Transport Phenomena:
10. Solving transport problems

1 Illustrations of thermal imaging

1.1 "Thermal" print on a note book

In one experiment the presenter hand was left on the cover of a notebook for $10\,s$. Right after the image of the hand could be identified at about 2 mm inside the book. Estimate the order of magnitude of the thermal diffusivity of paper.

Solution: $L \sim (\alpha t)^{1/2}$, which makes $\alpha \sim 4 \cdot 10^{-7}\,\mathrm{mm}^2/\mathrm{s}$. Data from literature: $\alpha \sim 2 \cdot 10^{-7} - 4 \cdot 10^{-7}\,\mathrm{mm}^2/\mathrm{s}$

1.2 Incandescent light bulbs

Until recently, incandescent light bulbs (the famous invention from Thomas Edison) were commonly used to provide light before being replaced by other light sources such as LEDs. The principle is very simple: an electrical current heats up a tungsten filament that behaves as a black body and thus emits radiations. To prevent a quick degradation, the filament is enclosed inside a glass bulb filled with inert gaz (Fig. 1).
We search to estimate the temperature of the filament and of the surface of the glass bulb for a lamp consuming an electrical power $P = 60W$.

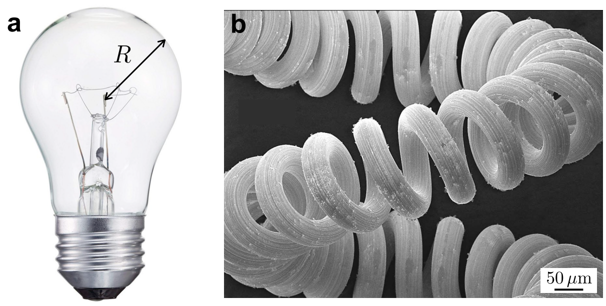

Figure 1: (a) Traditional incandescent lamp. (b) SEM imaging of a tungsten filament (source: Macisaac *et al.*, Basic physics of the incandescent lamp, *The Physics Teacher*, 37, 520, 1999).

As a quick estimate, we forget about the glass bulb and heat transfer from the environment. What would be the temperature of a filament of radius $r_w = 25\,\mu\mathrm{m}$ and length $L_w = 20\,\mathrm{cm}$ (curly filament) behaving as a perfect black body?
Which wavelength corresponds to the maximal emission? If the objective is to produce visible light, are such lamps optimal (Fig. 2)?

Solution: Quick estimate. Hypothesis: all electrical power radiated and wire perfect black body ($\epsilon = 0$). $P = 2\pi r_w L_w \sigma T_w^4$

$T_w = \left(\frac{60}{2\pi\times0.2\times25.10^{-6}\times5.7.10^{-8}}\right)^{1/4} = 2400°\text{K} = 2130°\text{C}$

$\lambda_{max} = \frac{2898}{2400} = 1.2\,\mu\text{m IR}$

Incadescent lamps, more energy in heat than in light. Advantage LED: more power into actual light.

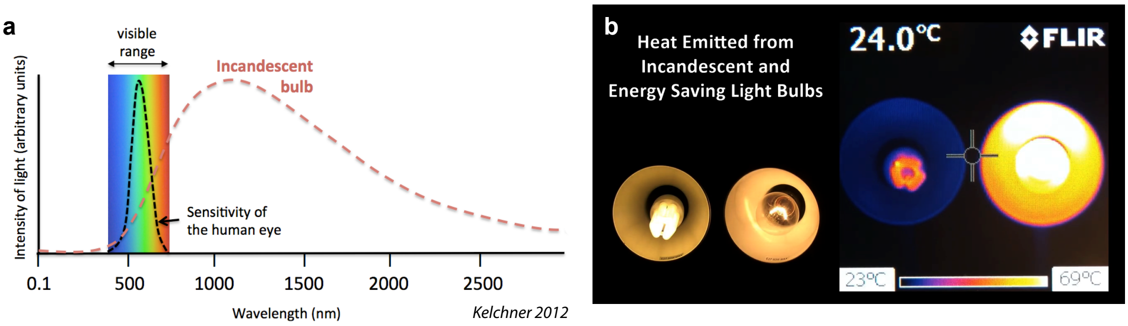

Figure 2: (a) Emission spectrum of an incandescent lamp. (b) Comparison of the heat emitted by a traditional incandescent lamp with an energy saving analog (source: https://youtu.be/CKo7gVjp_8s).

We now account for the glass bulb. As a first approximation glass absorbs infrared and is transparent to light. If the bulb sits in a room of temperature $T_{ext} = 25°\text{C}$, what is the balance of radiations fluxes? What is the additional self-induced convective flux? Estimate the temperature of the glass bulb (assuming that most of the light emitted from the filament is in the IR range). What are the relative contributions of radiation and conduction in the heat transfer?

Solution: Hyp: glass absorbs IR. Most power absorbed.

$P = 4\pi R^2\left(\sigma(T_b^4 - T_{ext}^4) + J_c\right)$ (radiated + convected)

$J_c = \frac{\kappa}{R}(T_b - Text)\text{Nu}$

Here free convection $\text{Nu} \sim \text{Ra}^{1/4} = \left(\frac{\beta g(T_b - T_{ext})R^3}{\alpha\nu}\right)^{1/4}$

$J_c \sim \kappa\left(\frac{\beta g}{\nu\alpha R}\right)^{1/4}(T_b - T_{ext})^{5/4}$

$\frac{P}{4\pi R^2} = \sigma(T_b^4 - T_{ext}^4) + \kappa\left(\frac{\beta g}{\nu\alpha R}\right)^{1/4}(T_b - T_{ext})^{5/4}$

$\frac{60}{4\pi 0.03^2} = 5.7.10^{-8}(T_b^4 - 302^4) + 0.026\left(\frac{3.4.10^{-3}\times9.8}{1.5.10^{-5}\times2.10^{-5}\times0.03}\right)^{1/4}(T_b - 302)^{5/4}$

$5300 = 5.7.10^{-8}(T_b^4 - 302^4) + 6.4(T_b - 302^4)^{5/4}$

$T_b = 452°\text{K} = 180°\text{C}$ Maybe overestimated, some power is transmitted through the bulb!

radiation: 36% free convection: 64%

Better estimate of T_w accounting for radiations from the bulb: $P = 2\pi r_w L_w \sigma(T_w^4 - T_b^4)$

$T_w = (2400^4 + 452^4)^{1/4} = 2400°K$. The correction does not change much T_w (effect of T^4).

1.3 Making fire

Thermal imaging is an excellent tool to assess friction effects. As a good demonstration, try to visualize the effect of rubbing a shoe on the floor.

An old way to make fire was to rub to pieces of dry wood together. One of the most efficient variant consists in rotating alternatively a wooden stick using some kind of bow while pushing it towards the opposive piece of wood with the other hand (Fig.3b).
The radius of the stick is typically 0.5 cm and the rotation velocity of 5 rotations per second. We may take 10 for the normal load and a friction coefficient of 0.4.
What is the heat flux produced at the contact between the stick and the substrate?
The temperature for spontaneous ignition of wood is about 500°C. How long would it take to make a fire?
Wood material properties: $\kappa = 0.15\,\mathrm{Wm^{-1}K^{-1}}$, $\rho = 600\,\mathrm{kg/m^3}$, $C_p = 1700\,\mathrm{J\,kg^{-1}K^{-1}}$.

Solution: The friction force is $\mu N \sim 4\,\mathrm{N}$.
The typical velocity is $R\omega \sim 2\pi \times 5 \times 0.005 \sim 0.15\,\mathrm{m/s}$.
The heat power produced by friction is therefore on the order of $4 \times 0.15 \sim 0.6\,\mathrm{W}$ an the heat flux $J \sim 1.2/\pi\, 0.01^2 \sim 8000\,\mathrm{W/m^2}$.
The region heated by the flux increases with $L \sim (\alpha t)^{1/2}$. At the contact, the temperature reaches T_c while around L, the temperature did not change much. The heat flux thus balances heat conduction $J \sim \kappa \frac{\Delta T}{L}$.
We thus expect $\Delta T \sim \frac{J\,L}{\kappa} \sim \frac{J\,(\alpha t)^{1/2}}{\kappa} \sim \frac{J}{(\kappa\rho C_p)^{1/2}} t^{1/2} \sim 20\, t^{1/2}$ in °C. We need to get $\Delta T \sim 470$°C, which requires 530s (8-10 min). The order of magnitude seems reasonable.

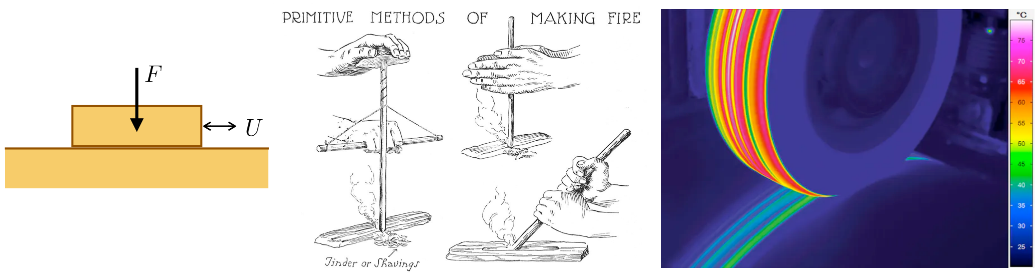

Figure 3: (a) Rubbing a piece of wood on another one. (b) Traditional ways of making fire (source: C.W. Jefferys, *The Picture Gallery of Canadian History*, 1942). (c) Using thermal imaging to monitor the friction on a rubber wheel (from Infratec.eu).

2 Solidification of a lead shot

Lead shots were traditionally manufactured in "shot towers". Molten lead were poured through a copper sieve at the top of the tower. Millimeter size droplets then formed under the effect of surface tension and would solidify as they fall down. A nice exemple of an old 70 m high shot tower can be visited in Couëron, Loire-Atlantique (Fig. 4). We propose to estimate the typical solidification time of a molten sphericule of lead under different heat exchange processes.

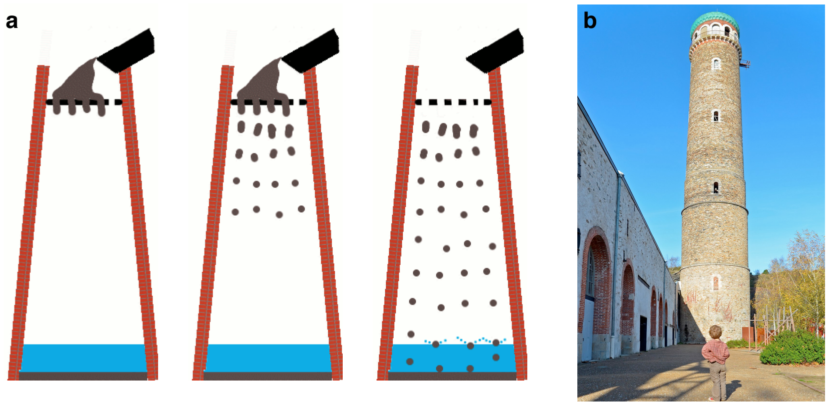

Figure 4: (a) Manufacturing process of lead shots in a "shot tower". (b) *Tour à plomb* in Couëron, Loire-Atlantique (source: Wikipedia).

We consider a sphere of molten lead of radius $R = 1\,\mathrm{mm}$, at the melting temperature $T_f = 327°\mathrm{C}$ and we will assume that the temperature remains uniform through the whole sphere during the solidification process. If the sphere cools down in a time scale τ_{cool}, what is the condition to validate this assumption?

Solution: Temperature will have the time to get uniform through the sphere if: $(\alpha\tau_{cool})^{1/2} \gg R$, ie $\tau_{cool} \gg \frac{R^2}{\alpha} \sim \frac{10^{-6}}{10^{-5}} = 0.1\,\mathrm{s}$.

Some material properties

- Lead:
emissivity $\varepsilon = 0.1$, density $\rho_{Pb} = 10.5 \cdot 10^3\,\mathrm{kg/m^3}$, latent heat $L_{Pb} = 23\,\mathrm{kJ/kg}$, thermal conductivity $\kappa_{Pb} = 16\,\mathrm{Wm^{-1}K^{-1}}$, thermal diffusivity $\alpha_{Pb} = 10^{-5}\,\mathrm{m^2/s}$,

- Air:
density $\rho_{air} = 1.2\,\mathrm{kg/m^3}$, kinematic viscosity $\nu_{air} = 1.5 \cdot 10^{-5}\,\mathrm{m^2/s}$, thermal conductivity $\kappa_{air} = 0.025\,\mathrm{Wm^{-1}K^{-1}}$, thermal diffusivity $\alpha_{air} = 2 \cdot 10^{-5}\,\mathrm{m^2/s}$, thermal expansion coefficient $\beta_{air} = 3.4 \cdot 10^{-3}\,\mathrm{K^{-1}}$

- Fundamental constants:
Absolute 0: $-273.15°\mathrm{C}$, Stefan-Bolztman constant $\sigma = 5.7 \cdot 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$

2.1 Monitoring the temperature of the droplet

If you watch the droplet with an infrared thermal camera, would the software indicate anything close to the correct surface temperature (argue your answer)?

Solution: No, because the estimation of the surface temperature is based on the emission of a black body. Since the emissivity is low, we are far from this approximation. Presumably, the measured temperature will be far below the actual one.

2.2 Droplet in sidereal space

Imagine Thomas Pesquet fixes something outside the space station and a droplet of molten lead solder gets ejected.
What is the heat flux through its surface?
How long would it take for the droplet to eventually solidify?

Solution: The only source of heat flux is in this case radiation: $j = \varepsilon\sigma T^4$. Since the temperature is uniform, the surface temperature is equal to $T_f = 600°$K. The heat to remove from the droplet is the latent heat $\frac{4}{3}\pi R^3 \rho_{Pb} L_{Pb}$, which leads to:

$$\tau_{rad} = \frac{\frac{4}{3}\pi R^3 \rho_{Pb} L_{Pb}}{4\pi R^2 \varepsilon\sigma T^4} = \frac{R\,\rho_{Pb}\,L_{Pb}}{3\varepsilon\sigma T^4} = 110\,\mathrm{s} \sim 2\,\mathrm{min}$$

2.3 Droplet in levitating in the absence of gravity

Now imagine that the repair was inside the space station maintained at 20°C. The molten droplet remains standing in the air due to the absence of gravity.
In terms of scaling laws, what is the order of magnitude for the solidification time if we do not account for radiation?

Solution: No gravity, no free convection.
In the absence of radiation, the remaining heat transfer process is diffusion through the air. The diffusion length scales as $\delta \sim (\alpha_{air} t)^{1/2}$ and the heat flux $j \sim \frac{\kappa_{air}}{\delta}(T_f - T_{amb})$. As a consequence, the solidification time is expected to follow:

$$\tau_{diff} \sim \frac{R\,\rho_{Pb}\,L_{Pb}\,(\alpha_{air}\tau_{diff})^{1/2}}{3\kappa_{air}(T_f - T_{amb})}$$

$$\tau_{diff} \sim \frac{\alpha_{air}}{9}\left(\frac{R\,\rho_{Pb}\,L_{Pb}}{\kappa_{air}(T_f - T_{amb})}\right)^2 = 2200\,\mathrm{s} \sim 40\,\mathrm{min}$$

As a consequence, radiation is significantly more efficient than pure diffusion through air of low conductivity.

2.4 Sessile droplet on Earth

We now come back to Earth gravity and assume that the droplet as been deposited on a thermally insulated surface such as a glass plate. We again neglect radiative processes.
What is now the order of magnitude of the solidification time?

Solution: Under the action of gravity, air heated up by the sphere become less dense and moves up resulting into a natural convection process. In that case, the heat flux is expected to follow:

$$j \sim \kappa_{air} \frac{Nu}{R}(T_f - T_{amb}), \quad \text{with} \quad Nu \sim Ra^{1/4} \sim \left(\frac{\beta_{air} g R^3 (T_f - T_{amb})}{\nu_{air} \alpha_{air}} \right)^{1/4}$$

which leads to

$$\tau_{natural} \sim \frac{R^2 \rho_{Pb} L_{Pb}}{3\kappa_{air} (T_f - T_{amb})} \frac{1}{Ra^{1/4}}$$

$$Ra \sim 35 \quad \tau_{natural} \sim 4\,\mathrm{s}$$

Radiative effects are negligible in this case.

2.5 Falling droplet

Now consider the case of a droplet falling from the tower. What is its terminal velocity U?
Just a quick recap from last year: drag force on a sphere: $6\pi\eta RU$, for $Re \ll 1$ and $\frac{1}{2}\rho U^2 \pi S C_d$, for $Re \gg 1$ with a drag coefficient $C_d \sim 0.5$ for a sphere.
What is now the order of magnitude of the solidification time?
Does the hight of the Couëron tower seem appropriate?

Solution: We are presumably at high Re, take for instance $U = 1\,\mathrm{m/s}$, we obtain $Re \sim UR/\nu_{air} \sim 67$. If we balance the drag force with the weight of the droplet, we get:

$$\frac{4}{3}\pi R^3 \rho_{Pb}\, g = \frac{1}{2}\rho_{air} U^2 \pi R^2 C_d$$

$$U = \left(\frac{8}{3} \frac{\rho_{Pb}}{\rho_{air}} \frac{R\,g}{C_d} \right)^{1/2} = 23\,\mathrm{m/s}$$

The high Re hypothesis is definitely valid.
Heat exchange now corresponds to forced convection. We thus expect:

$$\tau_{forced} \sim \frac{R^2 \rho_{Pb} L_{Pb}}{3\kappa_{air} (T_f - T_{amb})} \frac{1}{Nu} \quad \text{with} \quad Nu \sim Re^{1/2} Pr^{1/3} \sim \left(\frac{UR}{\nu_{air}} \right)^{1/2} \left(\frac{\nu_{air}}{\alpha_{air}} \right)^{1/3}$$

We finally obtain $\tau_{forced} \sim 0.3\,\mathrm{s}$. Much more efficient than the other processes. The travelled distance is on the order of $7\,\mathrm{m}$. The hight of the tower is of the good order of magnitude (we did not account for the acceleration phase of the droplet nor for the fact that the initial temperature of lead may be higher than T_f, which will increase the solidification time).