Some Hydrodynamic instabilities

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10×1h

Have a look at the National Committee for Fluid Mechanics Films :

http://web.mit.edu/hml/ncfmf.html

A more advanced course on hydrodynamic instabilities :

http://basilisk.fr/sandbox/easystab/M2DET/Instabilities.md

Outline

Example : 2D Poiseuille flow

$$u(y) = -\frac{a\,h^2}{2\,\rho\nu}\left(1-\left(\frac{y}{h}\right)^2\right), \qquad v=0, \qquad w=0, \quad a=\frac{\partial p}{\partial x}<0$$

Maximum velocity : $U_0 = u(y=0) = -\frac{a\,h^2}{2\,\rho\nu}$. Pressure : $P_0(x) = p_{\text{ref}} + a\,x$

Dimensionless equations : we choose U_0 as a characteristic velocity, h as a characteristic length, and ρU_0^2 as a characteristic pressure.

$$[x,y,z] = h\,[x^\star, y^\star, z^\star], [u,v,w] = U_0\,[u^\star, v^\star, w^\star], t = \frac{h\,t^\star}{U_0}, p = p^\star \rho U_0^2$$

$$\begin{aligned} \frac{\partial \boldsymbol{v}^\star}{\partial t^\star} + (\boldsymbol{v}^\star \cdot \boldsymbol{\nabla}^\star)\boldsymbol{v}^\star &= -\boldsymbol{\nabla}^\star p^\star + \frac{1}{\mathrm{Re}}\Delta^\star \boldsymbol{v}^\star \\ \boldsymbol{\nabla}^\star \cdot \boldsymbol{v}^\star &= 0 \end{aligned}$$

with $u^\star = v^\star = w^\star = 0$ in $y^\star = -1$ and $y^\star = 1$, $\mathrm{Re} = \frac{U_0 h}{\nu}$, and $a^\star = \frac{a\,h}{\rho U_0^2} = -\frac{2\,\rho\nu U_0}{h^2}\,\frac{h}{\rho U_0^2} = -\frac{2}{\mathrm{Re}}$.

Parallel flow solution : $\boldsymbol{v_0} = (1 - {y^\star}^2)\boldsymbol{e_x} \equiv U(y)\boldsymbol{e_x}$

Example : 2D Poiseuille flow

We look for a solution in the form (removing $\star$): $\boldsymbol{v} = \boldsymbol{v_0} + \varepsilon \boldsymbol{v'}$, $p = p_0 + \varepsilon p'$,

where $\varepsilon \ll 1$ and $\boldsymbol{v_0} = [1 - y^2, 0, 0]$, $p_0 = p_{\mathrm{ref}} - \frac{2x}{\mathrm{Re}}$.

The linearized dimensionless equations read :

$$\begin{aligned} \frac{\partial \boldsymbol{v'}}{\partial t} + v' \frac{dU(y)}{dy} \boldsymbol{e_y} + U(y) \frac{\partial \boldsymbol{v'}}{\partial x} &= -\boldsymbol{\nabla} p' + \frac{1}{\mathrm{Re}} \Delta \boldsymbol{v'} \\ \boldsymbol{\nabla} \cdot \boldsymbol{v'} &= 0 \end{aligned}$$

with $u' = v' = w' = 0$ in $y = 1$ and $y = -1$.

$\boldsymbol{v_0} = (1 - y^2)\boldsymbol{e_x}$ is invariant in x and z. Then we can define the Fourier Transform in both directions (Homogeneity in x and z):

$$\hat{\boldsymbol{v}}(k_x, y, k_z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \boldsymbol{v'}(x, y, z) e^{-i(k_x x + k_z z)} dx\, dz$$

$$\hat{p}(k_x, y, k_z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p'(x, y, z) e^{-i(k_x x + k_z z)} dx\, dz$$

Example: $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial p'}{\partial x} e^{-i(k_x x + k_z z)} dx\, dz = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} -i k_x p' e^{-i(k_x x + k_z z)} dx\, dz = i k_x \hat{p}$

Fourier Transform of the Navier–Stokes (NS) equations :

$$\begin{aligned}
\frac{\partial \hat{\boldsymbol{v}}}{\partial t} + \hat{v}\frac{dU(y)}{dy}\boldsymbol{e_y} + U(y) i k_x \hat{\boldsymbol{v}} &= -\begin{bmatrix} i k_x \\ \frac{\partial}{\partial y} \\ i k_z \end{bmatrix}\hat{p} + \frac{1}{\mathrm{Re}}\left(-k_x^2 - k_z^2 + \frac{\partial^2}{\partial y^2}\right)\hat{\boldsymbol{v}} \\
\left(i k_x \boldsymbol{e_x} + \boldsymbol{e_y}\frac{\partial}{\partial y} + i k_z \boldsymbol{e_z}\right)\cdot \hat{\boldsymbol{v}} &= 0
\end{aligned}$$

$$\frac{\partial \hat{\boldsymbol{v}}}{\partial t} + \hat{v}\frac{dU(y)}{dy}\boldsymbol{e_y} + U(y) i k_x \hat{\boldsymbol{v}} = -\begin{bmatrix} i k_x \\ \frac{\partial}{\partial y} \\ i k_z \end{bmatrix}\hat{p} + \frac{1}{\mathrm{Re}}\left(-k_x^2 - k_z^2 + \frac{\partial^2}{\partial y^2}\right)\hat{\boldsymbol{v}}$$

$$\left(i k_x \boldsymbol{e_x} + \boldsymbol{e_y}\frac{\partial}{\partial y} + i k_z \boldsymbol{e_z}\right)\cdot \hat{\boldsymbol{v}} = 0$$

+ Boundary conditions : $\hat{\boldsymbol{v}}(k_x, \pm 1, k_z) = 0$ + continuity of the pressure.

Inverse Fourier Transform :

$$\boldsymbol{v'}(x, y, z, t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{\boldsymbol{v}}(k_x, y, k_z, t) e^{i(k_x x + k_z z)} dx\, dz$$

= superposition of pure Fourier modes.

$$\frac{\partial \hat{\boldsymbol{v}}}{\partial t} + \hat{v}\frac{dU(y)}{dy}\boldsymbol{e_y} + U(y)ik_x\hat{\boldsymbol{v}} = -\begin{bmatrix} ik_x \\ \frac{\partial}{\partial y} \\ ik_z \end{bmatrix}\hat{p} + \frac{1}{\mathrm{Re}}\left(-k_x^2 - k_z^2 + \frac{\partial^2}{\partial y^2}\right)\hat{\boldsymbol{v}}$$

$$\left(ik_x\boldsymbol{e_x} + \boldsymbol{e_y}\frac{\partial}{\partial y} + ik_z\boldsymbol{e_z}\right)\cdot\hat{\boldsymbol{v}} = 0$$

System of linear differential equations with constant coefficient with respect to x, z, t.

Homogeneous : $x \to x + x_0$, $z \to z + z_0$ won't change the equations.

Then, looking for solutions is equivalent to looking for solutions in the form of normal modes :

$$\boldsymbol{v'} = \hat{\boldsymbol{v}}(k_x, y, k_z, t)\, e^{i(k_x x + k_z z)} = \tilde{\boldsymbol{v}}(k_x, y, k_z)\, e^{i(k_x x + k_z z) + st}$$

$$p' = \hat{p}(k_x, y, k_z, t)\, e^{i(k_x x + k_z z)} = \tilde{p}(k_x, y, k_z)\, e^{i(k_x x + k_z z) + st}$$

where k_x and k_z are wave numbers, and $s = \sigma + i\omega \in \mathbb{C}$.

Example : 2D Poiseuille flow

$$\boldsymbol{v'} = \tilde{\boldsymbol{v}}(k_x, y, k_z)\, e^{i(k_x x + k_z z) + st}, \, p' = \tilde{p}(k_x, y, k_z)\, e^{i(k_x x + k_z z) + st}$$

Plugging the normal modes into the equations gives :

$$\begin{aligned} s\tilde{\boldsymbol{v}}(y) + (-2y)\tilde{v}(y)\boldsymbol{e_y} + (1-y^2)ik_x\tilde{\boldsymbol{v}}(y) &= -\begin{bmatrix} ik_x \\ \dfrac{\partial}{\partial y} \\ ik_z \end{bmatrix}\tilde{p}(y) + \frac{1}{\mathrm{Re}}\left(-k_x^2 - k_z^2 + \frac{\partial^2}{\partial y^2}\right)\tilde{\boldsymbol{v}}(y) \\ ik_x\tilde{u}(y) + \frac{\partial \tilde{v}(y)}{\partial y} + ik_z\tilde{w}(y) &= 0 \end{aligned}$$

where $\tilde{\boldsymbol{v}}(k_x, \pm 1, k_z) = 0$.

We obtain a system of second-order differential equations according to y, that can only be solved numerically.

The present system cannot be solved analytically. We need to discretize according to y, leading to an eigenvalue problem :

$$s\,B\,\boldsymbol{X} = A(k_x, k_z)\boldsymbol{X}$$

where $\boldsymbol{X}$ is a vector containing the values of $\hat{\boldsymbol{v}}$ and $\hat{p}$ in discrete values of y. Matrix B exists because the continuity equation does not involve a time-derivative. The coefficients in matrix A involve k_x, k_z, $U(y)$, Re.

We suppose here that, for each couple $(k_x,\ k_z)$, and for each Re, there exists a family of eigenvalues :

$$s_j(k_x, k_z, \mathrm{Re}) = \sigma_j + i\omega_j, \text{for } j = 1, 2...$$

If there exists (j, k_x, k_z) such that $\mathrm{Re}(s_j(k_x, k_z)) = \sigma_j > 0$, then the perturbation will grow like:

$$e^{\sigma_j t}$$

Then, to find whether a system is stable or unstable, we seek the maximum value of σ_j for each couple (k_x, k_z) :

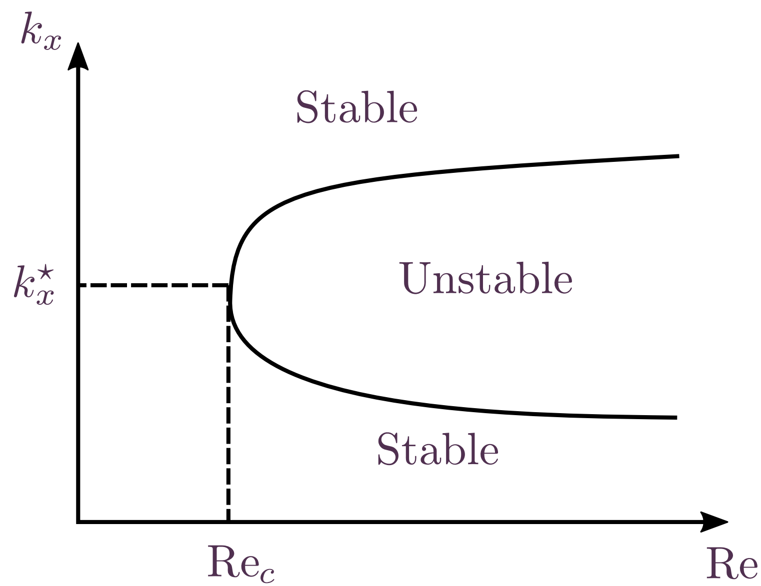
$$\sigma_m \equiv \mathrm{Max}_j(\sigma_j(k_x, k_z))$$

For a fixed value of Re, solving the eigenvalue problem, we may find a range of values of k_x (or k_z) for which $\exists j \mid \sigma_j(k_x, k_z, \mathrm{Re}) > 0$.

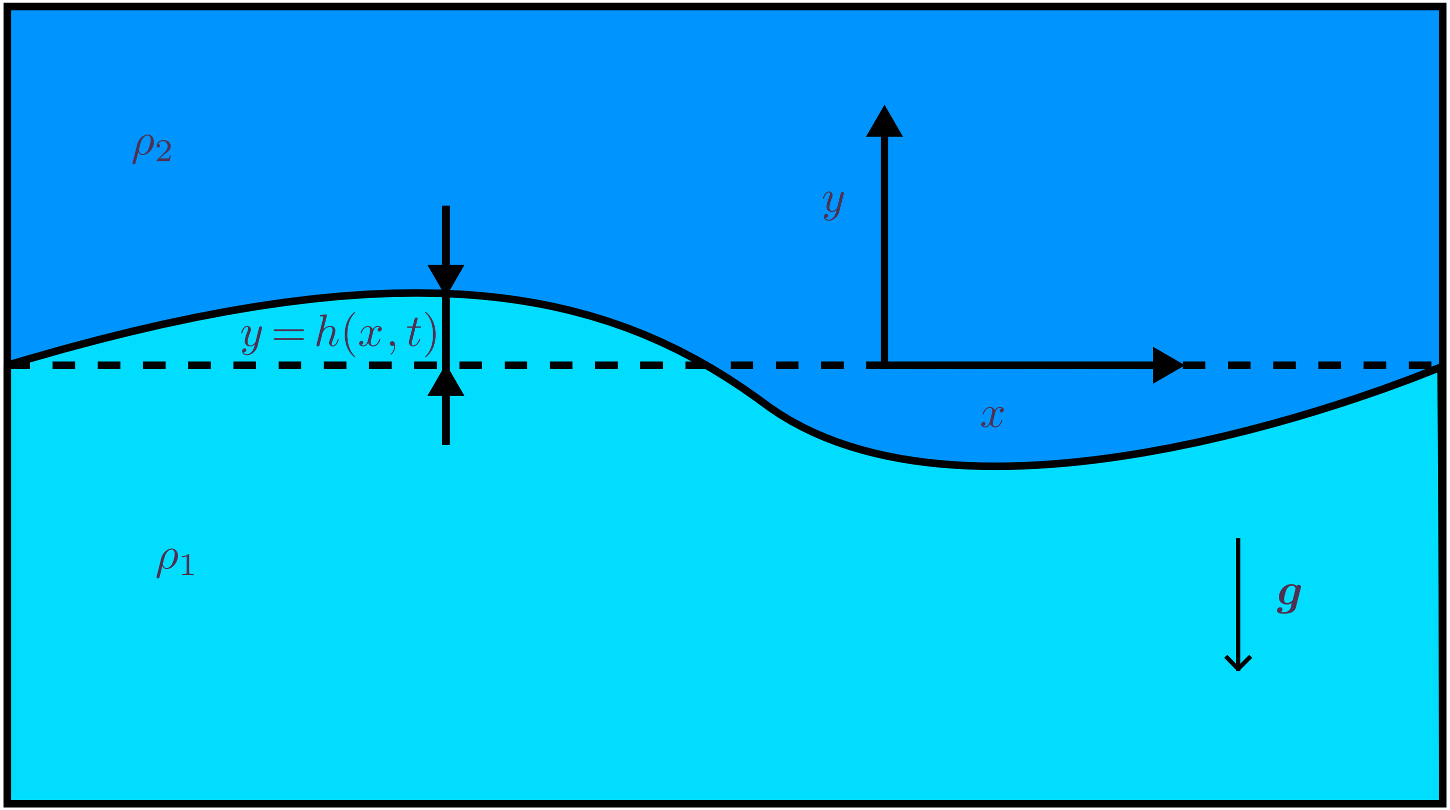
In the present case, we can define a critical Reynolds number :

$$\mathrm{Re}_c \equiv \mathrm{Min}\{\mathrm{Re} \mid \exists k_x, k_z \mid \sigma_{\max}(k_x, k_z, \mathrm{Re}) > 0\}$$

In the case of Poiseuille flow, we find : $\mathrm{Re}_c \simeq 5772$, for $k_x \simeq 1$.



The Rayleigh–Taylor Instability



Dimensional analysis

Hypothesis :

- A heavy fluid of density ρ_2 above a lighter fluid of density ρ_1
- no wall effects
- no viscosity effects
- gravity g, surface tension coefficient : $\gamma(N.m^{-1} = \mathrm{kg}.s^{-2})$

Π theorem :

- $\rho_1(\mathrm{kg}.m^{-3}), \rho_2(\mathrm{kg}.m^{-3}), g(m.s^{-2}), \gamma(\mathrm{kg}.s^{-2})$: $N = 4$ quantities
- kg, m, s : $M = 3$ dimensions (dimensional units)
- $\Rightarrow N - M = 1$ dimensionless numbers

$$\text{Atwood number: } \mathrm{At} = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} > 0 \text{ (RTI) or } < 0 \text{ (waves)}$$

Dimensional analysis

Let's consider a perturbation of the interface around $y=0$, of wavelength $\lambda=\frac{2\pi}{k}$, amplitude A.

Volume above $y=0$: $V=\int_0^{\lambda/2} A\sin kx\,\mathrm{dx}=\left[\frac{-A}{k}\cos kx\right]_0^{\pi/k}=\frac{2A}{k}$

Weight + Archimed's force : $F_g=\frac{2A}{k}g(\rho_2-\rho_1)>0$ in the case of RTI.

Surface tension is a stabilizing force.

Slope in $x=0$: $\tan\alpha=Ak\cos kx\,(x=0)=Ak\simeq\sin\alpha$

Vertical component of the surface tension force :

$$F_t=-2\gamma Ak$$

Model : spring with two forces :

$$M\ddot{A}=F_g+F_t=\frac{2A}{k}g(\rho_2-\rho_1)-2\gamma Ak=\frac{\rho_1+\rho_2}{k^2}\ \ddot{A}$$

Model : spring with two forces :

$$M\ddot{A} = F_g + F_t = \frac{2A}{k} g(\rho_2 - \rho_1) - 2\gamma A k = \frac{\rho_1 + \rho_2}{k^2}\ \ddot{A}$$

$$M\ddot{A} = K A = \left(\frac{2}{k} g(\rho_2 - \rho_1) - 2\gamma k\right) A$$

If $K > 0$, $A \sim e^{t\sqrt{K/M}}$ If $K < 0$, $A \sim \cos t\sqrt{K/M}$

Unstable iff

$$\frac{2}{k} g(\rho_2 - \rho_1) > 2\gamma k \Leftrightarrow \frac{1}{k^2} = \left(\frac{\lambda}{2\pi}\right)^2 > \frac{\gamma}{(\rho_2 - \rho_1) g} \Leftrightarrow \lambda > 2\pi\sqrt{\frac{\gamma}{(\rho_2 - \rho_1) g}} = 2\pi\ell_c$$

Remarks :

$$\ddot{A} = \left(2\,k\,g\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} - \frac{2\gamma k^3}{\rho_1 + \rho_2}\right) A = \frac{K}{M} A = \omega^2 A$$

$$\omega^2 = \frac{2\,k\,g(\rho_2 - \rho_1) - 2\gamma k^3}{\rho_1 + \rho_2}$$