 \rho_0 g \beta \frac{\theta}{a} \delta z = 6\pi\eta R U$$

$$U = \frac{d\delta z}{dt} = \frac{9}{2}\frac{\rho_0 g R^2}{\eta}\beta\frac{\theta}{a}\delta z$$

$\delta z(t) = \delta z_0 \exp(t/\tau_{conv})$ with $\boxed{\tau_{conv} \sim \frac{\eta}{\rho_0 g R^2}\frac{a}{\beta\theta}}$

As the blog rises up, thermal diffusion attenuates θ

$$\boxed{\tau_{diff} \sim \frac{R^2}{\alpha}}$$

Unstable if $\tau_{diff} \gg \tau_{conv}$ $$\frac{R^2}{\alpha} \gg \frac{\eta}{\rho_0 g R^2} \frac{a}{\beta\theta}$$

$$\frac{gR^4\beta\theta}{\nu\alpha a} \gg 1$$

The bigger the blob, the more instable ⇒ most unstable configuration $R \sim a$

Unstable if $$\boxed{Ra = \frac{ga^3\beta\theta}{\nu\alpha} > Ra_c}$$

Lecture from L. Duchemin $Ra_c \sim 1000$

estimate with $\theta = 50°\mathrm{C}$

	β [K^{-1}]	ν [m^2/s]	α [m^2/s]	$\beta/\nu\alpha$ [$\mathrm{s}^2\mathrm{m}^{-2}\mathrm{K}^{-1}$]	a_c[mm]
air	$3.4\ 10^{-3}$	$1.5\ 10^{-5}$	$2\ 10^{-5}$	10^{7}	6
water	$2\ 10^{-4}$	10^{-6}	$1.4\ 10^{-7}$	$1.4\ 10^{9}$	1
glycerin	$4.8\ 10^{-4}$	10^{-3}	10^{-7}	$5\ 10^{6}$	7