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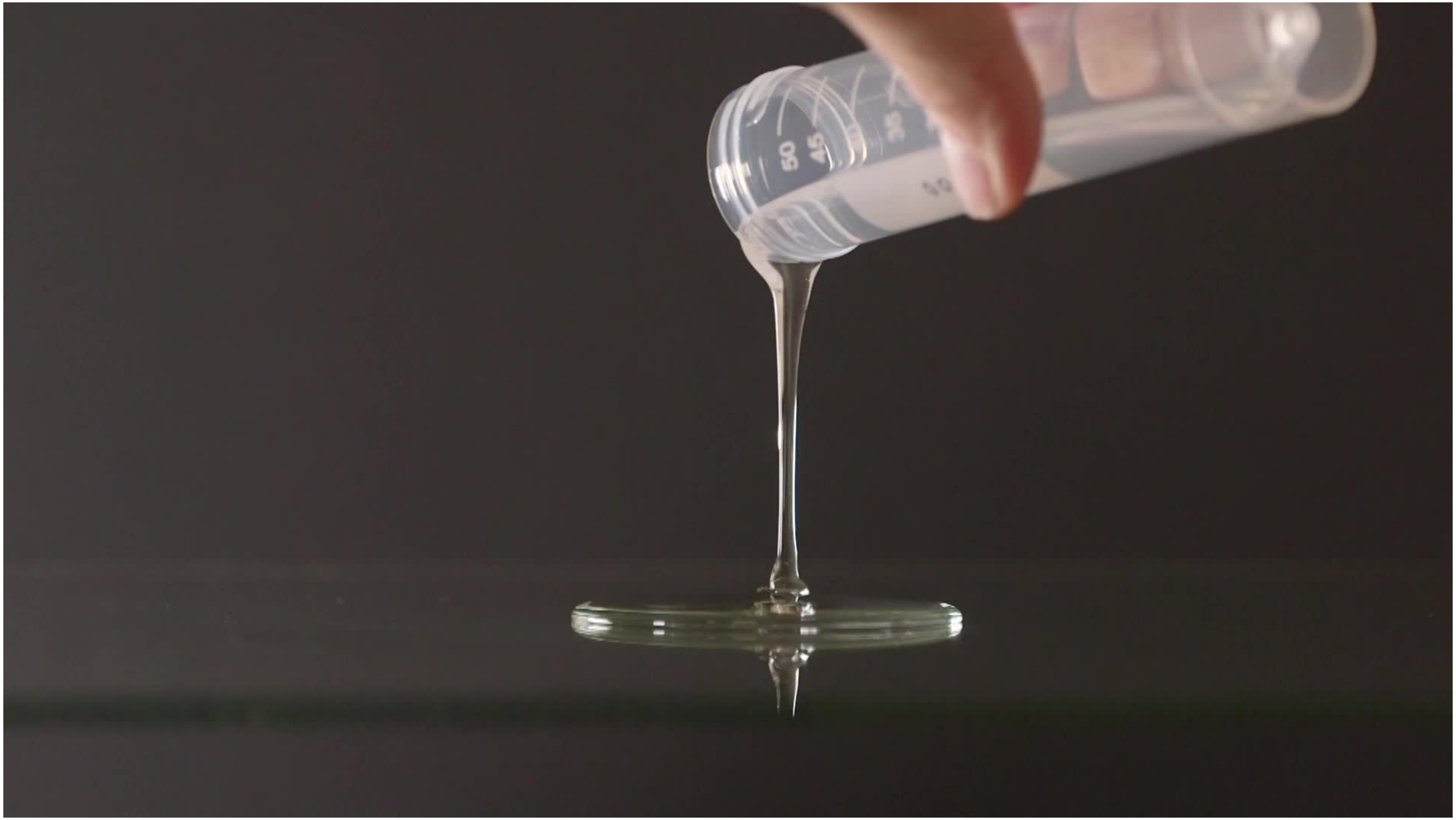

Figure 1. Rayleigh–Taylor Instability in a thin viscous film (Experiment : Christophe Clanet)

So far, we have not taken into account confinement effects and viscous effects.

In order to simplify, let's choose $\rho_1 = 0 \quad \Rightarrow \quad p_1 = C^{\text{ste}}$ and $p_1' = 0$ (free surface).

Dispersion relation :

$$\omega^2 = \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) g k + \frac{\gamma}{\rho_1 + \rho_2} k^3 \quad \Rightarrow \quad \omega^2 = -gk + \frac{\gamma k^3}{\rho_2}$$

Confinement : thickness H

$$\varphi_2' \sim e^{-kz} e^{i(kx - \omega t)}$$

We expect the confinement to become important when $H = \mathcal{O}(1/k)$

$$\text{New boundary condition: } \left.\frac{\partial \varphi_2'}{\partial z}\right|_{z=H} = 0$$

Equations and boundary conditions:

$$
\begin{aligned}
z \geqslant 0 &: \quad \Delta\varphi_2' = 0 \\
z \geqslant 0 &: \quad \rho_2 \frac{\partial \varphi_2'}{\partial t} + p_2' = 0 \\
z = 0 &: \quad -\rho_2 g h + p_2' = \gamma \frac{\partial^2 h}{\partial x^2} \\
z = 0 &: \quad \frac{\partial h}{\partial t} = \frac{\partial \varphi_2'}{\partial z} \\
z = H &: \quad \frac{\partial \varphi_2'}{\partial z} = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Normal modes: } \varphi_2' &= \delta\varphi_2(z) e^{i(kx-\omega t)} \\
p_2' &= \delta p_2(z) e^{i(kx-\omega t)} \\
h &= \delta h \, e^{i(kx-\omega t)}
\end{aligned}
$$

Laplace's equations give : $-k^2\delta\varphi_2(z)+\delta\varphi_2''(z)=0$

$$\delta\varphi_2(z) \;=\; A_2\,e^{kz}+B_2\,e^{-kz}$$

Boundary condition :

$$\left.\frac{\partial\delta\varphi_2}{\partial z}\right|_{z=H}=0 \quad\Rightarrow\quad A_2\,e^{kH}=B_2\,e^{-kH}$$

$$\delta\varphi_2(z)=A_2\,(e^{kz}+e^{2kH-kz})=A_2e^{kH}(e^{k(z-H)}+e^{-k(z-H)})=C_2\cosh k(z-H)$$

Kinematic boundary conditions :

$$-i\omega\delta h=-C_2\,k\sinh kH$$

$$\delta\varphi_2(0)=C_2\cosh kH=\frac{i\omega}{k}\delta h\frac{\cosh kH}{\sinh kH}$$

Dynamic boundary conditions :

$$\begin{aligned}-i\omega\rho_2\delta\varphi_2(0)+\delta p_2(0) &= 0\\ -\rho_2 g\delta h+\delta p_2(0) &= -\gamma k^2\delta h\end{aligned}$$

Dynamic boundary conditions :

$$
\begin{aligned}
-i\omega\rho_2\left(\frac{i\omega}{k}\delta h\frac{\cosh kH}{\sinh kH}\right)+\delta p_2(0) &= 0 \\
\delta p_2(0) &= \rho_2 g\delta h-\gamma k^2\delta h
\end{aligned}
$$

$$
i\omega\rho_2\left(\frac{i\omega}{k}\delta h\frac{\cosh kH}{\sinh kH}\right)=\rho_2 g\delta h-\gamma k^2\delta h
$$

$$
-\frac{\omega^2}{k}\rho_2\frac{\cosh kH}{\sinh kH}=\rho_2 g-\gamma k^2
$$

Dispersion relation :

$$
\omega^2=\left(-gk+\frac{\gamma}{\rho_2}k^3\right)\tanh kH
$$

to be compared with :

$$
\omega^2=-gk+\frac{\gamma}{\rho_2}k^3
$$

$$h = \mathrm{Re}(\delta h\, e^{i(kx-\omega t)}) = e^{\omega_I t}\,\mathrm{Re}(|\delta h|\, e^{i(kx-\omega_R t+\phi)}) = |\delta h|\, e^{\omega_I t}\cos(kx-\omega_R t+\phi)$$

$$\text{Unstable} \Leftrightarrow -gk+\frac{\gamma}{\rho_2}k^3<0 \Leftrightarrow \frac{2\pi}{\lambda}<\sqrt{\frac{\rho_2 g}{\gamma}}=\ell_c^{-1} \Leftrightarrow \lambda>2\pi\ell_c \;\text{ or }\; k=\ell_c^{-1}$$

The stability domain does not depend on H!

$$\omega=\omega_R+i\omega_I \Rightarrow \omega_I=\pm\sqrt{\left(gk-\frac{\gamma}{\rho_2}k^3\right)\tanh kH}$$

$$k^\star=k\ell_c=k\sqrt{\gamma/\rho_2 g} \;\text{ and }\; H^\star=H/\ell_c$$

$$\omega_I=\sqrt{\left(gk^\star\sqrt{\frac{\rho_2 g}{\gamma}}-\frac{\gamma}{\rho_2}{k^\star}^3\left(\frac{\rho_2 g}{\gamma}\right)^{3/2}\right)\tanh k^\star H^\star}=\left(\frac{g}{\ell_c}\right)^{1/2}\sqrt{(k^\star-{k^\star}^3)\tanh k^\star H^\star}$$

$$\omega^\star=\sqrt{(k^\star-{k^\star}^3)\tanh k^\star H^\star}$$

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         t '', sqrt((x-x**3)*tanh(3*x)) t '', sqrt((x-x**3)*tanh(4*x)) t '', sqrt((x-
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         t '', sqrt((x-x**3)*tanh(8*x)) t '', sqrt((x-x**3)*tanh(9*x)) t '', sqrt((x-
         x**3)*tanh(10*x)) t '', sqrt(x-x**3) lc "red" t ''
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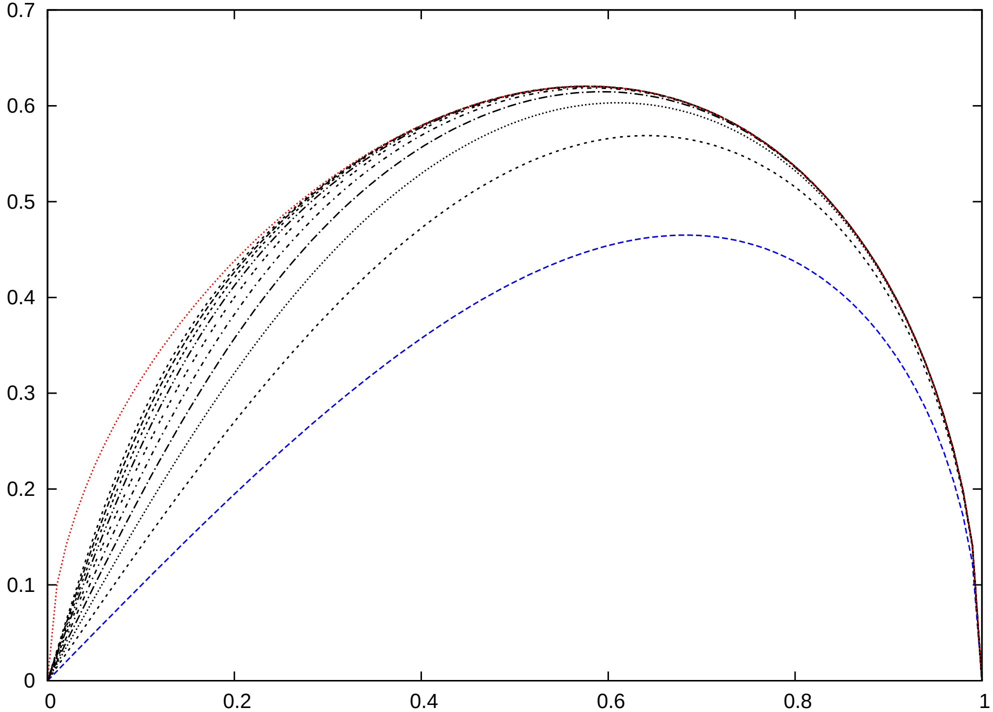


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Maximum such that $d\omega^\star / dk^\star = 0$:

$$\omega^{\star 2} = (k^\star - {k^\star}^3)\tanh k^\star H^\star$$

$$2\omega^\star \frac{d\omega^\star}{dk^\star} = (1 - 3k^{\star 2})\tanh k^\star H^\star + \frac{(k^\star - {k^\star}^3)}{\cosh^2 k^\star H^\star}$$

$$H^\star \to \infty \Rightarrow k^\star = \frac{1}{\sqrt{3}}$$

Close to $k^\star \to 0$, we have (dropping $\star$) :

$$\omega \sim \sqrt{k \tanh kH}$$

Then

$$\begin{aligned} H \to \infty &: \ \omega \sim \sqrt{k} \\ H \text{ finite} &: \ \omega \sim k\sqrt{H} \end{aligned}$$

Waves : $g \rightarrow -g$

$$\omega^2 = \left(gk + \frac{\gamma}{\rho_2} k^3 \right) \tanh kH > 0$$

$$\omega = \omega_R = \pm \sqrt{\left(gk + \frac{\gamma}{\rho_2} k^3 \right) \tanh kH}$$

Travelling wave

$$h \;=\; \mathrm{Re}(\delta h\, e^{i(kx-\omega t)}) = \mathrm{Re}(|\delta h|\, e^{i(kx-\omega_R t+\phi)}) = |\delta h| \cos(kx - \omega_R t + \phi)$$

$$\text{Phase velocity: } c = \frac{\omega}{k} = \sqrt{\left(\frac{g}{k} + \frac{\gamma}{\rho_2} k \right) \tanh kH}$$

$$\text{Infinite depth} \;\; : \;\; c = \sqrt{\frac{g}{k} + \frac{\gamma}{\rho_2} k}$$

$$\text{Shallow water: } kH \ll 1 \Rightarrow \tanh kH \simeq kH \;\; : \;\; c = \sqrt{\left(g + \frac{\gamma}{\rho_2} k^2 \right) H}$$

Thin viscous film of thickness H (lubrication)

Hypothesis : $\varepsilon' = kH \ll 1$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \Rightarrow Uk \sim \frac{W}{H} \Rightarrow W \sim \varepsilon' U$$

Slow + Quasi parallel flow $\Rightarrow$ lubrication equations :

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial z^2} \\ 0 &= -\frac{\partial p}{\partial z} - \rho_2 g \\ 0 &= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \end{aligned}$$

Basic state : $u = w = 0 \qquad p = p_0 - \rho_2 g z$

Perturbation :

$$\begin{aligned} u &= u' \\ w &= w' \\ p &= p_0 - \rho_2 g z + p' \Rightarrow \frac{\partial p'}{\partial z} = 0 \\ h &= h' \end{aligned}$$

We keep $'$ only for p'.

Stress tensor for a Newtonian fluid

$$\sigma_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Tangential and normal stresses at the interface :

$$\sigma_{ij} n_j = \gamma\kappa n_i$$

$$\begin{pmatrix} -p + 2\eta\frac{\partial u}{\partial x} & \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & -p + 2\eta\frac{\partial w}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} -\frac{\partial h}{\partial x} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\partial h}{\partial x}\left(-p + 2\eta\frac{\partial u}{\partial x}\right) + \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ -\frac{\partial h}{\partial x}\eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \left(-p + 2\eta\frac{\partial w}{\partial z}\right) \end{pmatrix}$$

$$\left|\frac{\partial h}{\partial x}\right| \sim \varepsilon \ll 1 \qquad \& \qquad W = \varepsilon' U$$

$$\begin{aligned} \frac{\partial u}{\partial z} &\simeq 0 \\ p' - \rho_2 g h &\simeq \gamma\frac{\partial^2 h}{\partial x^2} \end{aligned}$$

$$\frac{\partial p'}{\partial z} = 0 \quad \Rightarrow \quad p'(x, \not{z}) \Rightarrow \eta u = \frac{1}{2}\frac{\partial p'}{\partial x} z^2 + \alpha z + \beta$$

$$\alpha = 0 \text{ because } \left.\frac{\partial u}{\partial z}\right|_{z=0} = 0$$

$$\beta = -\frac{1}{2}\frac{\partial p'}{\partial x} H^2 \text{ because } u(H) = 0$$

$$u = \frac{1}{2\eta}\frac{\partial p'}{\partial x}(z^2 - H^2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \Rightarrow w = -\frac{1}{2\eta}\frac{\partial}{\partial x}\left[\left(\frac{z^3}{3} - zH^2 + \delta\right)\frac{\partial p'}{\partial x}\right]$$

$$w(H) = 0 \Rightarrow \delta = \frac{2H^3}{3}$$

$$\frac{\partial h}{\partial t} = w(0) = \frac{-H^3}{3\eta}\frac{\partial^2 p'}{\partial x^2} = \frac{-H^3}{3\eta}\left(\rho_2 g \frac{\partial^2 h}{\partial x^2} + \gamma\frac{\partial^4 h}{\partial x^4}\right)$$

$$\frac{\partial h}{\partial t} = w(0) = \frac{-H^3}{3\eta}\frac{\partial^2 p'}{\partial x^2} = \frac{-H^3}{3\eta}\left(\rho_2 g\,\frac{\partial^2 h}{\partial x^2} + \gamma\frac{\partial^4 h}{\partial x^4}\right)$$

Normal modes :

$$h \;=\; \delta h\, e^{i(kx-\omega t)}$$

$$-i\omega = -i(\omega_R + i\omega_I) = \frac{-H^3}{3\eta}(-\rho_2 g k^2 + \gamma k^4) \Rightarrow \omega_I = \frac{H^3}{3\eta}(\rho_2 g k^2 - \gamma k^4)$$

$$\omega_I = \frac{H^3}{3\eta}(\rho_2 g k^2 - \gamma k^4)$$

$$\omega_I = \frac{\rho_2 g H^3}{3\eta} k^2 \left(1 - \frac{\gamma}{\rho_2 g} k^2 \right)$$

$$\tau \equiv \frac{\eta}{\rho_2 g H} \;\&\; \ell_c = \sqrt{\frac{\gamma}{\rho_2 g}}$$

$$\omega_I = \frac{1}{3\tau} k^2 H^2 (1 - k^2 \ell_c^2)$$

$$k^\star = k\ell_c = k\sqrt{\gamma / \rho_2 g} \quad \text{and} \quad H^\star = H / \ell_c \quad \text{and} \quad \omega_I^\star = \omega_I \tau$$

$$\omega_I^\star = \frac{1}{3} k^{\star 2} H^{\star 2} (1 - k^{\star 2})$$

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gnuplot] plot [0:1][] x**2*(1-x**2)/3/4 t '', x**2*(1-x**2)/3 lc "blue" t '', 4*x**2*(1-
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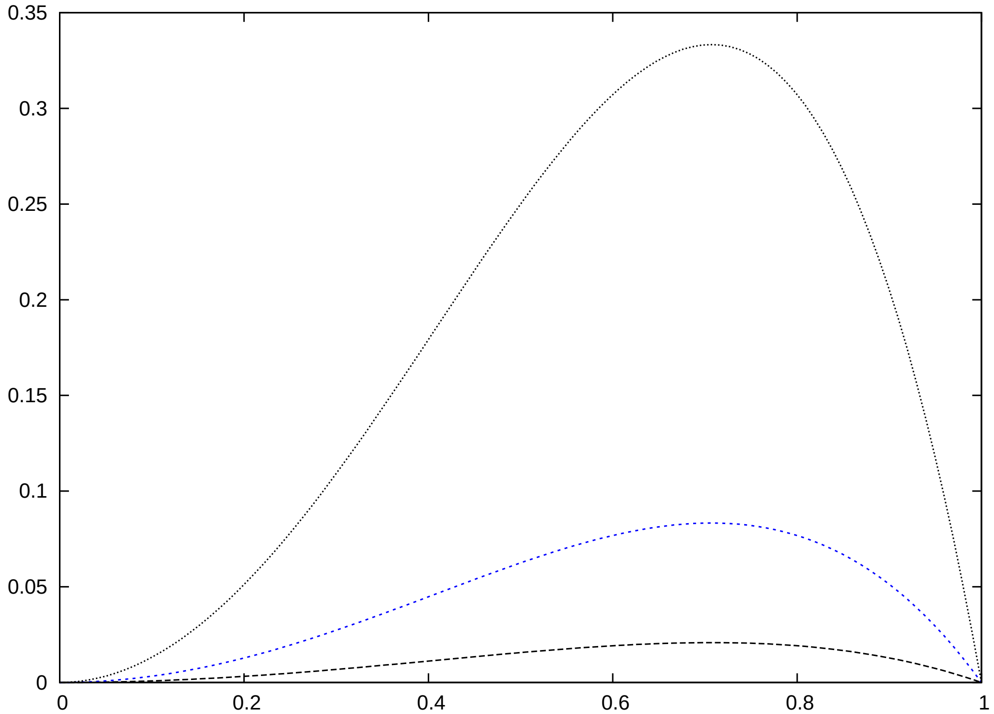


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