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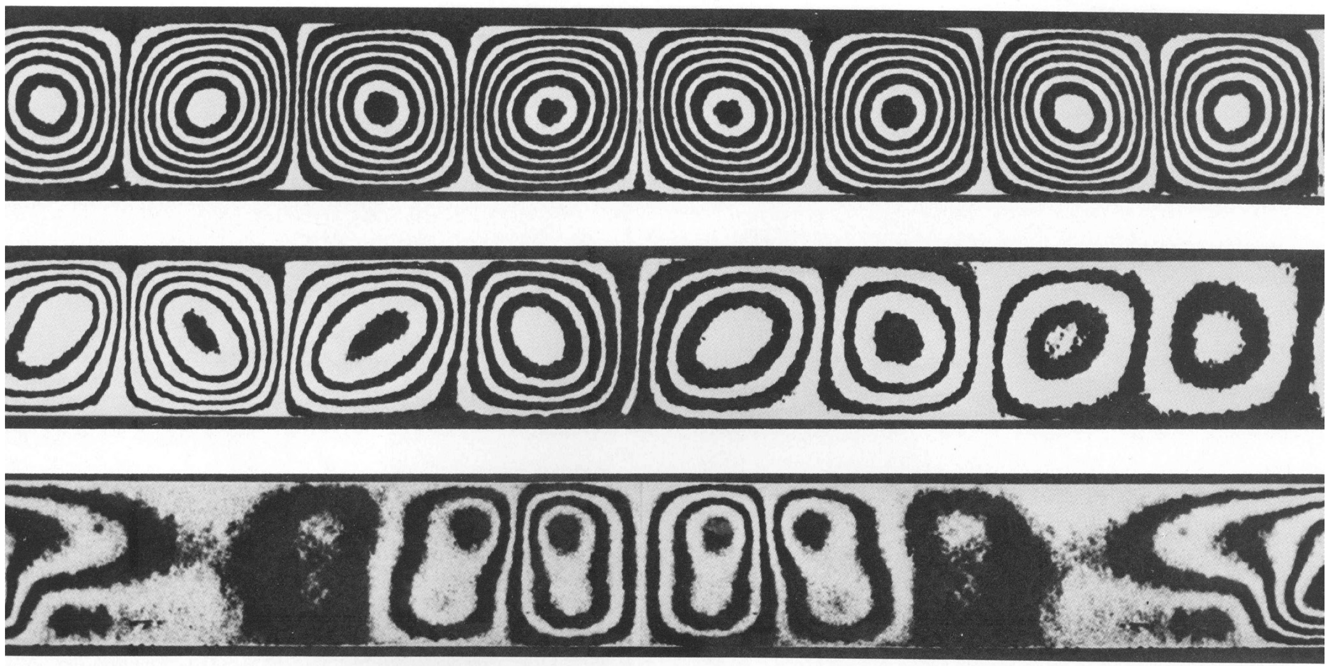
Rayleigh–Bénard convection



139. Buoyancy-driven convection rolls. Differential interferograms show side views of convective instability of silicone oil in a rectangular box of relative dimensions 10:4:1 heated from below. At the top is the classical Rayleigh-Bénard situation: uniform heating produces rolls parallel to the shorter side. In the middle photograph the temperature difference and hence the amplitude of motion increase from right to left. At the bottom, the box is rotating about a vertical axis. *Oertel & Kirchartz 1979, Oertel 1982*a

Figure 1. Van Dyke, M., An Album of Fluid Motion, Stanford, CA, The Parabolic Press, 1982, p. 82.

Rayleigh–Bénard convection

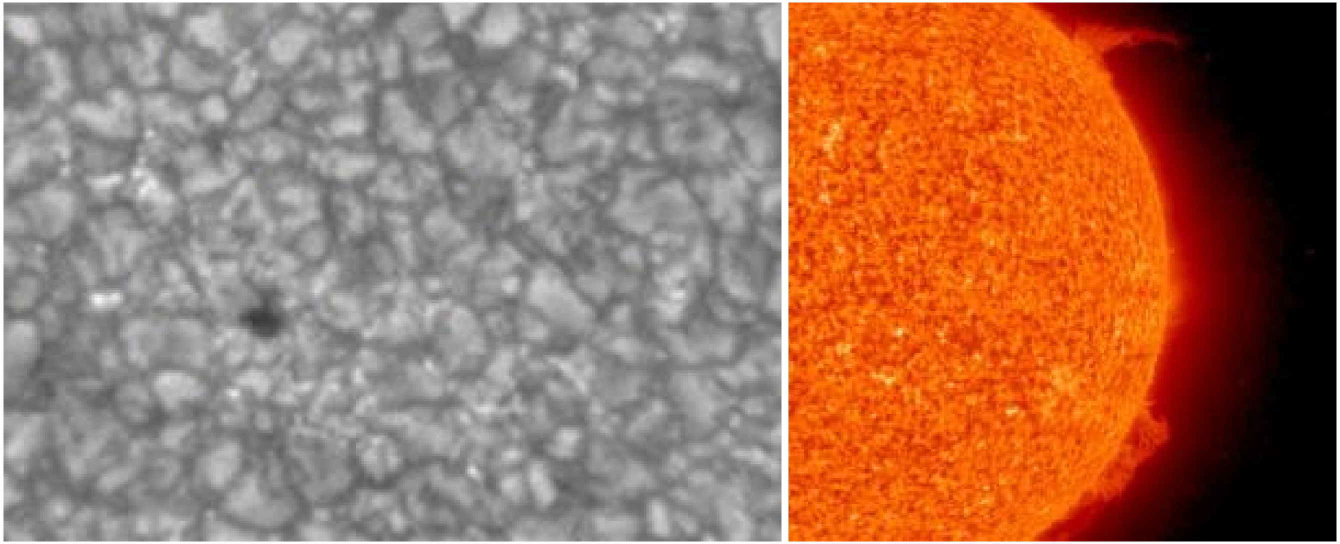
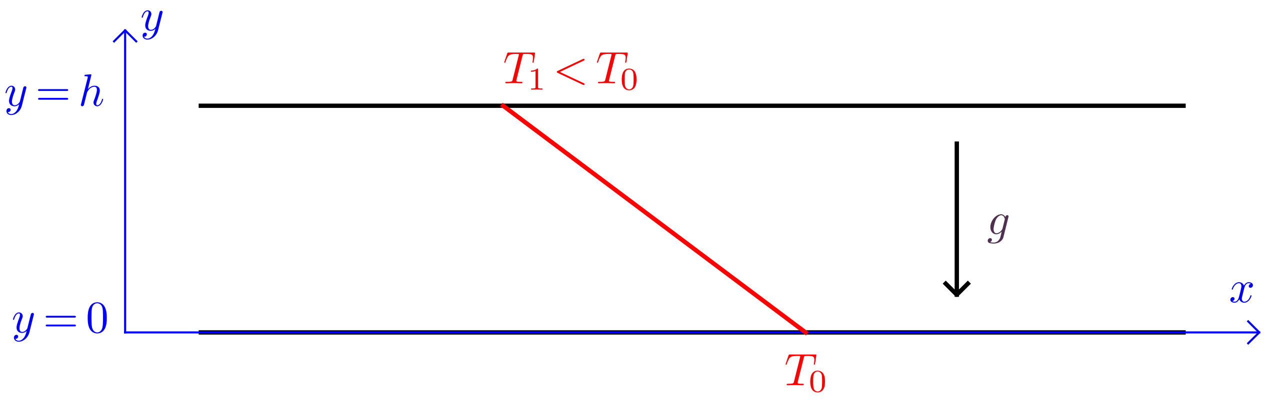


Figure 2. Convection cells can be seen on the surface of the Sun.

Rayleigh–Bénard convection

Figure 3. Rayleigh–Bénard convection in a heated fluid below its isothermal solid phase.

Rayleigh–Bénard convection



Parameters

- T_1 is the temperature of the top plate
- $T_0 > T_1$ is the temperature of the bottom plate
- h is the height of the container
- g is the acceleration due to gravity
- ρ is the density
- ν is the kinematic viscosity ($m^2 s^{-1}$)
- κ is the Thermal diffusivity ($m^2 s^{-1}$)
- $\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$ is the Thermal expansion coefficient ($\simeq 2 \times 10^{-4}\, K^{-1}$ for water at $20°C$).

Boussinesq approximation

Let's call ρ_0 the density at temperature T_0.

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

$$M = C^{\text{ste}} = \rho V \Rightarrow \beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{-1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \simeq -\frac{\rho - \rho_0}{\rho_0(T - T_0)}$$

$$\rho \simeq \rho_0(1 - \beta(T - T_0)) \;\text{(simplified equation of state)}$$

Boussinesq approximation :

- Density variations are retained only in the gravity term in Navier–Stokes equations and depend only on temperature fluctuations (not the pressure ones)
- Viscous dissipation and pressure effect are neglected in enthalpy equation
- Thermodynamic coefficients are assumed constant, compressibility is neglected

Boussinesq approximation

$$\begin{aligned}
\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} &= -\frac{1}{\rho_0}\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{v} - \frac{\rho - \rho_0}{\rho_0} g\, \boldsymbol{e_y} \\
\boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 \\
\frac{\partial T}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) T &= \kappa \Delta T
\end{aligned}$$

$$H = c_p T\,,\ \kappa = \frac{\lambda}{\rho_0 c_p}\ (H \sim J\,/\,\mathrm{kg} \Rightarrow c_p \sim J\,/\,(\mathrm{kg}\,K))$$

$$\rho_0 \frac{dH}{dt} = \rho_0 c_p \frac{dT}{dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{Q} = -\boldsymbol{\nabla} \cdot (-\lambda \boldsymbol{\nabla} T)$$

where λ is the thermal conductivity :

$$\text{Heat flux: } Q = -\lambda \boldsymbol{\nabla} T \sim J\,/\,(m^2\, s)$$

$$\lambda \sim J\,/\,(m\,K\,s) \Rightarrow \kappa \sim \frac{\cancel{J}\,/\,(m\,\cancel{K}\,s)}{\cancel{\mathrm{kg}}\,m^{-3}\cancel{J}\,/\,(\cancel{\mathrm{kg}}\,\cancel{K})} \sim m^2\, s^{-1}$$

Basic state

$$\boldsymbol{v} = \boldsymbol{0}, T(x,y,z,t) = \bar{T}(x,y,z), p(x,y,z,t) = \bar{p}(x,y,z), \bar{\rho}, \text{etc...}$$

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} &= -\frac{1}{\rho_0}\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{v} - \frac{\rho-\rho_0}{\rho_0} g\, \boldsymbol{e_y} \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 \\ \frac{\partial T}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})T &= \kappa \Delta T \end{aligned}$$

$$\Delta \bar{T} = 0 \Rightarrow \bar{T}(y) = T_0 + \frac{y}{h}(T_1 - T_0)$$

$$\frac{\bar{\rho}-\rho_0}{\rho_0} \simeq -\beta(T - T_0) \Rightarrow \bar{\rho}(y) = \rho_0\left(1 - \beta\frac{y}{h}(T_1 - T_0)\right)$$

$$\boldsymbol{0} = -\frac{1}{\rho_0}\boldsymbol{\nabla} p - \frac{\rho-\rho_0}{\rho_0} g\, \boldsymbol{e_y} \Rightarrow \bar{p} = p_0 + \rho_0\, g \beta \frac{y^2}{2h}(T_1 - T_0)$$

Perturbation

$$T = \bar{T} + T', \rho = \bar{\rho} + \rho', \boldsymbol{v} = \mathbf{0} + \boldsymbol{v}, p = \bar{p} + p'$$

$$\rho \simeq \rho_0(1 - \beta(T - T_0))$$

$$\bar{\rho} \simeq \rho_0(1 - \beta(\bar{T} - T_0))$$

$$\rho - \bar{\rho} \simeq -\rho_0\beta(T - \bar{T}) \Rightarrow \rho' = -\rho_0\beta T'$$

$$\begin{aligned}
\frac{\partial \boldsymbol{v}}{\partial t} &= -\frac{1}{\rho_0}\boldsymbol{\nabla} p' + \nu\Delta\boldsymbol{v} + \beta g T'\,\boldsymbol{e_y} \\
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 \\
\frac{\partial T'}{\partial t} + v\frac{T_1 - T_0}{h} &= \kappa\Delta T'
\end{aligned}$$

Non-dimensional equations

$$
\begin{aligned}
T^\star &= \frac{T'}{T_0 - T_1} \\
(x^\star, y^\star, z^\star) &= \frac{(x, y, z)}{h} \\
\boldsymbol{v}^\star &= \frac{\boldsymbol{v}}{\kappa / h} \\
t^\star &= \frac{t}{h^2 / \kappa} \\
p^\star &= \frac{p'}{p_0}
\end{aligned}
$$

Removing *, we get :

$$
\frac{\kappa^2}{h^3}\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{p_0}{\rho_0 h}\boldsymbol{\nabla} p + \frac{\nu\kappa}{h^3}\Delta \boldsymbol{v} + \beta g (T_0 - T_1)\, T \boldsymbol{e_y}
$$

$$
\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{p_0 h^2}{\rho_0 \kappa^2}\boldsymbol{\nabla} p + \frac{\nu}{\kappa}\Delta \boldsymbol{v} + \frac{\beta g (T_0 - T_1) h^3}{\kappa^2} T \boldsymbol{e_y}
$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{p_0 h^2}{\rho_0 \kappa^2} \boldsymbol{\nabla} p + \frac{\nu}{\kappa} \Delta \boldsymbol{v} + \frac{\beta g (T_0 - T_1) h^3}{\kappa^2} T \boldsymbol{e_y}$$

We choose : $p_0 = \frac{\rho_0 \kappa^2}{h^2}$, and define :

$$\mathrm{Pr} = \frac{\nu}{\kappa} \text{ and } \mathrm{Ra} = \frac{\beta g (T_0 - T_1) h^3}{\nu \kappa}$$

Then :

$$\boxed{\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla} p + \mathrm{Pr}\, \Delta \boldsymbol{v} + \mathrm{Pr}\, \mathrm{Ra}\, T \boldsymbol{e_y}}$$

$$\frac{\partial T'}{\partial t} + v \frac{T_1 - T_0}{h} = \kappa \Delta T' \Rightarrow \frac{\kappa}{h^2} \frac{\partial T}{\partial t} - \frac{\kappa}{h} \frac{v}{h} = \frac{\kappa}{h^2} \Delta T$$

$$\boxed{\frac{\partial T}{\partial t} - v = \Delta T}$$

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0}$$

Stability analysis

$$\boldsymbol{\nabla} \wedge \left(\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla} p + \mathrm{Pr}\, \Delta \boldsymbol{v} + \mathrm{Pr}\, \mathrm{Ra}\, T \boldsymbol{e_y} \right)$$

$$\boldsymbol{\nabla} \wedge \left[\frac{\partial \boldsymbol{\nabla} \wedge \boldsymbol{v}}{\partial t} = \mathrm{Pr}\, \Delta (\boldsymbol{\nabla} \wedge \boldsymbol{v}) + \mathrm{Pr}\, \mathrm{Ra} \begin{pmatrix} -\frac{\partial T}{\partial z} \\ 0 \\ \frac{\partial T}{\partial x} \end{pmatrix} \right]$$

$$\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{v}) = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{v}) - \Delta \boldsymbol{v}$$

$$-\frac{\partial \Delta \boldsymbol{v}}{\partial t} = -\mathrm{Pr}\, \Delta^2 \boldsymbol{v} + \mathrm{Pr}\, \mathrm{Ra} \begin{pmatrix} \frac{\partial^2 T}{\partial x \partial y} \\ -\frac{\partial^2 T}{\partial z^2} - \frac{\partial^2 T}{\partial x^2} \\ \frac{\partial^2 T}{\partial y \partial z} \end{pmatrix}$$

Stability analysis

$$\frac{\partial T}{\partial t} - v \;=\; \Delta T$$

Heat equation is only coupled with $v \Rightarrow y$ component of Navier–Stokes equations :

$$\frac{\partial \Delta v}{\partial t} = \mathrm{Pr}\,\Delta^2 v + \mathrm{Pr}\,\mathrm{Ra}\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2}\right)$$

Normal modes :

$$v = \hat{v}(y) e^{i(k_x x + k_z z - \omega t)}, T = \hat{T}(y) e^{i(k_x x + k_z z - \omega t)}$$

Let's call $k^2 = k_x^2 + k_z^2$.

$$-i\omega\left(\frac{d^2}{dy^2} - k^2\right)\hat{v} = \mathrm{Pr}\left(\frac{d^2}{dy^2} - k^2\right)^2 \hat{v} - \mathrm{Pr}\,\mathrm{Ra}\,k^2 \hat{T}$$

$$-i\omega\hat{T} - \hat{v} = \left(\frac{d^2}{dy^2} - k^2\right)\hat{T}$$

Boundary conditions

Temperature imposed in $y=0,1$. Therefore $\hat{T}(0)=\hat{T}(1)=0$.

No-slip boundary conditions complicated...

Free-slip boundary conditions in $y=0,1$:

$$\frac{\partial u}{\partial y}=\frac{\partial w}{\partial y}=0 \;\text{ and } v=0 \text{ in } y=0,1$$

Mass conservation :

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\right)$$

$$\frac{\partial^2 v}{\partial y^2}=0$$

$$\hat{v}=0 \;\text{ and }\; \frac{d^2\hat{v}}{dy^2}=0 \text{ in } y=0,1$$

Boundary conditions

$$\hat{v} = 0, \frac{d^2\hat{v}}{dy^2} = 0 \text{ and } \hat{T} = 0 \text{ in } y = 0, 1$$

$$-i\omega\left(\frac{d^2}{dy^2} - k^2\right)\hat{v} = \mathrm{Pr}\left(\frac{d^2}{dy^2} - k^2\right)^2\hat{v} - \mathrm{Pr}\,\mathrm{Ra}\,k^2\hat{T}$$

$$y = 0, 1: \qquad \mathrm{Pr}\frac{d^4\hat{v}}{dy^4} = 0$$

$$-i\omega\left(\frac{d^2}{dy^2}-k^2\right)\hat{v}=\mathrm{Pr}\left(\frac{d^2}{dy^2}-k^2\right)^2\hat{v}-\mathrm{Pr}\,\mathrm{Ra}\,k^2\hat{T}$$

$$-\hat{v}=\left(\frac{d^2}{dy^2}-k^2+i\omega\right)\hat{T}$$

Let's call $\Delta_k=\frac{d^2}{dy^2}-k^2$. Then

$$\left(\frac{d^2}{dy^2}-k^2+i\omega\right)\left(\frac{d^2}{dy^2}-k^2\right)\left(-i\omega\,\hat{v}-\mathrm{Pr}\left(\frac{d^2}{dy^2}-k^2\right)\hat{v}\right)=\mathrm{Pr}\,\mathrm{Ra}\,k^2\hat{v}$$

$$(\Delta_k+i\omega)\Delta_k(-i\omega\,\hat{v}-\mathrm{Pr}\,\Delta_k\,\hat{v})=\mathrm{Pr}\,\mathrm{Ra}\,k^2\hat{v}$$

$$[\omega^2\Delta_k+\omega(-i\Delta_k^2-i\,\mathrm{Pr}\,\Delta_k^2)-(\mathrm{Pr}\,\Delta_k^3+\mathrm{Pr}\,\mathrm{Ra}\,k^2)]\hat{v}=0$$

$$s\equiv -i\omega\Rightarrow[-s^2\Delta_k+s(1+\mathrm{Pr})\Delta_k^2-\mathrm{Pr}\,(\Delta_k^3+\mathrm{Ra}\,k^2)]\hat{v}=0$$

$$\hat{v}=\sin(\pi j y),\, j=1,2\ldots\,(\text{such that }\hat{v}=\hat{v}''=\hat{v}''''=0\text{ in }y=0,1)$$

$$\Delta_k\sin(\pi j y)=(-\pi^2 j^2-k^2)\sin(\pi j y)$$

$$[-s^2\Delta_k + s(1+\mathrm{Pr})\Delta_k^2 - \mathrm{Pr}\,(\Delta_k^3 + \mathrm{Ra}\,k^2)]\hat{v} = 0$$

$$\Delta_k \sin(\pi j y) = -(\pi^2 j^2 + k^2)\sin(\pi j y) \equiv -A\sin(\pi j y)$$

Dispersion relation :

$$s^2 A + s(1+\mathrm{Pr})A^2 + \mathrm{Pr}\,(A^3 - \mathrm{Ra}\,k^2) = 0$$

$$s = \frac{-(1+\mathrm{Pr})A^2 \pm \sqrt{(1+\mathrm{Pr})^2 A^4 - 4A\mathrm{Pr}\,(A^3 - \mathrm{Ra}\,k^2)}}{2\,A}$$

$$s = \frac{-1}{2}(1+\mathrm{Pr})A \pm \sqrt{\frac{(1+\mathrm{Pr})^2 A^2}{4} - \frac{\mathrm{Pr}\,(A^3 - \mathrm{Ra}\,k^2)}{A}}$$

$$s = 0 \Leftrightarrow \frac{(1+\mathrm{Pr})^2 A^2}{4} = \frac{(1+\mathrm{Pr})^2 A^2}{4} - \frac{\mathrm{Pr}\,(A^3 - \mathrm{Ra}\,k^2)}{A}$$

$$\boxed{s = 0 \Leftrightarrow \mathrm{Ra} = \frac{A^3}{k^2} = \frac{(\pi^2 j^2 + k^2)^3}{k^2}}$$

$$s = 0 \Leftrightarrow \mathrm{Ra} = \frac{A^3}{k^2} = \frac{(\pi^2 j^2 + k^2)^3}{k^2}$$

$$s \geqslant 0 \Leftrightarrow \mathrm{Ra} \geqslant \frac{(\pi^2 j^2 + k^2)^3}{k^2}$$

For each k, the minimum value of $\frac{(\pi^2 j^2 + k^2)^3}{k^2}$ is obtained for $j = 1 \Rightarrow \hat{v} = \sin(\pi y)$

Critical curves or Marginal stability curves : $\mathrm{Ra}_{c_j}(k) = \frac{(\pi^2 j^2 + k^2)^3}{k^2}$

Then, for a given k, the critical Rayleigh number is :

$$\mathrm{Ra}_c = \frac{(\pi^2 + k^2)^3}{k^2}$$

$$\frac{d\mathrm{Ra}_c}{dk} = 0 \Leftrightarrow \frac{6(\pi^2 + k^2)^2 k^3 - 2(\pi^2 + k^2)^3 k}{k^4} = 0 \Leftrightarrow 3k^2 = (\pi^2 + k^2)$$

$$\boxed{\frac{d\mathrm{Ra}_c}{dk} = 0 \Leftrightarrow k = \frac{\pi}{\sqrt{2}}}$$

gnuplot 4.6 patchlevel 7

```
gnuplot] set multiplot; set log; do for [j=1:4] {plot [0.1:100][1e2:1e6]
         (pi**2*j**2+x**2)**3/x**2 lt 3 lw 4 lc 'red' t ''}
```

```
gnuplot]
```

gnuplot 4.6 patchlevel 7

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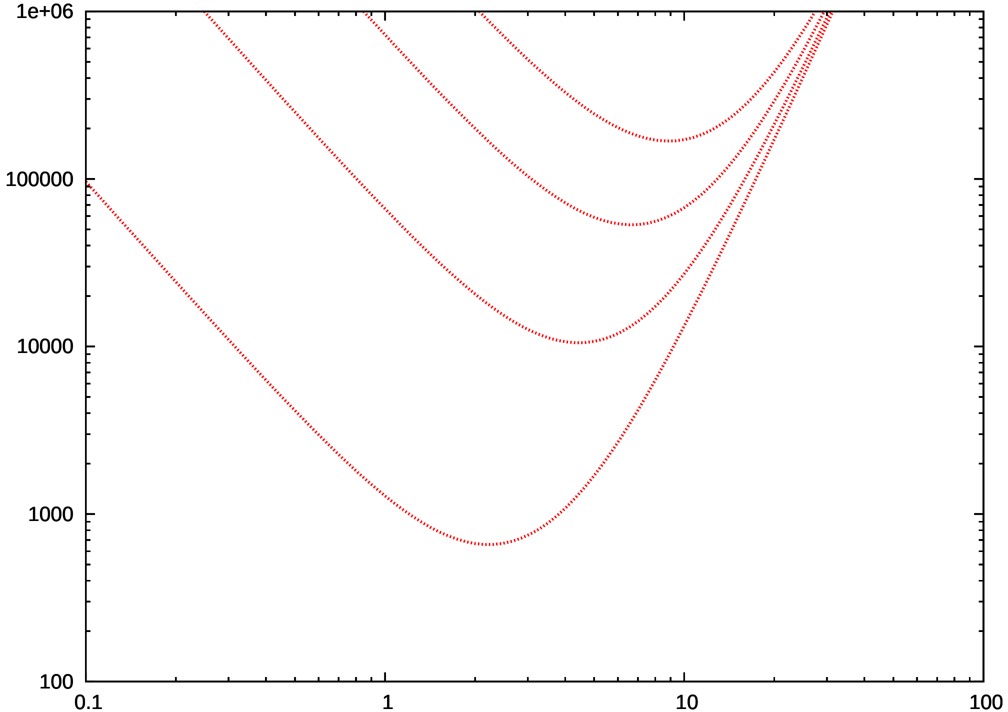
```




```
gnuplot]
```

$$\boxed{\frac{d\mathrm{Ra}_c}{dk} = 0 \Leftrightarrow k = \frac{\pi}{\sqrt{2}}}$$

Example :

$$k_z = 0 \Rightarrow k_c = k_x = \frac{2\pi}{\lambda^\star} = \frac{2\pi h}{\lambda_c} = \frac{\pi}{\sqrt{2}}$$

$$\lambda_c = 2\sqrt{2}\,h$$

$$\mathrm{Ra}_c = \frac{\left(\pi^2 + \frac{\pi^2}{2}\right)^3}{\frac{\pi^2}{2}} = \frac{27}{4}\pi^4 \simeq 657.5$$

With 2 no-slip boundary conditions :

$$\frac{2\pi h}{\lambda_c} \simeq 3.117 \Rightarrow \lambda_c \simeq 2h$$

$$\mathrm{Ra}_c \simeq 1707$$

$$\mathrm{Ra} = \frac{\beta g (T_0 - T_1) h^3}{\nu \kappa}$$

For air :

$$\beta \simeq 4 \times 10^{-3}\, K^{-1}, \nu \simeq 1.5 \times 10^{-5}\, m^2 s^{-1}, \kappa \simeq 2 \times 10^{-5}\, m^2 s^{-1}$$

Then :

$$\mathrm{Ra} \simeq 10^8 (T_0 - T_1) h^3$$

With free-slip boundary conditions :

$$\mathrm{Ra} \geqslant 657.5 \Leftrightarrow (T_0 - T_1) h^3 \geqslant 6 \times 10^{-6}$$

With no-slip boundary conditions :

$$\mathrm{Ra} \geqslant 1707 \Leftrightarrow (T_0 - T_1) h^3 \geqslant 1.7 \times 10^{-5}$$

$$-\hat{v} = -\sin(\pi j y) = \left(\frac{d^2}{dy^2} - k^2 + i\omega\right)\hat{T} = (\Delta_k - s)\hat{T}$$

$$(\Delta_k - s)\sin(\pi j y) = (-\pi^2 j^2 - k^2 - s)\sin(\pi j y)$$

$$(\Delta_k - s)^{-1} = \frac{1}{-\pi^2 j^2 - k^2 - s}$$

$$\hat{T} = \frac{\sin(\pi j y)}{\pi^2 j^2 + k^2 + s}$$

The critical Rayleigh number does not depend on the orientation of the wavevector $\vec{k}$, but only on its norm $|\vec{k}|$.

$$T = \hat{T}(y) e^{i(k_x x + k_z z - \omega t)}$$

$$T = A(\cos k_c x + \cos k_c z)\sin(\pi y) \Rightarrow \text{Squares}$$

$$\vec{k} = k_c \begin{pmatrix} 1 \\ 0 \end{pmatrix}, k_c \begin{pmatrix} \cos 2\pi/3 \\ \sin 2\pi/3 \end{pmatrix}, k_c \begin{pmatrix} \cos 4\pi/3 \\ \sin 4\pi/3 \end{pmatrix} \Rightarrow \text{Hexagones}$$