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Landau's weakly non-linear approach

Lev Landau (1908-1968), Nobel Prize 1962.

In linear stability analysis, each eigenmode can be written :

$$u(\mathbf{x}, t) = A(t) \mathbf{f}(\mathbf{x}) + A^*(t) \mathbf{f}^*(\mathbf{x}),$$

where A describes the temporal evolution and $\mathbf{f}(\mathbf{x})$ the spatial structure (Fourier mode).

The linear stability analysis is limited to small perturbations, for which non-linear terms can be neglected. A general non-linear stability theory does not exist. Nevertheless, a class of solutions, called weakly non-linear, is based on perturbation techniques.

In a linear stability analysis, each eigenmode's amplitude is solution of the equation :

$$\frac{dA}{dt} = \sigma A$$

If $\text{Re}(\sigma)$ is positive, A grows exponentially, otherwise it decays exponentially.

For $A \ll 1$, this equation can be seen as the first-order Taylor expansion of dA/dt as a power series according to the amplitude A . We can therefore complete this expansion with higher-order terms A^2, AA^*, A^{*2} , etc...

If the system is invariant in the symmetry $A \rightarrow -A$, then the lowest-order term that shares this property is $|A|^2 A$ and the equation, at this order of truncation, reads :

$$\frac{dA}{dt} = \sigma A - \kappa |A|^2 A,$$

which is called Landau's amplitude equation, where the complex number κ is called Landau coefficient.

Neither the value, nor the sign of κ are given by symmetry arguments.

Highly damped oscillator

Let's start with a damped oscillator for which the inertial term d^2A/dt^2 can be neglected :

$$\frac{dA}{dt} = -V'(A), \text{ where } V'(A) = -\mu A + \kappa A^3 + \lambda A^5$$

It is a model for phase transitions.

Suppose that $\kappa = 1$.

$$V(A) = -\mu \frac{A^2}{2} + \frac{A^4}{4} + \lambda \frac{A^6}{6}$$

- $\mu < 0$: $V'(A) = 0 \Leftrightarrow A(-\mu + A^2 + \lambda A^4) = 0 \Leftrightarrow A = 0$
- $\mu > 0$: $V'(A) = 0 \Leftrightarrow A(-\mu + A^2 + \lambda A^4) = 0 \Leftrightarrow A = 0$ or $A \simeq \pm \sqrt{\mu}$ for $\mu \ll 1$
- $A^2 = \frac{-1 \pm \sqrt{1 + 4\lambda\mu}}{2\lambda} \simeq \frac{2\lambda\mu}{2\lambda} = \mu$
- Stability of $A_0 = 0$: $A = A_0 + A' \Rightarrow \frac{dA'}{dt} \simeq \mu A' \Rightarrow$ stable for $\mu < 0$, otherwise unstable.

Highly damped oscillator

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```
gnuplot 5.4 patchlevel 4
```

```
gnuplot] set multiplot; do for [j=0:2] {mu = 1./4.*(j-1); plot [-1:1] [-0.02:0.02] -  
mu*x**2/2+x**4/4 lt 1 lw 4 lc 'blue' t ''}
```

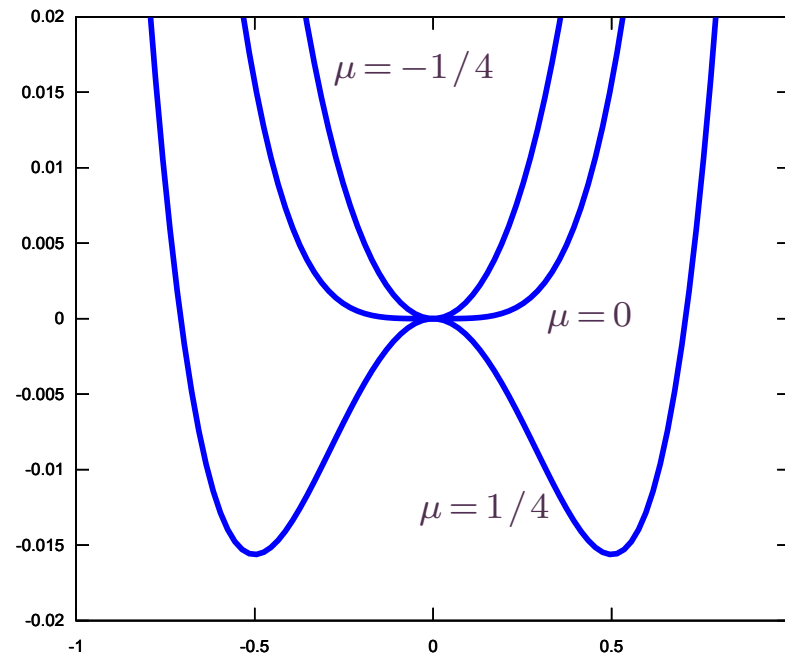
```
gnuplot]
```

Highly damped oscillator

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gnuplot 5.4 patchlevel 4

```
gnuplot] set multiplot; do for [j=0:2] {mu = 1./4.*(j-1); plot [-1:1][-0.02:0.02] -  
mu*x**2/2+x**4/4 lt 1 lw 4 lc 'blue' t ''}
```



gnuplot]

Supercritical pitchfork bifurcation

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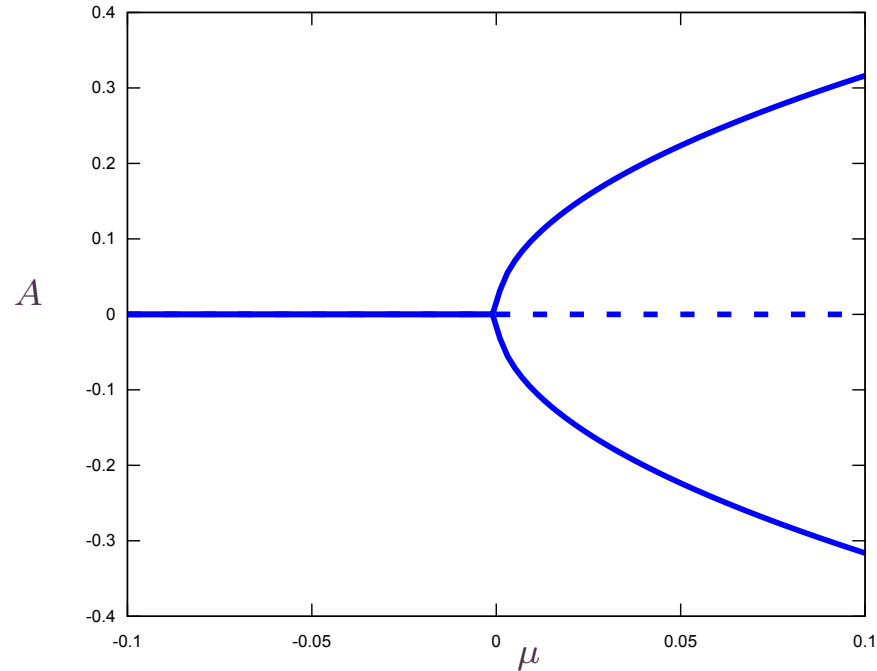
```
gnuplot] f(x) = x<0 ? 0 : sqrt(x); plot [-0.1:0.1][-0.4:0.4] 0 lt 1 lw 4 dt 2 lc 'blue' t  
'', f(x) lt 1 lw 4 lc 'blue' t '', -f(x) lt 1 lw 4 lc 'blue' t ''
```

```
gnuplot]
```

Supercritical pitchfork bifurcation

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```
gnuplot] f(x) = x<0 ? 0 : sqrt(x); plot [-0.1:0.1][-0.4:0.4] 0 lt 1 lw 4 dt 2 lc 'blue' t  
'', f(x) lt 1 lw 4 lc 'blue' t '', -f(x) lt 1 lw 4 lc 'blue' t ''
```



```
gnuplot]
```


$$\frac{dA}{dt} = -V'(A), \text{ where } V'(A) = -\mu A + \kappa A^3 + \lambda A^5$$

Suppose now that $\kappa = -1$ and $\lambda = 1$.

$$V(A) = -\mu \frac{A^2}{2} - \frac{A^4}{4} + \frac{A^6}{6}$$

for $\mu > 0$, the second term is not stabilising. The third term will saturate the instability.

- $V'(A) = 0 \Leftrightarrow A(-\mu - A^2 + A^4) = 0 \Leftrightarrow A = 0$ or $A^2 = \frac{1 \pm \sqrt{1 + 4\mu}}{2} \simeq \mu$
- If $\mu \leq -1/4$: $A = 0$
- If $\mu > 0$: $A_{\min} = \pm \sqrt{\mu}$
- If $-1/4 < \mu \leq 0$: $A_{\min} = 0, \pm \sqrt{\frac{1 + \sqrt{1 + 4\mu}}{2}}$
- $V(A_{\min}) = 0 \Leftrightarrow -6\mu - 3A^2 + 2A^4 = 0 \Leftrightarrow -6\mu - 3\frac{1 + \sqrt{1 + 4\mu}}{2} + 2\left(\frac{1 + \sqrt{1 + 4\mu}}{2}\right)^2 = 0$

Highly damped oscillator

$$-6\mu - 3\frac{1 + \sqrt{1 + 4\mu}}{2} + 2\left(\frac{1 + \sqrt{1 + 4\mu}}{2}\right)^2 = 0$$

$$-12\mu - 3\sqrt{1 + 4\mu} - 1 + 4\mu + 2\sqrt{1 + 4\mu} = 0$$

$$\sqrt{1 + 4\mu} = -8\mu - 1$$

$$1 + 4\mu = 1 + 64\mu^2 + 16\mu$$

$$64\mu^2 + 12\mu = 0$$

$$\mu = -\frac{3}{16}$$

Highly damped oscillator

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gnuplot 5.4 patchlevel 4

```
gnuplot] set multiplot; do for [j=0:6] {mu = (j-5)/16.; plot [-1.5:1.5][-0.12:0.08] -  
mu*x**2/2-x**4/4+x**6/6 lt 1 lw 4 lc 'blue' t '',0 dt 2 lc 'red' t ''}
```

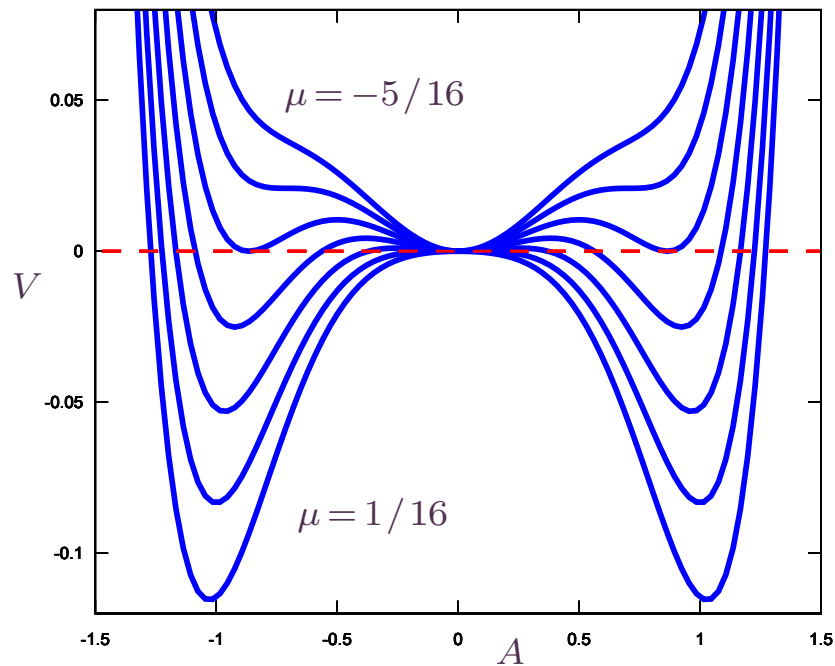
```
gnuplot]
```

Highly damped oscillator

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gnuplot 5.4 patchlevel 4

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mu*x**2/2-x**4/4+x**6/6 lt 1 lw 4 lc 'blue' t '',0 dt 2 lc 'red' t ''}
```



gnuplot]

Hysteresis

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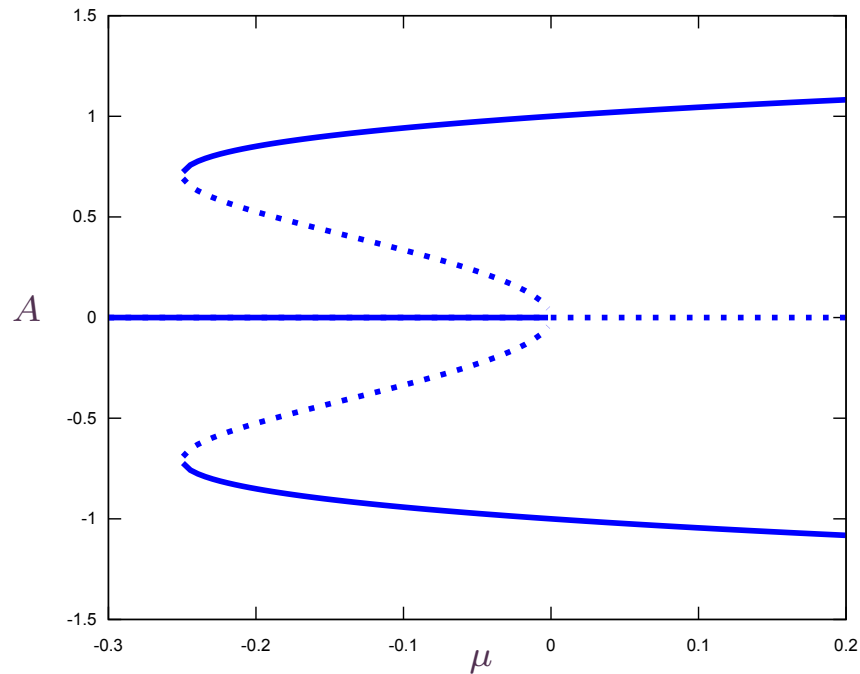
```
gnuplot] f(x) = x>-1./4. ? sqrt((1+sqrt(1+4*x))/2.) : 1/0; g(x) = x>-1./4. ? sqrt((1-  
sqrt(1+4*x))/2.) : 1/0; plot [-0.3:0.2][-1.5:1.5] 0 lt 1 lw 4 dt 3 lc 'blue' t  
'', f(x) lt 1 lw 4 lc 'blue' t '', -f(x) lt 1 lw 4 lc 'blue' t '', x<0 ? 0 : 1/0  
w l lt 1 lw 4 lc 'blue' t '', g(x) lt 1 lw 4 dt 3 lc 'blue' t '', -g(x) lt 1 lw  
4 dt 3 lc 'blue' t ''
```

```
gnuplot]
```

Hysteresis

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```
gnuplot] f(x) = x>-1./4. ? sqrt((1+sqrt(1+4*x))/2.) : 1/0; g(x) = x>-1./4. ? sqrt((1-  
sqrt(1+4*x))/2.) : 1/0; plot [-0.3:0.2][-1.5:1.5] 0 lt 1 lw 4 dt 3 lc 'blue' t  
'', f(x) lt 1 lw 4 lc 'blue' t '', -f(x) lt 1 lw 4 lc 'blue' t '', x<0 ? 0 : 1/0  
w l lt 1 lw 4 lc 'blue' t '', g(x) lt 1 lw 4 dt 3 lc 'blue' t '', -g(x) lt 1 lw  
4 dt 3 lc 'blue' t ''
```



```
gnuplot]
```

Non-linear pendulum :

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0, \text{ where } \omega_0^2 = \frac{g}{L}$$

Exact solution :

$$\dot{\theta}\ddot{\theta} + \omega_0^2 \dot{\theta}\sin\theta = 0 \Rightarrow \frac{\dot{\theta}^2}{2} - \omega_0^2 \cos\theta = -\omega_0^2 \cos\theta_{\max}$$

$$\frac{d\theta}{dt} = \omega_0 \sqrt{2(\cos\theta - \cos\theta_{\max})} \Rightarrow \omega_0 T = \int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_{\max})}}$$

Linear period : $T_0 = 2\pi\sqrt{L/g} = 2\pi/\omega_0$

$$T = T_0 \frac{\sqrt{2}}{\pi} \int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_{\max}}} \Rightarrow T \text{ is a function of } \theta_{\max}$$

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0, \text{ where } \omega_0^2 = \frac{g}{L}$$

Let's define a dimensionless time : $\tilde{t} = \omega_0 t$

We get (removing $\tilde{\cdot}$) :

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

A naive expansion :

$$\frac{d^2\theta}{dt^2} + \theta - \frac{\theta^3}{6} + \dots = 0$$

Let's call ε a typical value of the amplitude and look for an expansion according to ε :

$$\theta(t) = \varepsilon\theta_1(t) + \varepsilon^2\theta_2(t) + \varepsilon^3\theta_3(t) + \dots$$

$$\frac{d^2\theta}{dt^2} + \theta - \frac{\theta^3}{6} + \dots = 0$$

$$\theta(t) = \varepsilon\theta_1(t) + \varepsilon^2\theta_2(t) + \varepsilon^3\theta_3(t) + \dots$$

Plugging the expansion in the equation gives :

$$\varepsilon \frac{d^2\theta_1}{dt^2} + \varepsilon^2 \frac{d^2\theta_2}{dt^2} + \varepsilon^3 \frac{d^2\theta_3}{dt^2} + \dots + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3\theta_3 + \dots - \frac{1}{6}(\varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3\theta_3 + \dots)^3 + \dots = 0$$

At order ε and ε^2 , we have :

$$\frac{d\theta_1}{dt} + \theta_1 = 0 \quad (\theta_1 = A \cos t) \quad \text{and} \quad \frac{d\theta_2}{dt} + \theta_2 = 0 \quad (\theta_2 = B \cos t)$$

$$\frac{d^2\theta}{dt^2} + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots = 0$$

Let's write $\theta = \sqrt{\varepsilon}A(t)$:

$$\frac{d^2A}{dt^2} + \frac{1}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon}A) = 0 \Rightarrow \frac{d^2A}{dt^2} + A - \varepsilon \frac{A^3}{3!} + \varepsilon^2 \frac{A^5}{5!} \dots = 0$$

and expand $A(t)$ as a power series in ε :

$$A(t) = A_0(t) + \varepsilon A_1(t) + \varepsilon^2 A_2(t) + \dots$$

Plugging the expansion in the equation gives :

$$\begin{aligned} & \frac{d^2A_0}{dt^2} + \varepsilon \frac{d^2A_1}{dt^2} + \varepsilon^2 \frac{d^2A_2}{dt^2} + \varepsilon^3 \frac{d^2A_3}{dt^2} + \dots \\ & \quad + A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \dots \\ & - \frac{\varepsilon}{6} (A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \dots)^3 + \dots = 0 \end{aligned}$$

$$\begin{aligned} & \frac{d^2 A_0}{dt^2} + \varepsilon \frac{d^2 A_1}{dt^2} + \varepsilon^2 \frac{d^2 A_2}{dt^2} + \varepsilon^3 \frac{d^2 A_3}{dt^2} + \dots \\ & \quad + A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \dots \\ & - \frac{\varepsilon}{6} (A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \dots)^3 + \dots = 0 \end{aligned}$$

At first order, we have :

$$\frac{d^2 A_0}{dt^2} + A_0 = 0 \Rightarrow A_0 = a_0 \cos(t + \phi)$$

At order ε , we have :

$$\frac{d^2 A_1}{dt^2} + A_1 = \frac{A_0^3}{6} = \frac{a_0^3}{6} \cos^3(t + \phi)$$

$$\cos 3x = \cos x \cos 2x - \sin x \sin 2x = \cos x (2\cos^2 x - 1) - 2 \cos x (1 - \cos^2 x) = 4\cos^3 x - 3\cos x$$

$$\cos^3 x = \frac{1}{4} (\cos 3x + 3 \cos x) \Rightarrow \frac{d^2 A_1}{dt^2} + A_1 = \frac{a_0^3}{24} (3\cos(t + \phi) + \cos 3(t + \phi))$$

$$\frac{d^2 A_1}{dt^2} + A_1 = \frac{a_0^3}{24}(3\cos(t + \phi) + \cos 3(t + \phi))$$

The second term's contribution to the solution gives $-\frac{a_0^3}{8 \times 24}\cos 3(t + \phi)$, which is bounded.

But the first one gives $\alpha(t + \phi) \sin(t + \phi)$:

$$\frac{d^2 A_1}{dt^2} + A_1 = \alpha[2\cos(t + \phi) - (t + \phi) \sin(t + \phi)] + \alpha(t + \phi) \sin(t + \phi) \Rightarrow \alpha = \frac{a_0^3}{16}$$

which grows in time. Then, for time of order $1/\varepsilon$, A_1 becomes comparable to A_0 and the expansion loses its coherence.

This is because the first term of the RHS is resonant with the free oscillations described by the equation.

There are two timescales in the problem : the natural frequency 1 and this long-time $1/\varepsilon$.

The idea of the multiscale method is to postulate that there are two independent time variables, a rapid one t and a slow one $T = \varepsilon t$.

Then

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}, \quad \text{and} \quad \frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} \right) \left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} \right) = \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial T} + \varepsilon^2 \frac{\partial^2}{\partial T^2}$$

$$\frac{d^2 \theta}{dt^2} + \sin \theta = 0$$

Let's write $\theta = \sqrt{\varepsilon} A(t)$:

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial T} + \varepsilon^2 \frac{\partial^2}{\partial T^2} \right) A + \frac{1}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon} A) = 0$$

and

$$A = A_0(t, T) + \varepsilon A_1(t, T) + \varepsilon^2 A_2(t, T) + \dots$$

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial T} + \varepsilon^2 \frac{\partial^2}{\partial T^2} \right) A + \frac{1}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon} A) = 0$$

$$A = A_0(t, T) + \varepsilon A_1(t, T) + \varepsilon^2 A_2(t, T) + \dots$$

At first order :

$$\frac{d^2 A_0}{dt^2} + A_0 = 0 \Rightarrow A_0 = a_0(T) \cos(t + \phi(T))$$

At next order :

$$\frac{\partial^2 A_1}{\partial t^2} + A_1 = \frac{A_0^3}{6} - 2 \frac{\partial^2 A_0}{\partial t \partial T}$$

$$\frac{\partial^2 A_1}{\partial t^2} + A_1 = \frac{a_0^3}{24} (3 \cos(t + \phi) + \cos 3(t + \phi)) + 2 \frac{da_0}{dT} \sin(t + \phi) + 2a_0 \frac{d\phi}{dT} \cos(t + \phi)$$

$$\frac{\partial^2 A_1}{\partial t^2} + A_1 = \frac{a_0^3}{24}(3\cos(t + \phi) + \cos 3(t + \phi)) + 2\frac{da_0}{dT}\sin(t + \phi) + 2a_0\frac{d\phi}{dT}\cos(t + \phi)$$

We can now require that the secular terms do not appear in the solution of the equation, by setting :

$$\begin{aligned}\frac{da_0}{dT} &= 0 \\ \frac{d\phi}{dT} &= -\frac{a_0^2}{16} \Rightarrow \phi = \phi_0 - \frac{a_0^2}{16}T\end{aligned}$$

Then the solution for A_0 reads :

$$\theta = \sqrt{\varepsilon}a_0 \cos\left(t + \phi_0 - \frac{a_0^2}{16}\varepsilon t + \dots\right) \text{ and } \theta_0 = a_0\sqrt{\varepsilon} \Rightarrow \omega = 1 - \frac{\theta_0^2}{16}$$

$$T = \frac{2\pi}{\omega} \simeq 2\pi\left(1 + \frac{\theta_0^2}{16}\right) = T_{\text{lin}}\left(1 + \frac{\theta_0^2}{16} + \dots\right)$$