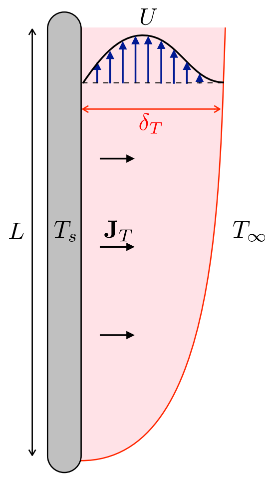
Transport Phenomena:
6. Natural convection

Take home message:



Thermal expansion coefficient:

$$\rho = \rho_0 \left(1 - \beta(T - T_0)\right)$$

Rayleigh number:

$$\mathrm{Ra} = \frac{\beta g L^3 \Delta T}{\nu \alpha}$$

Heat transfer coefficient:

$$\mathbf{J}_T = -h_T(T_\infty - T_s)$$

Nusselt number:

$$\mathrm{Nu} = \frac{h_T L}{\kappa} \sim \mathrm{Ra}^{1/4}$$ (vertical plate, Re_x>>1)

different expressions for other configurations

same formalism for mass transport

1 Self-induced flow along a vertical wall

We consider a vertical hot plate immersed in a colder fluid (Fig. 1). In the vicinity of the plate, the fluid becomes warmer and therefore less dense. As a first approximation, the variation of density in proportional to the temperature difference, $\rho = \rho_0 \left(1 - \beta(T - T_\infty)\right)$, where β is the (volume) thermal expansion coefficient. We define the temperature mismatch as $\theta = T - T_\infty$. One usual simplification consist in considering the temperature dependance of ρ solely in the gravity term (Boussinesq approximation). We propose to describe the characteristics of the rising boundary layer in terms of velocity flow and heat flux.

In contrast with forced convection, fluid flow and heat transfer are deeply intricate. The flow in the self-induced boundary layer differs from the case of forced convection. Describe qualitatively the distribution of velocity in this boundary layer.
The coupling between viscous flow and buoyancy forces can be interpreted in terms of force balance. If $\delta(x)$ is the width of the boundary layer, what is the typical viscous shear force acting along the wall? What is the buoyancy force acting on the same boundary layer? Deduce a first relation between U, θ and δ.
Using the heat equation deduce independent scalings for δ and U.

We now need to estimate the heat exchange between the plate and the surrounding air. Express the heat flux J_T, as a scaling law. An important non-dimensional number for natural convection is the Rayleigh number defined as:

$$Ra = \frac{\beta g \theta L^3}{\nu \alpha}$$

How does Nusselt number scales with Ra in the present configuration?

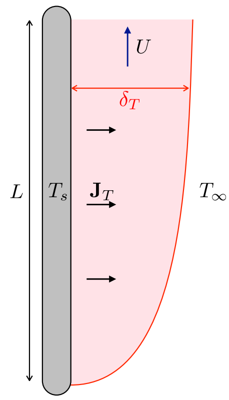
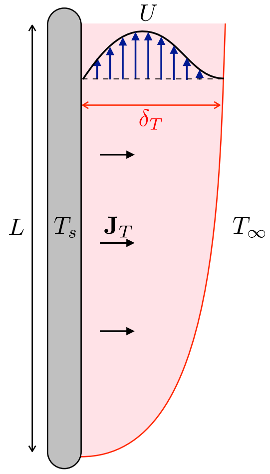


Figure 1: Convection flow along a vertical wall characterized by a boundary layer self-induced by buoyancy forces.

Transport Phenomena:
7. Natural convection & Convective instabilities

Take home message:



Thermal expansion coefficient:

$$\rho = \rho_0 \left(1 - \beta(T - T_0)\right)$$

Rayleigh number:

$$\mathrm{Ra} = \frac{\beta g L^3 \Delta T}{\nu \alpha}$$

Heat transfer coefficient:

$$\mathbf{J}_T = -h_T(T_\infty - T_s)$$

Nusselt number:

$$\mathrm{Nu} = \frac{h_T L}{\kappa} \sim \mathrm{Ra}^{1/4}$$ (vertical plate, Re_x>>1)

different expressions for other configurations

same formalism for mass transport

1 Human plumes

Figure 1a illustrates the thermal plume emanating from a human body ("schlieren" optical technique). A quick estimate of the velocity of the plume with ImageJ indicates 0.7 m/s. We would like to verify if the previous scaling laws are consistent with standard physiological data (a person at rest typically produces 100 W of heat for a body area on the order of 1.6 m^2).

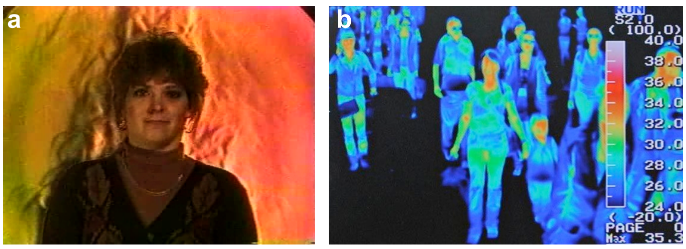


Figure 1: **a** Thermal plume emanating from a human being visualized through the optical "Schlieren" technique (source: https://youtu.be/1MA-zEUepvs).

Can we obtain this heat flux from personal experience? Without clothes, we feel thermally comfortable for an exterior temperature of about 27°C (under moderate humidity). If our skin has a temperature of 37°C, what are the corresponding values for the Rayleigh and Nusselt numbers? Do we recover a heat flux consistent with physiological data?
If the heat flux remains the same, what temperature would you recommend for the bath of a baby?

In the illustrated exemple, the person wears a rather thick sweater, a reasonable estimate of the exterior temperature may be 16°C. Using the same value for the heat flux, is it possible to estimate the temperature at the surface of the clothes? Is the value consistent with IR imaging (Fig 1b)? Is the plume velocity in agreement with the prediction?

Material constants for air:
$\beta = 3.4 \cdot 10^{-3}\,\mathrm{K}^{-1}$, $\nu = 1.5 \cdot 10^{-5}\,\mathrm{m}^2\mathrm{s}^{-1}$, $\alpha = 2 \cdot 10^{-5}\,\mathrm{m}^2\mathrm{s}^{-1}$, $\kappa = 0.026\,\mathrm{Wm}^{-1}\mathrm{K}^{-1}$
for water:
$\beta = 4 \cdot 10^{-4}\,\mathrm{K}^{-1}$, $\nu = 1 \cdot 10^{-6}\,\mathrm{m}^2\mathrm{s}^{-1}$, $\alpha = 0.14 \cdot 10^{-6}\,\mathrm{m}^2\mathrm{s}^{-1}$, $\kappa = 0.6\,\mathrm{Wm}^{-1}\mathrm{K}^{-1}$

2 Dissolution of a lollipop

How long does it take to dissolve a lollipop in a bath of water? About 200 min, following an experiment conducted with a lollipop of initial radius $R_0 = 3\,\mathrm{cm}$ (Fig. 2). The dissolution mechanism relies on a self-induced convective flow. A lollipop is basically composed of sugar. In contact with water, sugar tends to dissolve and reaches a saturating (mass) concentration c_{sat} in the vicinity of the lollipop. As a consequence, the solution is denser close to the lollipop than far away, which results in a convective flow.

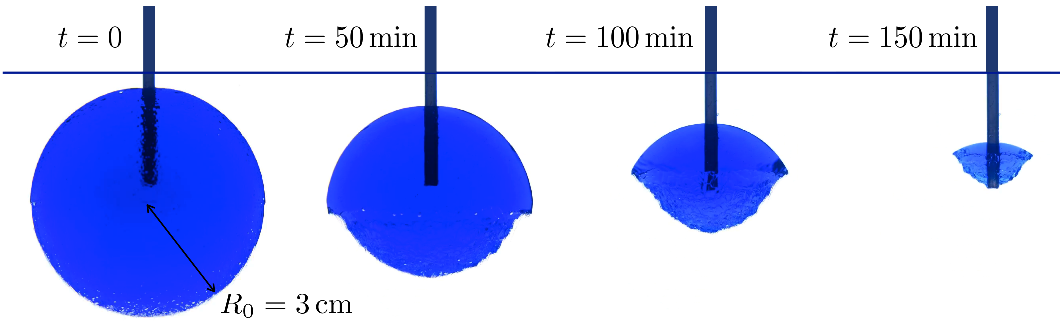


Figure 2: Successive snapshots of the dissolution of a lollipop in a bath of water (from M.S. Davies Wykes, *Physical Review Fluids*, **3**, 043801 (2018)).

The liquid density varies linearly with the sugar mass concentration c: $\rho = \rho_0 + c$. What is the analogous of Rayleigh number in the problem? Estimate the mass flux through the surface of the lollipop.

Using mass conservation, predict the time evolution of the radius of the lollipop $R(t)$ (assuming that the lollipop remains roughly spherical). Estimate the dissolution time of the lollipop t_f. Is the prediction for the time evolution in agreement with the experiment (Fig. 3)?

Numerical data
$R_0 = 3\,\mathrm{cm}$, $c_{sat} = 0.3\,\mathrm{g/cm}^3$, lollipop density $\rho_S = 1.43\,\mathrm{g/cm}^3$, $D = 4.3 \cdot 10^{-10}\,\mathrm{m}^2/\mathrm{s}$ (sucrose in pure water, lower in concentrated sucrose solution), $\nu_{water} = 10^{-6}\,\mathrm{m}^2/\mathrm{s}$ (for saturated sucrose $\nu_s = 8 \cdot 10^{-4}\,\mathrm{m}^2/\mathrm{s}$).

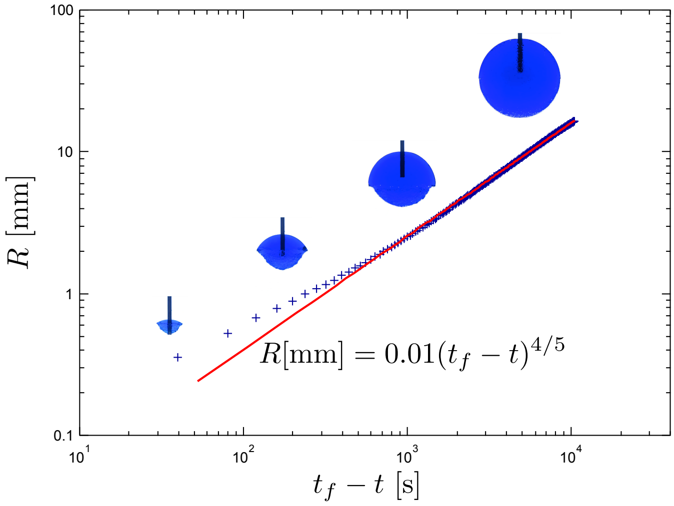


Figure 3: Evolution of the effective radius (estimated from the area of the lollipop in the pictures $\mathcal{A}$ as $R = \mathcal{A}/\pi$) as a function of $t_f - t$ and fit by a power law.

3 Convective instabilities

3.1 Rayleigh-Bénard instability

Here is an experiment you can try at home: pour a millimetric layer of oil in a frying pan, sprinkle it with some fine powder such as flour and place it on a hotplate (Fig. 4a). If the heating temperature is low, nothing happens. However, you should observe the particles moving beyond a critical temperature and gradually forming cellular patterns (Fig. 4c). A pedagogical video describing the phenomenon is available on YouTube: https://youtu.be/Eud7uG5JHng This instability was first described by Rayleigh and Bénard although the exact origin of the observed patterns have been the subject of scientific debates for almost 50 years.

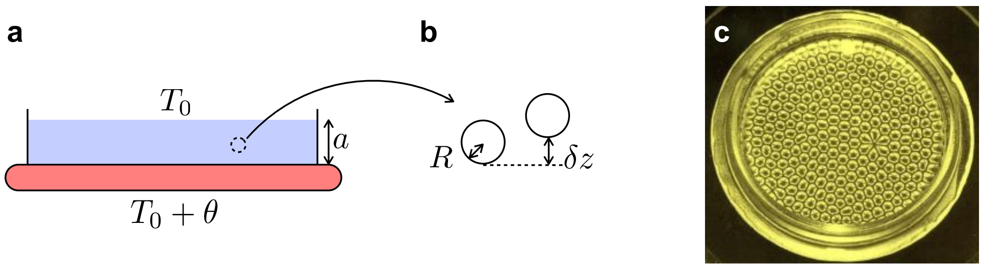


Figure 4: **a.** Liquid layer heated from underneath. **b.** What does happen to sphere of fluid of radius R whose vertical position is perturbed by a quantity δz? **c.** Convection cells in a millimetric layer of viscous oil (image E. Koschmieder).

We fist consider the situation where the liquid remains still. If the difference in temperature between bottom and top is θ, what is the temperature profile across the layer of thickness a? What is the resulting profile in density for a fluid of a thermal expansion coefficient β?

Imagine that due to some perturbation, a bubble of fluid of radius R moves by a quantity

δz (Fig. 4b). We will first assume that the sphere is thermally isolated from the surrounding liquid. What is the resulting buoyancy force acting on the sphere? Is this situation stable?

If we ignore inertial effects, what term balances this force? Deduce an expression of the velocity of the bubble. Integrate the velocity to obtain the position of the bubble as a function of time. What is the characteristic convection time for the bubble?

In reality, the bubble can exchange heat with the surrounding fluid. What would be the diffusion time required to equilibrate the inner temperature of the bubble with the outer temperature?

In which case is the situation unstable? What is the most unstable situation in terms of bubble size?