

Transport phenomena

2. Solving transport equations

mass:

Diffusion

$$\mathbf{j}_d = -D \nabla c$$

Convection

$$\mathbf{j}_c = c \mathbf{u}$$

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = D \Delta c + r$$

[m².s⁻¹]

source/sink
eg: chemical reaction

heat:

Conduction: Fourier's law

$$\mathbf{j}_{cond} = -\kappa \nabla T$$

thermal flux
[W.m⁻²]

thermal conductivity
[W.m⁻¹.K⁻¹]

Convection

$$\mathbf{j}_{conv} = \rho C_p T \mathbf{u}$$

density
[kg.m⁻³]

specific heat capacity
[J.kg⁻¹.K⁻¹]

Heat equation:

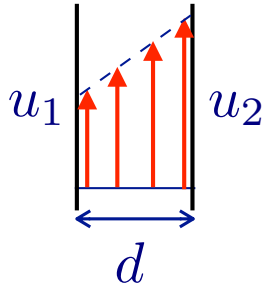
$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \alpha \Delta T + \frac{r}{\rho C_p}$$

$$\alpha = \kappa / \rho C_p$$

thermal diffusivity
[m².s⁻¹]

source/sink
eg: Joule effect

momentum:



“Conduction”

$$\sigma_v = \eta(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

viscous stress [Pa]
flux of momentum

dynamic viscosity [Pa.s]
momentum “conductivity”

“Convection”

$$\rho \mathbf{u} \otimes \mathbf{u}$$

Navier & Stokes equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{u} - \frac{1}{\rho} \nabla p + \mathbf{g}$$

$\nu = \eta / \rho$ kinematic viscosity [$\text{m}^2 \cdot \text{s}$]
momentum “diffusion”

boundary conditions: $\mathbf{j}_c \cdot \mathbf{n}$, $\mathbf{j}_T \cdot \mathbf{n}$, $\boldsymbol{\sigma} \cdot \mathbf{n}$, c , T , \mathbf{u} continuous
exception: latent heat

Transport phenomena

2. Solving transport equations

2.1 From mass to heat and viscous stress transport

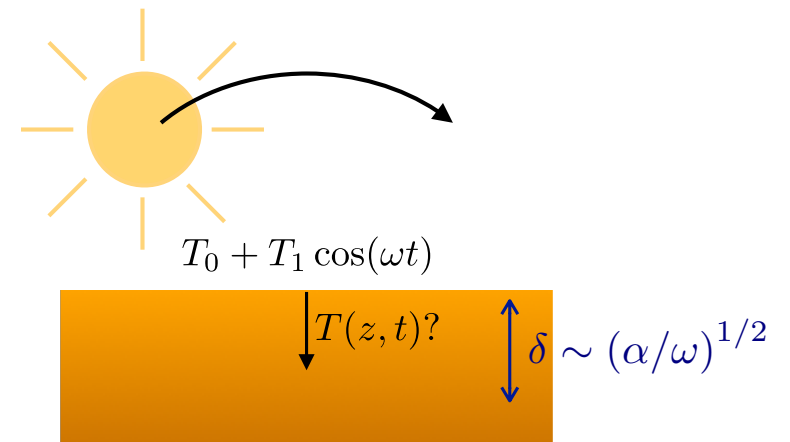
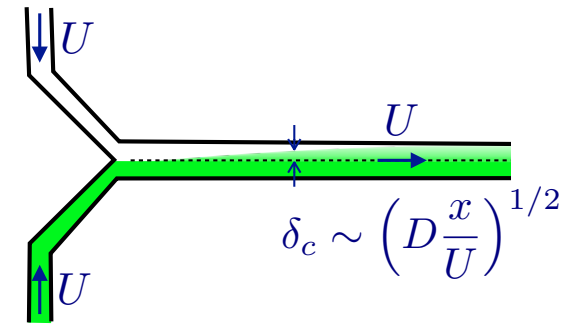
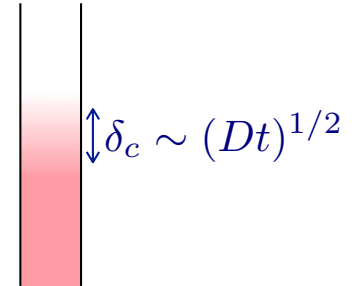
2.2 Homogenization of a cocktail (exact solution?)

2.3 Measuring a diffusion coefficient ?

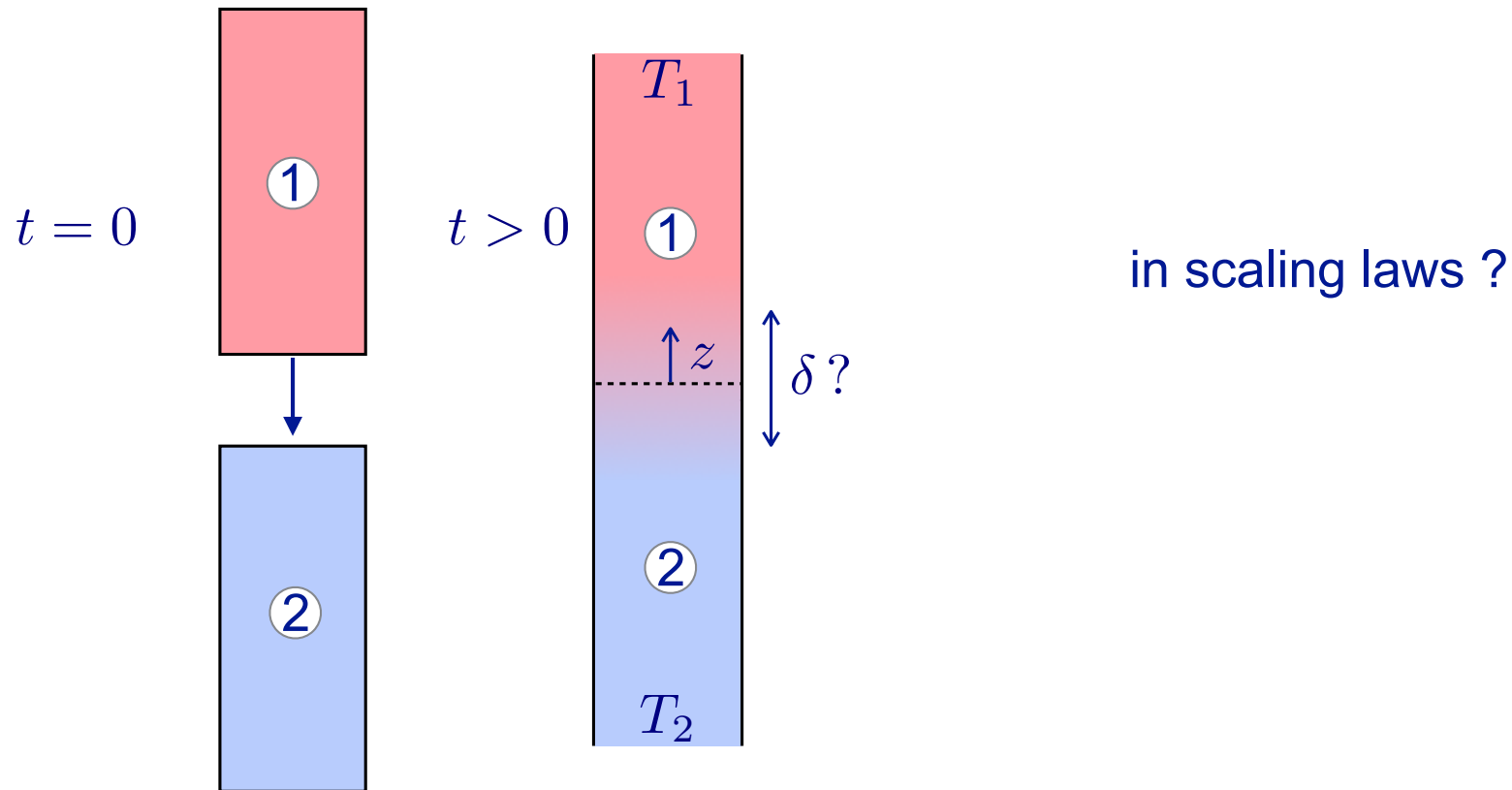
2.4 Depth of penetration

2.5 Contact temperature

2.6 Melting of an ice cube



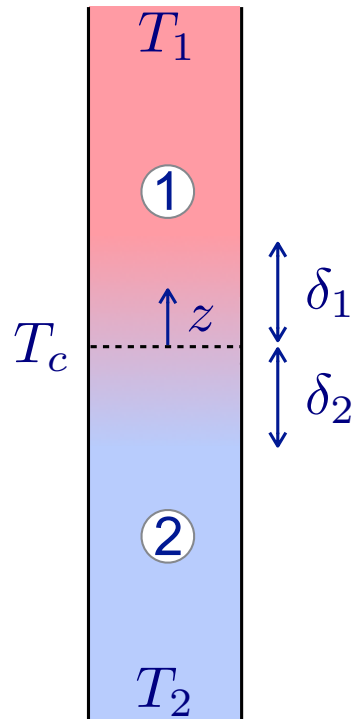
2.5 Contact temperature



We need to solve:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \alpha_i \Delta T + \frac{r_i}{\rho_i C_{p_i}}$$

with T and \mathbf{j} continuous



2 boundary layers:

$$\delta_1 \sim (\alpha_1 t)^{1/2}$$

$$\delta_2 \sim (\alpha_2 t)^{1/2}$$

flux continuity:

$$\kappa_1 \left. \frac{\partial T}{\partial z} \right|_{0+} = \kappa_2 \left. \frac{\partial T}{\partial z} \right|_{0-}$$

$$\kappa_1 \frac{T_1 - T_c}{\delta_1} \sim \kappa_2 \frac{T_c - T_2}{\delta_2} \quad \text{contact temperature}$$

$$\alpha_i = \frac{\kappa_i}{\rho_i C_{p_i}}$$

$$\sqrt{\kappa_1 \rho_1 C_{p_1}} (T_1 - T_c) \sim \sqrt{\kappa_2 \rho_2 C_{p_2}} (T_c - T_2)$$

$$T_c = \frac{\sqrt{\kappa_1 \rho_1 C_{p1}} T_1 + \sqrt{\kappa_2 \rho_2 C_{p2}} T_2}{\sqrt{\kappa_1 \rho_1 C_{p1}} + \sqrt{\kappa_2 \rho_2 C_{p2}}}$$

$\sqrt{\kappa \rho C_p}$ effusivity [J.K.m⁻².s^{1/2}]

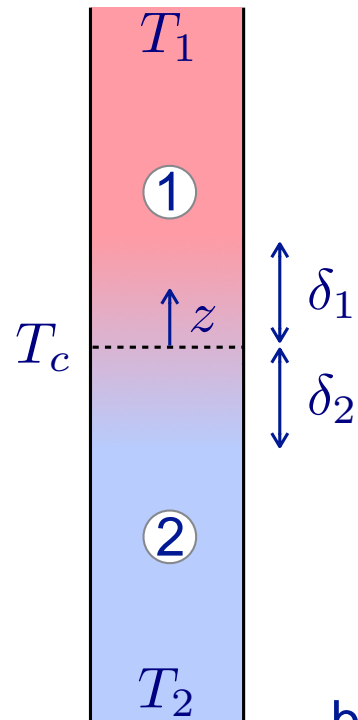
Numerical estimate:

	κ [W.m ⁻¹ .K ⁻¹]	ρ [kg.m ⁻³]	C_p [J.kg ⁻¹ .K ⁻¹]	$\sqrt{\kappa \rho C_p}$ [J.K.m ⁻² .s ^{1/2}]
skin (~water)	0.6	1000	4180	~ 1500
wood	0.15	600	1700	~ 400
aluminium	100	2700	900	~ 16000
glass	1.8	2200	840	~ 1800

skin at 37°C + wood at 20°C $\Rightarrow T_c \sim 33^\circ\text{C}$

skin at 37°C + aluminium at 20°C $\Rightarrow T_c \sim 21^\circ\text{C}$

Detailed solution



general solution found for the diffusion in the cocktail applied to materials 1 and 2:

material 1

for $z > 0$

$$T(z, t) = A \operatorname{erf} \left(\frac{z}{2(\alpha_1 t)^{1/2}} \right) + B$$

material 2

for $z < 0$

$$T(z, t) = C \operatorname{erf} \left(\frac{z}{2(\alpha_2 t)^{1/2}} \right) + D$$

boundary conditions

$$T(0^+) = T(0^-) \Rightarrow B = D = T_c$$

$$T(z \rightarrow \infty) = T_1 = A + T_c$$

$$T(z \rightarrow -\infty) = T_2 = -C + T_c$$

$$\kappa_1 \left. \frac{\partial T}{\partial z} \right|_{0^+} = \kappa_2 \left. \frac{\partial T}{\partial z} \right|_{0^-}$$

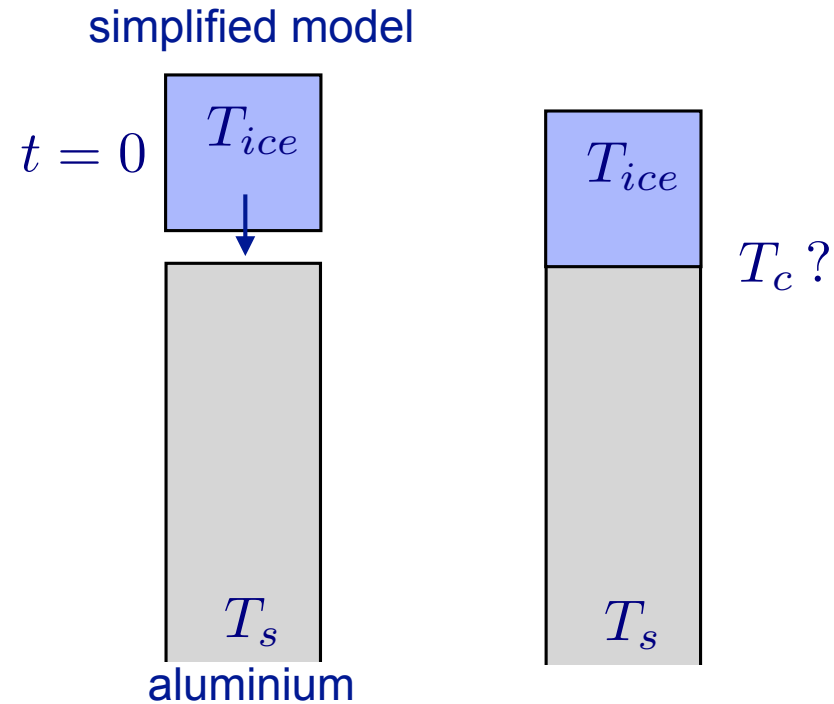
$$\frac{\partial}{\partial z} \operatorname{erf} \left(\frac{z}{2(\alpha t)^{1/2}} \right) = \frac{1}{2(\alpha t)^{1/2}} \frac{2}{\sqrt{\pi}} \exp \left(-\frac{z^2}{4\alpha t} \right)$$

$$\kappa_1 \left. \frac{\partial T}{\partial z} \right|_{0^+} = \kappa_2 \left. \frac{\partial T}{\partial z} \right|_{0^-} \Rightarrow \frac{\kappa_1}{\sqrt{\alpha_1}} A = \frac{\kappa_2}{\sqrt{\alpha_2}} C$$

$$\frac{\kappa}{\sqrt{\alpha}} = (\kappa \rho C_p)^{1/2} \Rightarrow T_c = \frac{(\kappa_1 \rho_1 C_{p_1})^{1/2} T_1 + (\kappa_2 \rho_2 C_{p_2})^{1/2} T_2}{(\kappa_1 \rho_1 C_{p_1})^{1/2} + (\kappa_2 \rho_2 C_{p_2})^{1/2}}$$

same expression as scaling law!

2.6 Melting of an ice cube



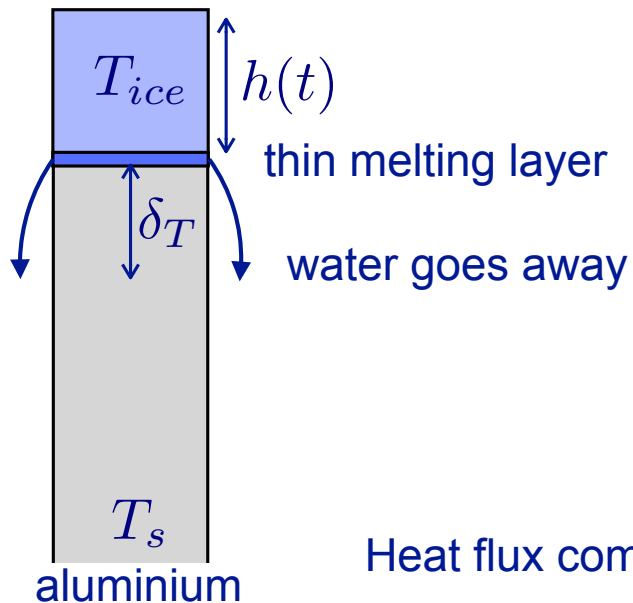
if $T_{ice} = -20^\circ\text{C}$ and $T_s = 20^\circ\text{C}$

	κ [$\text{W}\cdot\text{m}^{-1}\cdot\text{K}^{-1}$]	ρ [$\text{kg}\cdot\text{m}^{-3}$]	C_p [$\text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$]	$\sqrt{\kappa\rho C_p}$ [$\text{J}\cdot\text{K}\cdot\text{m}^{-2}\cdot\text{s}^{1/2}$]
ice	2	920	2000	~ 1900
aluminium	100	2700	900	~ 16000

$\Rightarrow T_c \sim 15^\circ\text{C}$ not compatible with $T_f = 0^\circ\text{C} \dots$

2.6 Melting of an ice cube

even more simplified model: $T_{ice} = T_f = 0^\circ\text{C}$



$$\delta_T \sim (\alpha t)^{1/2}$$

$$j_T \sim \kappa_s \frac{T_s - T_f}{\delta_T}$$

$$j_T \sim \left(\frac{\kappa_s \rho_s C_p}{t} \right)^{1/2} (T_s - T_f)$$

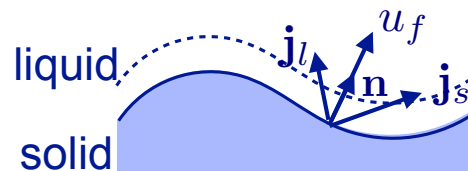
Heat flux compensates for latent heat:

$$j_T S dt = -\rho_{ice} L S dh$$

↙ latent heat [$\text{J}\cdot\text{kg}^{-1}$]

latent heat \Rightarrow jump of heat flux at the boundary

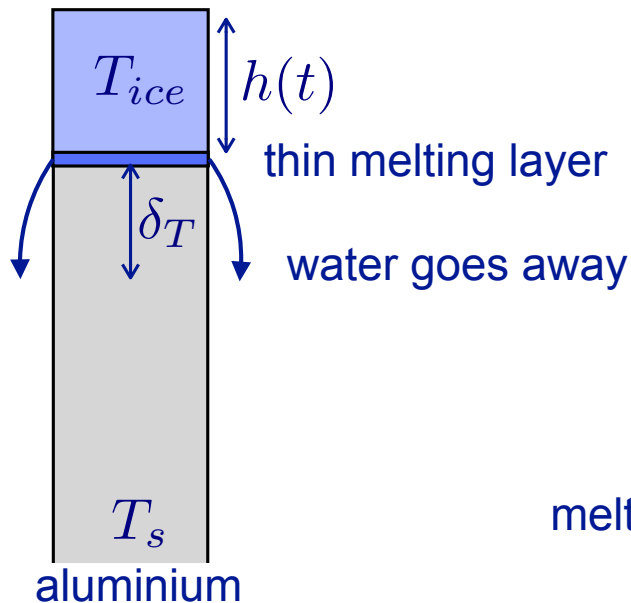
Stefan problem: moving front



$$u_f \rho_s L_f = \mathbf{j}_s \cdot \mathbf{n} - \mathbf{j}_l \cdot \mathbf{n}$$

2.6 Melting of an ice cube

even more simplified model: $T_{ice} = 0^\circ\text{C}$



$$\frac{dh}{dt} = -\frac{j_T}{\rho_{ice}L} \sim -\left(\frac{\kappa_s \rho_s C_P}{t}\right)^{1/2} \frac{T_s - T_f}{\rho_{ice}L}$$

$$h_0 - h(t) \sim \frac{T_s - T_f}{\rho_{ice}L} (\kappa_s \rho_s C_p t)^{1/2}$$

melting time

$$\tau \sim \left(\frac{\rho_{ice}L h_0}{T_s - T_f}\right)^2 \frac{1}{\kappa_s \rho_s C_p}$$

Numerical estimate:

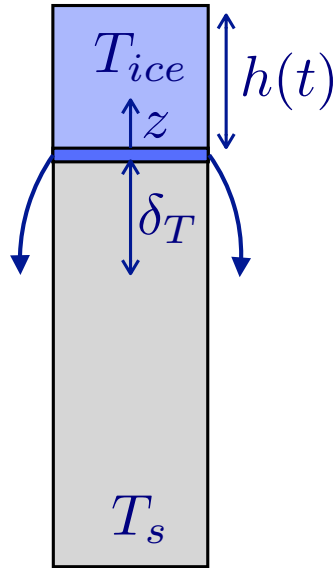
$$h_0 \sim 3 \text{ cm} \quad \rho_{ice} \sim 920 \text{ kg/m}^3 \quad L \sim 334 \text{ kJ/kg}$$

$$\text{aluminium } (\kappa \rho_s C_P)^{1/2} \sim 16000 \text{ J.K.m}^{-2}\text{s}^{1/2} \Rightarrow \tau \sim 800\text{s} \sim 13 \text{ min}$$

$$\text{wood } (\kappa \rho_s C_P)^{1/2} \sim 400 \text{ J.K.m}^{-2}\text{s}^{1/2} \Rightarrow \tau \sim 10^6\text{s} \sim 15 \text{ days !}$$

other heat exchange mechanisms ?

Detailed solution



in the solid: $T(z, t) = A \operatorname{erf} \left(\frac{z}{2(\alpha_1 t)^{1/2}} \right) + B$

$$T(z = 0) = B = T_f$$

$$T(z \rightarrow -\infty) = T_s = -A + B$$

$$T(z, t) = (T_f - T_s) \operatorname{erf} \left(\frac{z}{2(\alpha_2 t)^{1/2}} \right) + T_f$$

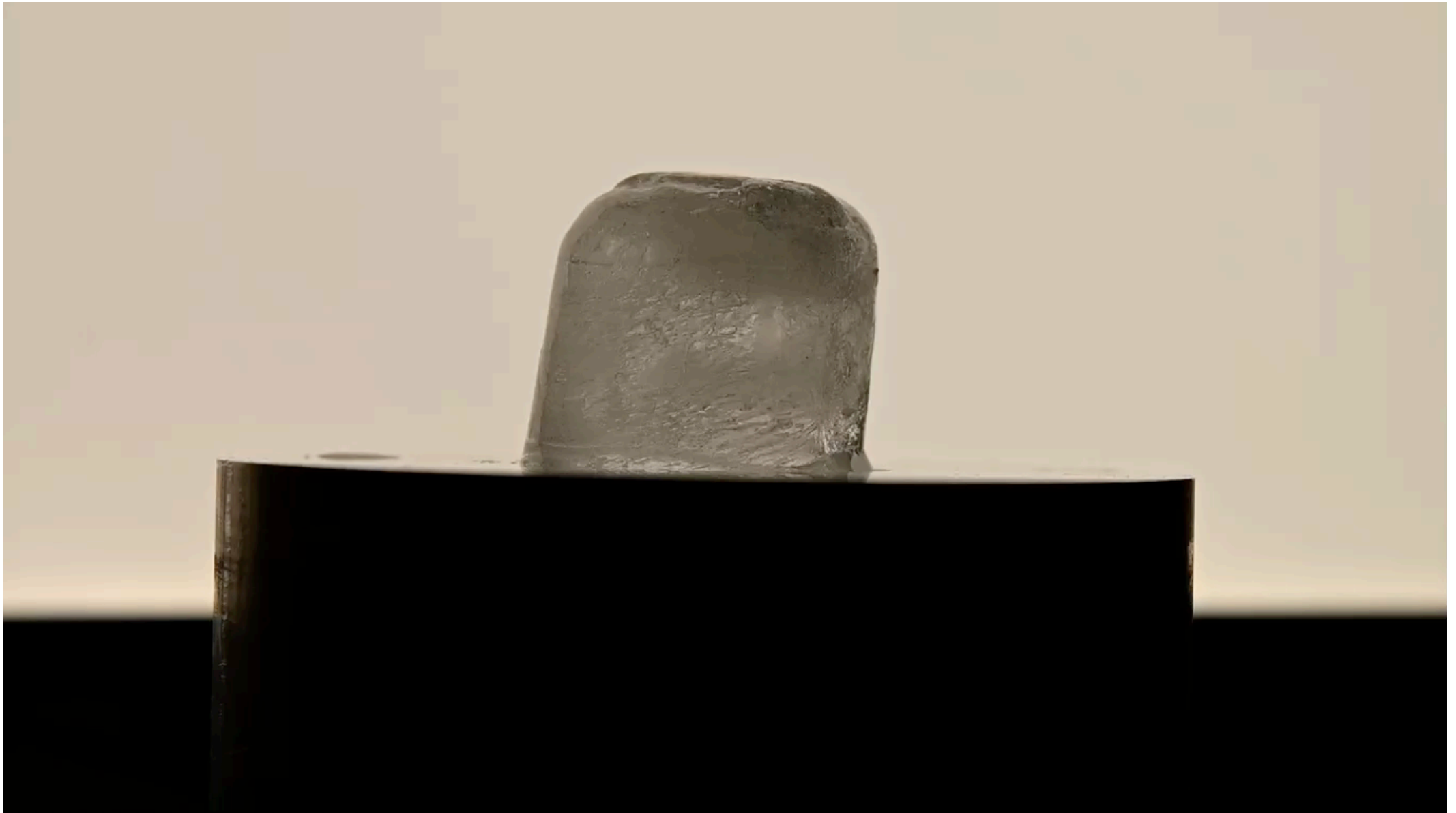
$$j_T = -\kappa_s \left. \frac{\partial T}{\partial z} \right|_{z=0} = \frac{\kappa_s (T_s - T_f)}{2(\alpha t)^{1/2}} = \left(\frac{\kappa_s \rho_s C_p}{t} \right)^{1/2} \frac{T_s - T_f}{2}$$

$$\frac{dh}{dt} = -\frac{j_T}{\rho_{ice} L} = - \left(\frac{\kappa_s \rho_s C_p}{t} \right)^{1/2} \frac{T_s - T_f}{2\rho_{ice} L}$$

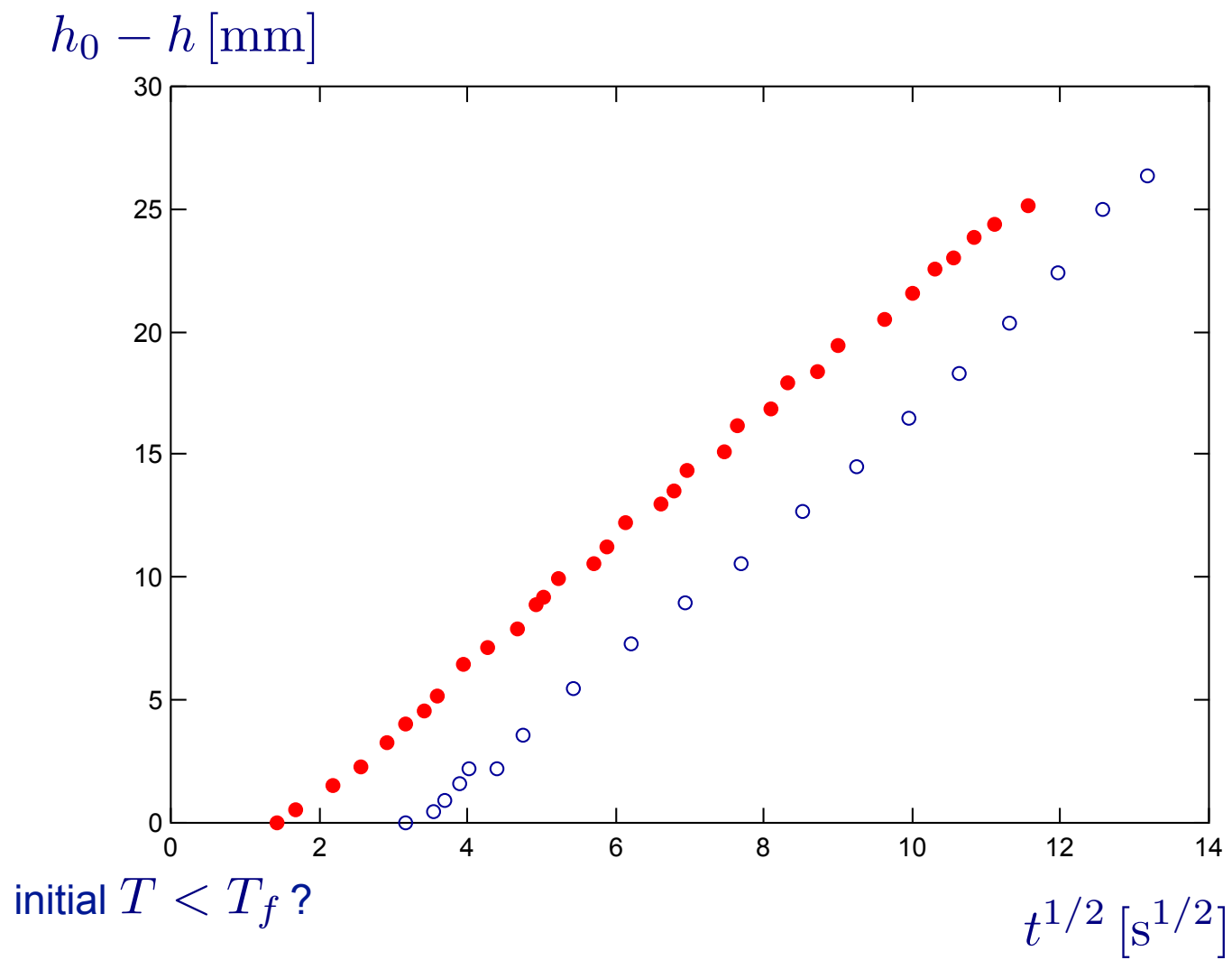
$$h_0 - h(t) = (\kappa_s \rho_s C_p t)^{1/2} \frac{T_s - T_f}{\rho_{ice} L}$$

same expression as scaling law!

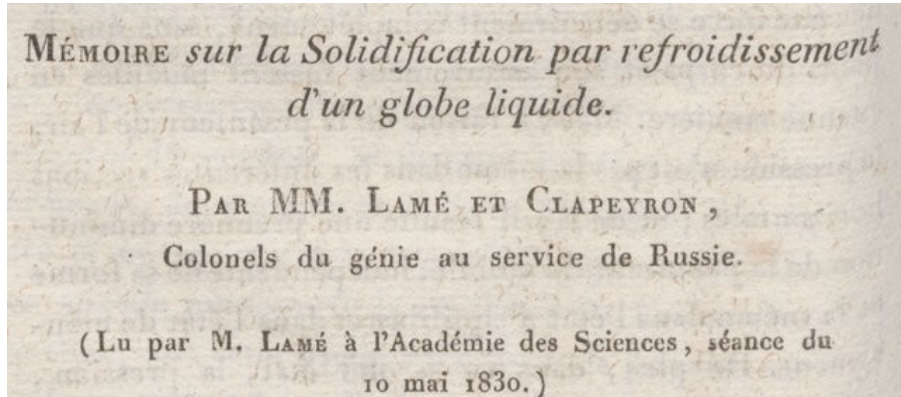
Experiment in the lab



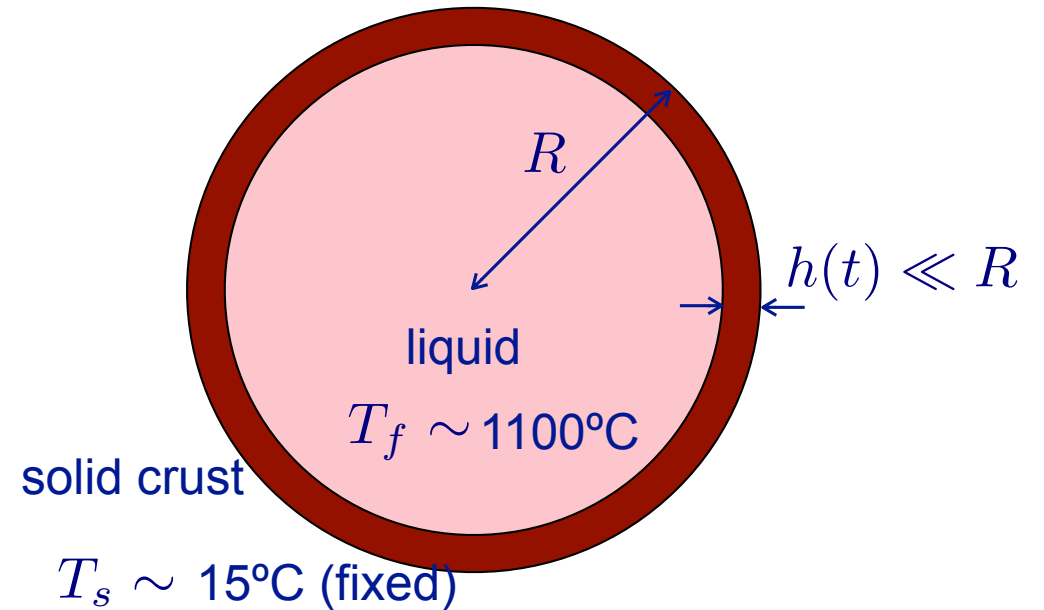
Experiment in the lab



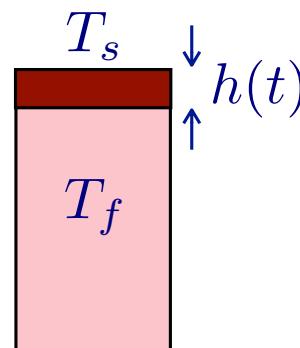
2.7 Estimating the age of the Earth ?

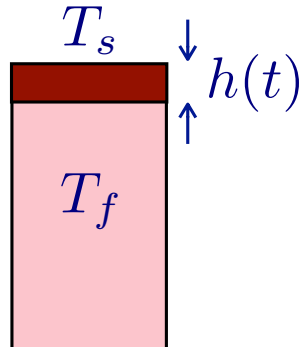


very simplified model



even more simplified model: 1D





uniform T in the liquid \Rightarrow no heat flux inside the liquid

heat flux through the solid:

$$j_T = \kappa_s \frac{T_f - T_s}{h}$$

heat flux compensates latent heat:

$$\rho_s L \frac{dh}{dt} = j_T = \kappa_s \frac{T_f - T_s}{h}$$

$$h \frac{dh}{dt} = \frac{\kappa_s}{\rho_s L} (T_f - T_s)$$

$$h(t) = \left(2 \frac{\kappa_s}{\rho_s L} (T_f - T_s) t \right)^{1/2}$$

Numerical estimate:

$$h \sim 35 \text{ km} \quad \kappa_s \sim 2 \text{ W.m}^{-1}\text{K}^{-1} \quad \rho_s \sim 3.10^3 \text{ kg.m}^{-3} \quad L \sim 4.10^5 \text{ J.kg}^{-1} \quad T_s \sim 15^\circ\text{C} \quad T_f \sim 1100^\circ\text{C}$$

$\Rightarrow \tau \sim 10$ million years ... model too simple!