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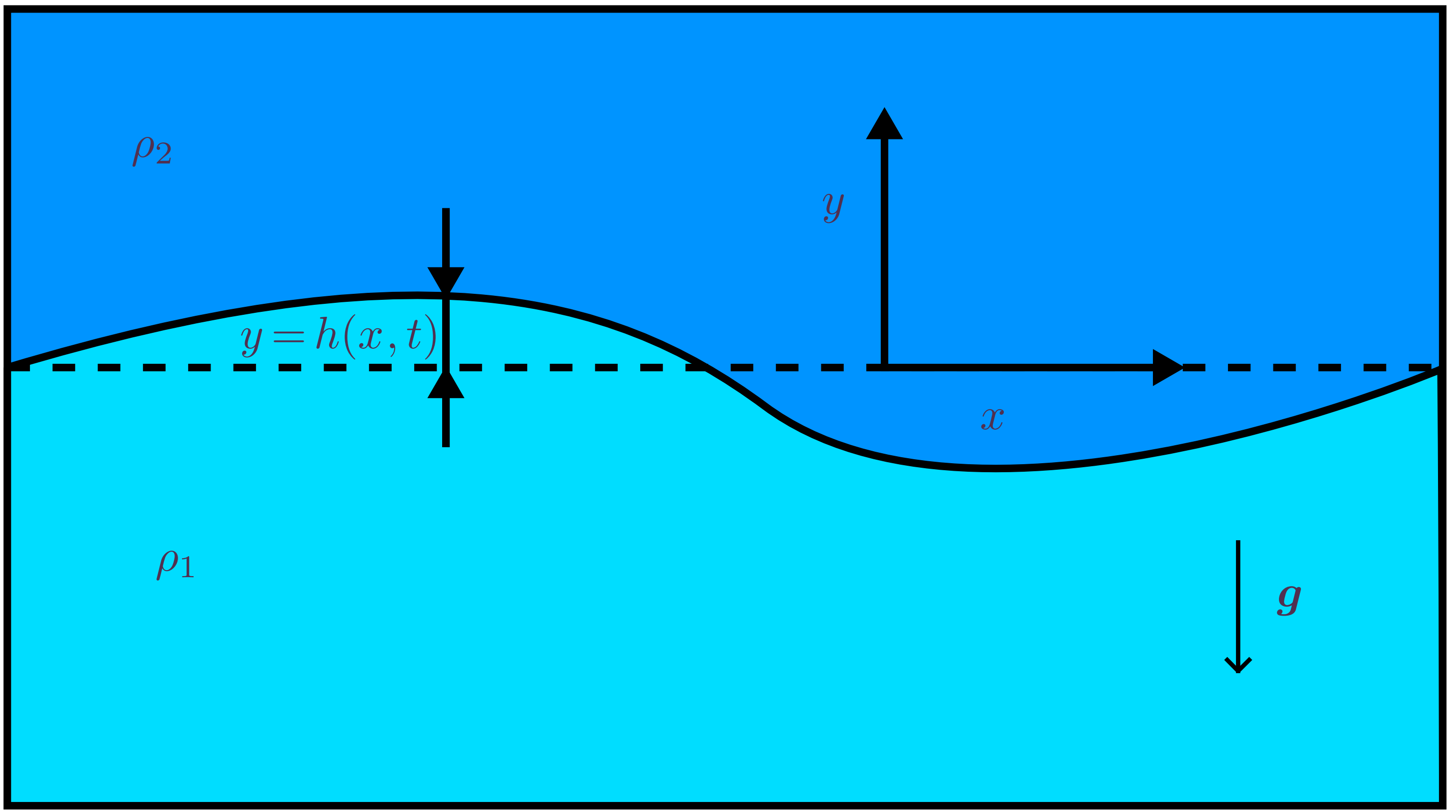
The Rayleigh–Taylor Instability



Rayleigh–Taylor Instability (RTI)

Dimensional analysis

Hypothesis :

- A heavy fluid of density ρ_2 above a lighter fluid of density ρ_1
- no wall effects
- no viscosity effects
- gravity g, surface tension coefficient : $\gamma(N.m^{-1} = \mathrm{kg}.s^{-2})$

Π theorem :

- $\rho_1(\mathrm{kg}.m^{-3}), \rho_2(\mathrm{kg}.m^{-3}), g(m.s^{-2}), \gamma(\mathrm{kg}.s^{-2})$: $N = 4$ quantities
- kg, m, s : $M = 3$ dimensions (dimensional units)
- $\Rightarrow N - M = 1$ dimensionless numbers

$$\text{Atwood number: } \mathrm{At} = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} > 0 \text{ (RTI) or } < 0 \text{ (waves)}$$

Dimensional analysis

Let's consider a perturbation of the interface around $y=0$, of wavelength $\lambda = \frac{2\pi}{k}$, amplitude A.

Volume above $y=0$: $V = \int_0^{\lambda/2} A \sin kx \, \mathrm{dx} = \left[\frac{-A}{k} \cos kx\right]_0^{\pi/k} = \frac{2A}{k}$

Weight + Archimed's force : $F_g = \frac{2A}{k} g(\rho_2 - \rho_1) > 0$ in the case of RTI.

Surface tension is a stabilizing force.

Slope in $x=0$: $\tan\alpha = Ak\cos kx\,(x=0) = Ak \simeq \sin\alpha$

Vertical component of the surface tension force :

$$F_t = -2\gamma A k$$

Model : spring with two forces :

$$M\ddot{A} = F_g + F_t = \frac{2A}{k} g(\rho_2 - \rho_1) - 2\gamma A k = \frac{\rho_1 + \rho_2}{k^2}\,\ddot{A}$$

Model : spring with two forces :

$$M\ddot{A} = F_g + F_t = \frac{2A}{k} g(\rho_2 - \rho_1) - 2\gamma A k = \frac{\rho_1 + \rho_2}{k^2}\ \ddot{A}$$

$$M\ddot{A} = K A = \left(\frac{2}{k} g(\rho_2 - \rho_1) - 2\gamma k\right) A$$

If $K > 0$, $A \sim e^{t\sqrt{K/M}}$ If $K < 0$, $A \sim \cos t\sqrt{K/M}$

Unstable iff

$$\frac{2}{k} g(\rho_2 - \rho_1) > 2\gamma k \Leftrightarrow \frac{1}{k^2} = \left(\frac{\lambda}{2\pi}\right)^2 > \frac{\gamma}{(\rho_2 - \rho_1) g} \Leftrightarrow \lambda > 2\pi\sqrt{\frac{\gamma}{(\rho_2 - \rho_1) g}} = 2\pi \ell_c$$

Remarks :

$$\ddot{A} = \left(2\,k\,g\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} - \frac{2\gamma k^3}{\rho_1 + \rho_2}\right) A = \frac{K}{M} A = \omega^2 A$$

$$\omega^2 = \frac{2\,k\,g(\rho_2 - \rho_1) - 2\gamma k^3}{\rho_1 + \rho_2}$$

Euler equations :

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p - g\,\boldsymbol{e_z}$$

A fluid particle moves on the a distance a over a time interval τ.

Then its typical velocity is $v \sim a/\tau$.

This velocity varies on a length scale equal to λ.

$$\frac{\partial \boldsymbol{v}}{\partial t} \sim \frac{a}{\tau^2} \qquad (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \sim \frac{1}{\lambda}\left(\frac{a}{\tau}\right)^2$$

$$\frac{\partial \boldsymbol{v}}{\partial t} \gg (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \quad \Leftrightarrow \qquad a \ll \lambda$$

$$\text{We define} \quad \varepsilon \equiv \frac{a}{\lambda}$$

$$\frac{\partial \boldsymbol{\nabla} \wedge \boldsymbol{v}}{\partial t} = \boldsymbol{0} \quad \Rightarrow \qquad \boldsymbol{\nabla} \wedge \boldsymbol{v} = \boldsymbol{0} \quad \Rightarrow \qquad \boldsymbol{v} = \boldsymbol{\nabla}\varphi$$

$\varphi(x, y, z, t)$ is the velocity potential.

Linearized Euler equations (Hypothesis : $\rho = C^{\text{ste}}$) :

$$\boldsymbol{\nabla}\left[\frac{\partial\varphi}{\partial t} + \frac{p}{\rho} + g\,z\right] = 0 \quad \Rightarrow \qquad \frac{\partial\varphi}{\partial t} + \frac{p}{\rho} + g\,z = C(t)$$

$$\Delta\varphi = 0 = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad\qquad (2D)$$

Boundary conditions :

$$\boldsymbol{\nabla}\varphi \xrightarrow[z\to\pm\infty]{} 0$$

$$\boldsymbol{\nabla}\varphi_1 \xrightarrow[z\to-\infty]{} 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\varphi_2 \xrightarrow[z\to+\infty]{} 0$$

Kinematic boundary condition in $z = h(x,t)$

Normal to the interface, pointing from fluid 1 to fluid 2 :

$$\boldsymbol{n} = \begin{pmatrix} \dfrac{-h'}{\sqrt{1+h'^2}} \\ \dfrac{1}{\sqrt{1+h'^2}} \end{pmatrix}, \text{where } h' = \frac{\partial h}{\partial x}$$

General case : surface described by $S(x,z,t) = C^{\text{ste}}$

Normal to the interface :

$$\boldsymbol{n} = \frac{\boldsymbol{\nabla S}}{|\boldsymbol{\nabla S}|}$$

$$dS = 0 = S(x+\mathrm{dx}, z+\mathrm{dz}, t+\mathrm{dt}) - S(x,z,t) = \frac{\partial S}{\partial x}\mathrm{dx} + \frac{\partial S}{\partial z}\mathrm{dz} + \frac{\partial S}{\partial t}\mathrm{dt}$$

$$\frac{\partial S}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla S} = 0,$$

where $\boldsymbol{V} = \frac{\mathrm{dx}}{\mathrm{dt}}\boldsymbol{e_x} + \frac{\mathrm{dz}}{\mathrm{dt}}\boldsymbol{e_z}$ is the velocity of the interface.

Normal velocity of the interface = Normal velocity of the fluid particles :

$$\boldsymbol{V}\cdot\boldsymbol{n} = \frac{\boldsymbol{V}\cdot\boldsymbol{\nabla S}}{|\boldsymbol{\nabla S}|} = \frac{\boldsymbol{v}\cdot\boldsymbol{\nabla S}}{|\boldsymbol{\nabla S}|} = -\frac{1}{|\boldsymbol{\nabla S}|}\frac{\partial S}{\partial t}$$

$$\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla S} = 0$$

$$S(x,z,t) = z - h(x,t) \Rightarrow -\frac{\partial h}{\partial t} + \begin{pmatrix} u \\ w \end{pmatrix}\cdot\begin{pmatrix} -h' \\ 1 \end{pmatrix} = 0$$

Kinematic boundary condition: $\frac{\partial h}{\partial t} = w_i(x,h) - u_i(x,h)\frac{\partial h}{\partial x}, i = 1, 2$

Dynamic boundary condition in $z = h(x,t)$

Thomas Young (1773–1829)

Pierre–Simon de Laplace (1749–1827)

Young–Laplace equation :

$$p_2 - p_1 = \gamma\kappa \,(\text{Static interface})$$

$$2D\,\text{curvature}: \kappa = \frac{\partial^2 h / \partial x^2}{[1 + (\partial h / \partial x)^2]^{3/2}} \quad (m^{-1})$$

$$z = h(x,t) \colon p_2(x,h) - p_1(x,h) = \gamma \frac{\partial^2 h / \partial x^2}{[1 + (\partial h / \partial x)^2]^{3/2}}$$

Equations and boundary conditions

$$\frac{\partial \varphi_i}{\partial t} + \frac{p_i}{\rho_i} + gz = 0, i = 1, 2$$

$$\Delta \varphi_i = 0 = \frac{\partial^2 \varphi_i}{\partial x^2} + \frac{\partial^2 \varphi_i}{\partial z^2} = 0, i = 1, 2$$

$$z = h(x, t) \colon \frac{\partial h}{\partial t} = \frac{\partial \varphi_i}{\partial z}(x, h) - \frac{\partial \varphi_i}{\partial x}(x, h)\frac{\partial h}{\partial x}, i = 1, 2$$

$$z = h(x, t) \colon p_2(x, h) - p_1(x, h) = \gamma \frac{\partial^2 h / \partial x^2}{[1 + (\partial h / \partial x)^2]^{3/2}}$$

$$\boldsymbol{\nabla} \varphi_1 \xrightarrow[z \to -\infty]{} 0 \qquad \text{and} \qquad \boldsymbol{\nabla} \varphi_2 \xrightarrow[z \to +\infty]{} 0$$

Basic state : $(\varphi_{0_i}, p_{0_i}), i=1,2$, flat interface : $0=-\frac{1}{\rho}\frac{\partial p}{\partial x}$ and $0=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g$

$$p_{0_i}=p_{\text{ref}}-\rho_i g z, i=1,2$$

Continuity of the pressure in $z=0 \Rightarrow p_{\text{ref}}=p_{0_1}(0)=p_{0_2}(0)$

$$\varphi(x,z,t)=\varphi_0(x,z)+\varphi'(x,z,t) \qquad p(x,z,t)=p_0(x,z)+p'(x,z,t) \qquad h(x,t)$$

$$\frac{\partial \varphi_i'}{\partial t}+\frac{p_i'}{\rho_i}=0$$

$$\frac{\partial h}{\partial x}\sim\varepsilon\ll 1 \quad \Rightarrow \quad \kappa\simeq\frac{\partial^2 h}{\partial x^2} \text{ and } \left|u(x,h)\frac{\partial h}{\partial x}\right|\ll|w(x,h)|$$

$$z=h(x,t): \frac{\partial h}{\partial t}=\frac{\partial \varphi_1'}{\partial z}=\frac{\partial \varphi_2'}{\partial z}$$

$$z=h(x,t): -\rho_2 g h+p_2'(x,h)+\rho_1 g h-p_1'(x,h)=\gamma\frac{\partial^2 h}{\partial x^2}$$

Equations and boundary conditions

$$\begin{aligned}
z \geqslant h(x,t) &: \quad \Delta \varphi_2' = 0 \\
z \leqslant h(x,t) &: \quad \Delta \varphi_1' = 0 \\
z \leqslant h(x,t) &: \quad \rho_1 \frac{\partial \varphi_1'}{\partial t} + p_1' = 0 \\
z \geqslant h(x,t) &: \quad \rho_2 \frac{\partial \varphi_2'}{\partial t} + p_2' = 0 \\
z = h(x,t) &: \quad -\rho_2 g h + p_2'(x,h) + \rho_1 g h - p_1'(x,h) = \gamma \frac{\partial^2 h}{\partial x^2} \\
z = h(x,t) &: \quad \frac{\partial h}{\partial t} = \frac{\partial \varphi_1'}{\partial z} = \frac{\partial \varphi_2'}{\partial z} \\
z \to +\infty &: \quad \boldsymbol{\nabla} \varphi_2' \to 0 \\
z \to -\infty &: \quad \boldsymbol{\nabla} \varphi_1' \to 0
\end{aligned}$$

Problem : Boundary conditions in $z = h(x,t)$.

Example :

$$\frac{\partial \varphi'}{\partial t}(x,h,t) = \frac{\partial \varphi'}{\partial t}(x,0,t) + h \frac{\partial}{\partial z}\left(\frac{\partial \varphi'}{\partial t}\right)(x,0,t) + \cdots$$

Equations and boundary conditions:

$$
\begin{aligned}
z \geqslant 0 &: \Delta \varphi_2' = 0 \\
z \leqslant 0 &: \Delta \varphi_1' = 0 \\
z \leqslant 0 &: \rho_1 \frac{\partial \varphi_1'}{\partial t} + p_1' = 0 \\
z \geqslant 0 &: \rho_2 \frac{\partial \varphi_2'}{\partial t} + p_2' = 0 \\
z = 0 &: -\rho_2 g h + p_2' + \rho_1 g h - p_1' = \gamma \frac{\partial^2 h}{\partial x^2} \\
z = 0 &: \frac{\partial h}{\partial t} = \frac{\partial \varphi_1'}{\partial z} = \frac{\partial \varphi_2'}{\partial z} \\
z \to +\infty &: \boldsymbol{\nabla} \varphi_2' \to 0 \\
z \to -\infty &: \boldsymbol{\nabla} \varphi_1' \to 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Normal modes: } \varphi_1' &= \delta\varphi_1(z) e^{i(kx-\omega t)} \\
\varphi_2' &= \delta\varphi_2(z) e^{i(kx-\omega t)} \\
p_1' &= \delta p_1(z) e^{i(kx-\omega t)} \\
p_2' &= \delta p_2(z) e^{i(kx-\omega t)} \\
h &= \delta h\, e^{i(kx-\omega t)}
\end{aligned}
$$

Laplace's equations give :

$$
\begin{aligned}
-k^2\delta\varphi_1(z) + \delta\varphi_1''(z) &= 0 \\
-k^2\delta\varphi_1(z) + \delta\varphi_1''(z) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\delta\varphi_1(z) &= A_1\, e^{kz} + B_1\, \cancel{e}^{-kz} \\
\delta\varphi_2(z) &= A_2 \cancel{e}^{kz} + B_2\, e^{-kz}
\end{aligned}
$$

Kinematic boundary conditions :

$$-i\omega\delta h = A_1 k = -B_2 k$$

$$\delta\varphi_1(0) = A_1 = -\frac{i\omega}{k}\delta h \qquad \delta\varphi_2(0) = B_2 = \frac{i\omega}{k}\delta h$$

Dynamic boundary conditions :

$$
\begin{aligned}
-i\omega\rho_1\delta\varphi_1(0) + \delta p_1(0) &= 0 \\
-i\omega\rho_2\delta\varphi_2(0) + \delta p_2(0) &= 0 \\
-\rho_2 g\delta h + \delta p_2(0) + \rho_1 g\delta h - \delta p_1(0) &= -\gamma k^2\delta h
\end{aligned}
$$

Dynamic boundary conditions :

$$
\begin{aligned}
-i\omega\rho_1\left(-\frac{i\omega}{k}\delta h\right)+\delta p_1(0) &= 0 \\
-i\omega\rho_2\left(\frac{i\omega}{k}\delta h\right)+\delta p_2(0) &= 0 \\
\delta p_2(0)-\delta p_1(0) &= \rho_2 g\delta h-\rho_1 g\delta h-\gamma k^2\delta h
\end{aligned}
$$

$$
i\omega\rho_2\left(\frac{i\omega}{k}\delta h\right)-i\omega\rho_1\left(-\frac{i\omega}{k}\delta h\right)=\rho_2 g\delta h-\rho_1 g\delta h-\gamma k^2\delta h
$$

$$
-\frac{\omega^2}{k}(\rho_1+\rho_2)=(\rho_2-\rho_1)g-\gamma k^2
$$

Dispersion relation :

$$
\omega^2=\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)gk+\frac{\gamma}{\rho_1+\rho_2}k^3
$$

$$
\begin{aligned}
\rho_1>\rho_2 &\Rightarrow \omega\in\mathbb{R},\text{ no instability}\Rightarrow\text{travelling waves} \\
\rho_1<\rho_2 &\Rightarrow \text{unstable if }\gamma k^3<(\rho_2-\rho_1)gk
\end{aligned}
$$

$$\rho_2 > \rho_1$$

$$\omega^2 = \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) gk + \frac{\gamma}{\rho_1 + \rho_2} k^3$$

$$h \;=\; \mathrm{Re}(\delta h\, e^{i(kx-\omega t)}) = e^{\omega_I t}\,\mathrm{Re}(|\delta h|\, e^{i(kx-\omega_R t+\phi)}) = |\delta h|\, e^{\omega_I t}\cos(kx - \omega_R t + \phi)$$

Hypothesis : $\gamma = 0$

$$\omega = \pm i\sqrt{\left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right) gk} = \omega_R + i\omega_I$$

$$\omega_I = \pm\sqrt{\left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right) gk}$$

Remark : There is no length scale.

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gnuplot] plot [0:1][] sqrt(x) w l lt 3 lw 4 t '', -sqrt(x) w l lt 10 lw 4 t ''
```

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gnuplot]
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gnuplot] plot [0:1][] sqrt(x) w l lt 3 lw 4 t '', -sqrt(x) w l lt 10 lw 4 t ''
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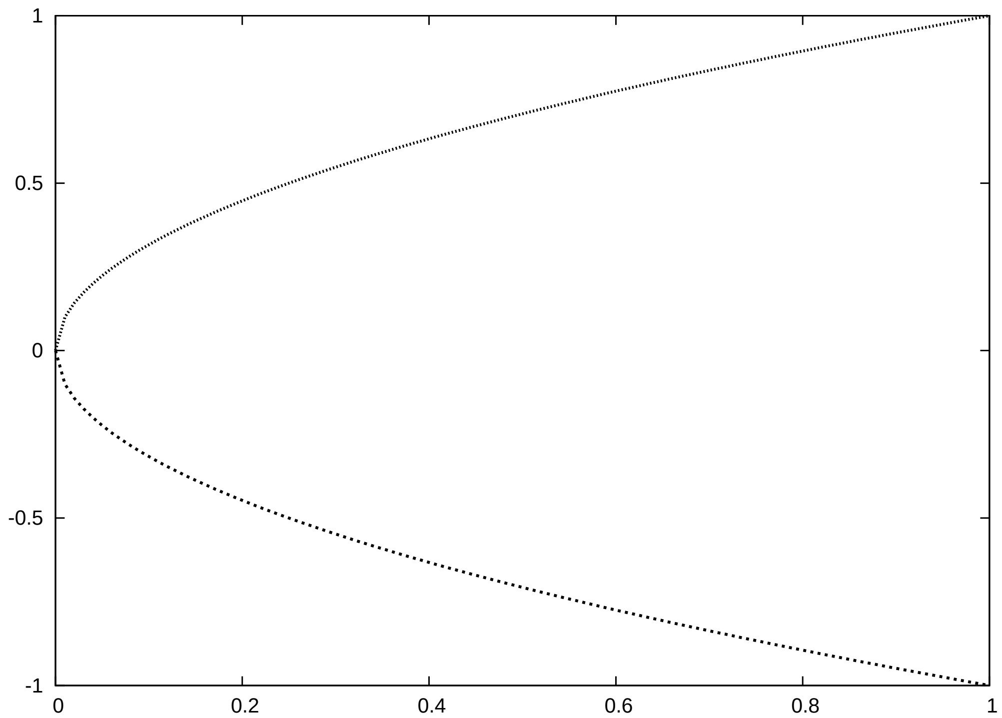


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$$\rho_2 > \rho_1$$

$$\omega^2 = \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) gk + \frac{\gamma}{\rho_1 + \rho_2} k^3$$

Hypothesis : $\gamma \neq 0$

$$\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) gk + \frac{\gamma}{\rho_1 + \rho_2} k^3 < 0 \Leftrightarrow \quad k^2 < \frac{g(\rho_2 - \rho_1)}{\gamma} \Leftrightarrow \lambda > 2\pi \sqrt{\frac{\gamma}{g(\rho_2 - \rho_1)}} = 2\pi \ell_c$$

$$\omega = \pm i \sqrt{\left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right) gk - \frac{\gamma}{\rho_1 + \rho_2} k^3} = \omega_R + i\omega_I$$

$$k^\star = k \sqrt{\frac{\gamma}{g(\rho_2 - \rho_1)}}$$

$$\omega_I^2 = \left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right) gk^\star \sqrt{\frac{g(\rho_2 - \rho_1)}{\gamma}} - \frac{\gamma}{\rho_1 + \rho_2} k^{\star 3} \left(\frac{g(\rho_2 - \rho_1)}{\gamma}\right)^{3/2}$$

$$\omega_I^2 = g\left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right) \sqrt{\frac{g(\rho_2 - \rho_1)}{\gamma}} [k^\star - k^{\star 3}] \Rightarrow \omega_I^\star = \pm\sqrt{k^\star - k^{\star 3}}$$

```
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gnuplot] plot [0:1][] sqrt(x) w l lt 3 lw 4 t '', -sqrt(x) w l lt 10 lw 4 t '', sqrt(x-
         x**3) w l lt 3 lw 4 lc rgbcolor "red" t '', -sqrt(x-x**3) w l lt 10 lw 4 lc
         rgbcolor "red" t ''
```

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gnuplot]
```
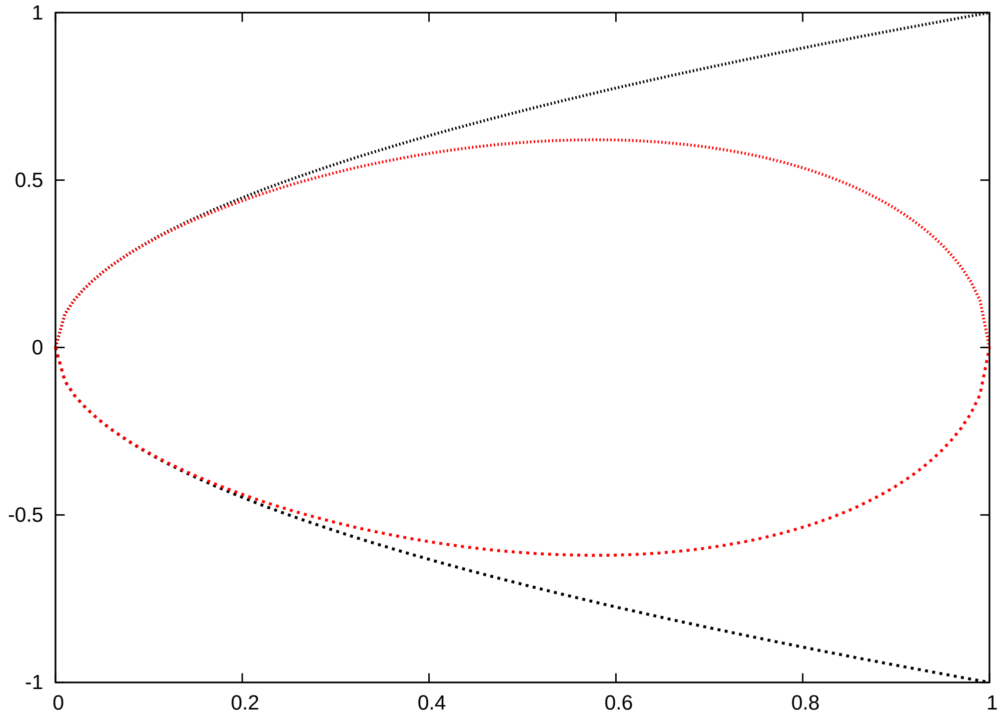

Remark: if $\rho_1 = 0$, there is only one length scale in the problem : $\ell_c = \sqrt{\gamma / \rho_2 g}$.

gnuplot 4.6 patchlevel 7

```
gnuplot] plot [0:1][] sqrt(x) w l lt 3 lw 4 t '', -sqrt(x) w l lt 10 lw 4 t '', sqrt(x-
         x**3) w l lt 3 lw 4 lc rgbcolor "red" t '', -sqrt(x-x**3) w l lt 10 lw 4 lc
         rgbcolor "red" t ''
```




```
gnuplot]
```

Remark: if $\rho_1 = 0$, there is only one length scale in the problem : $\ell_c = \sqrt{\gamma / \rho_2 g}$.

Waves : $\rho_2 < \rho_1$

$$\omega^2 = \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) g k + \frac{\gamma}{\rho_1 + \rho_2} k^3 > 0$$

$$\omega = \omega_R = \pm \sqrt{\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) g k + \frac{\gamma}{\rho_1 + \rho_2} k^3}$$

Travelling wave

$$h \;=\; \mathrm{Re}(\delta h\, e^{i(kx - \omega t)}) = \mathrm{Re}(|\delta h|\, e^{i(kx - \omega_R t + \phi)}) = |\delta h| \cos(kx - \omega_R t + \phi)$$

$$\text{Phase velocity: } c = \frac{\omega}{k} = \sqrt{\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right) \frac{g}{k} + \frac{\gamma}{\rho_1 + \rho_2} k} \neq \frac{\partial \omega}{\partial k} \Rightarrow \text{Dispersive waves}$$

Minimum phase velocity : $c^\star = \sqrt{1/k^\star + k^\star} \Rightarrow \frac{dc^\star}{dk^\star} = 0 \Leftrightarrow \frac{-1}{k^{\star 2}} + 1 = 0$

$$k^\star_{\min} = 1 \Rightarrow c^\star_{\min} = \sqrt{2}$$

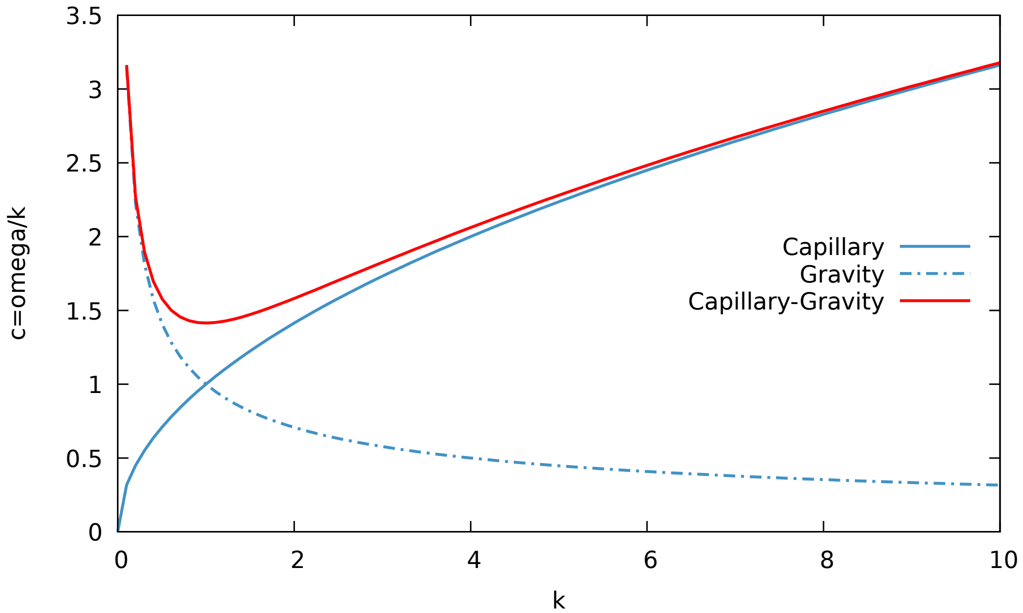
gnuplot 4.6 patchlevel 7

```
gnuplot] set term pdf dashed enhanced; set key center right; set xlabel ’k’; set ylabel
         ’c=omega/k’; plot [0:10][0:] sqrt(x) w l lt 1 lw 2 dt 1 t ’Capillary’, sqrt(1./
         x) w l lt 1 lw 2 dt 4 t ’Gravity’, sqrt(1./x+x) w l lt 1 lw 2 lc rgbcolor ’red’
         t ’Capillary-Gravity’
```

```
gnuplot]
```

gnuplot 4.6 patchlevel 7

```
gnuplot] set term pdf dashed enhanced; set key center right; set xlabel 'k'; set ylabel
         'c=omega/k'; plot [0:10][0:] sqrt(x) w l lt 1 lw 2 dt 1 t 'Capillary', sqrt(1./
         x) w l lt 1 lw 2 dt 4 t 'Gravity', sqrt(1./x+x) w l lt 1 lw 2 lc rgbcolor 'red'
         t 'Capillary-Gravity'
```

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