Some Hydrodynamic instabilities

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10×1h

Have a look at the National Committee for Fluid Mechanics Films :

http://web.mit.edu/hml/ncfmf.html

A more advanced course on hydrodynamic instabilities :

`http://basilisk.fr/sandbox/easystab/M2DET/Instabilities.md`

Outline

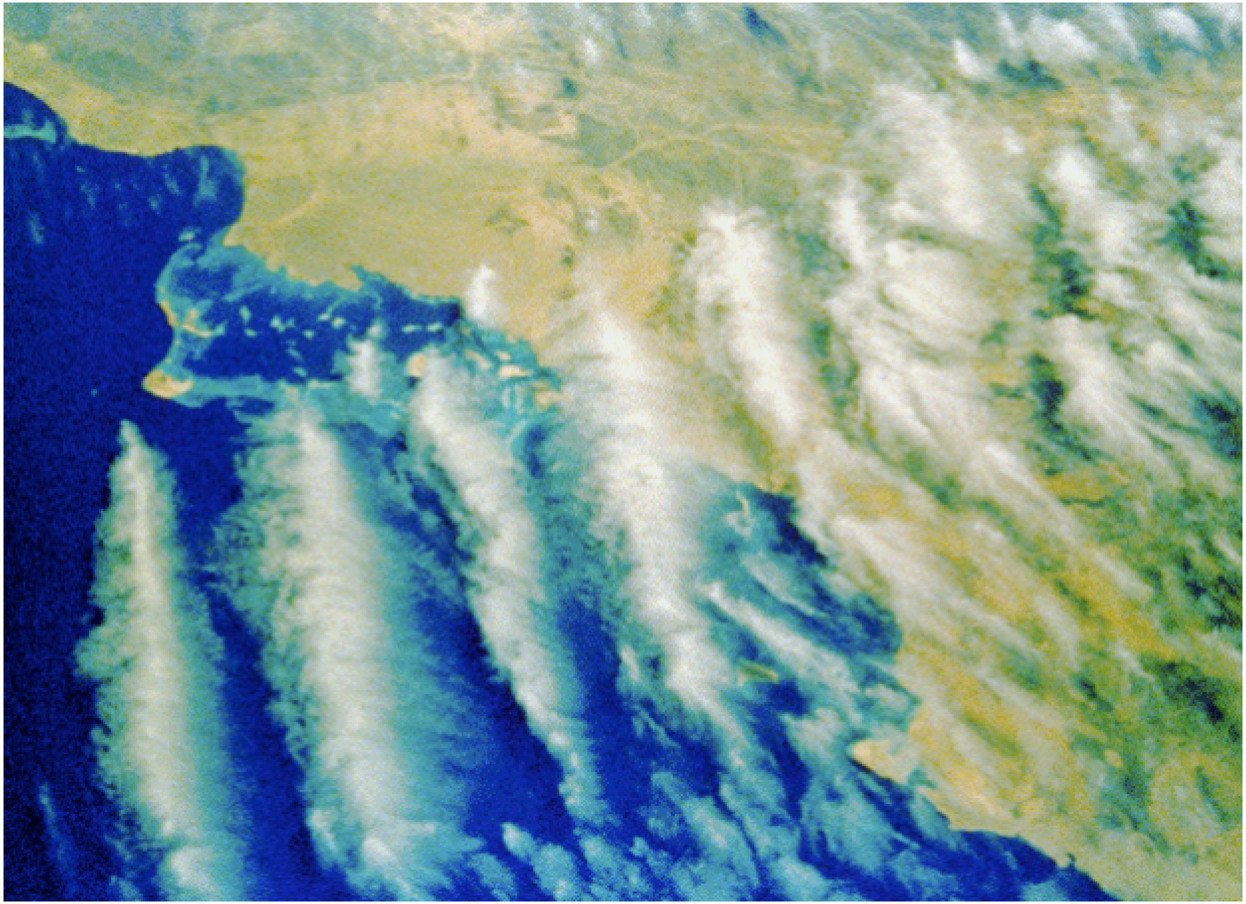

Figure 1. Cirrus clouds developing in a jet stream over Saudi Arabia and the Red Sea. The picture was taken from the Space Shuttle (NASA).

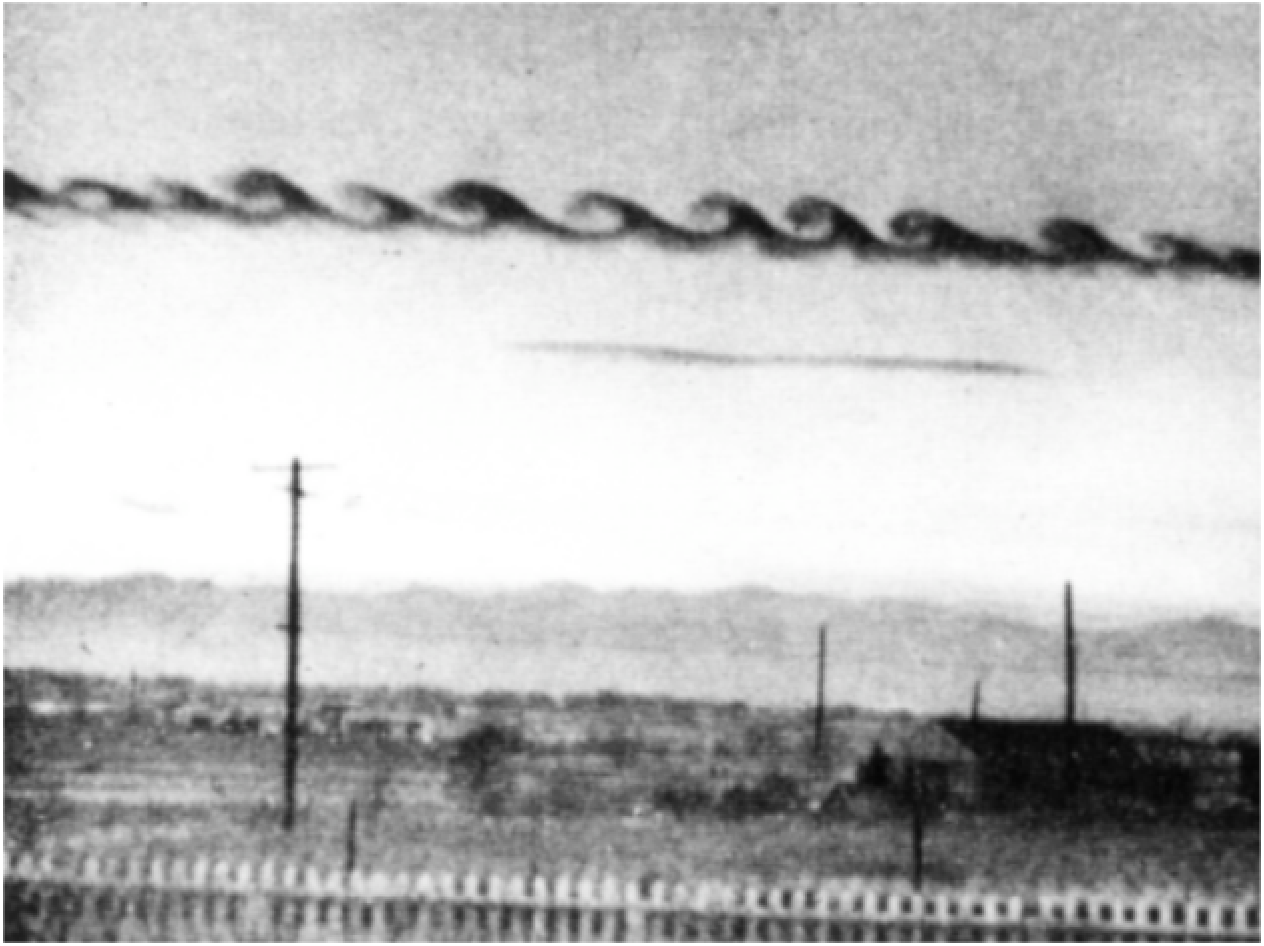

Figure 2. An example of the Kelvin–Helmholtz shear instability.

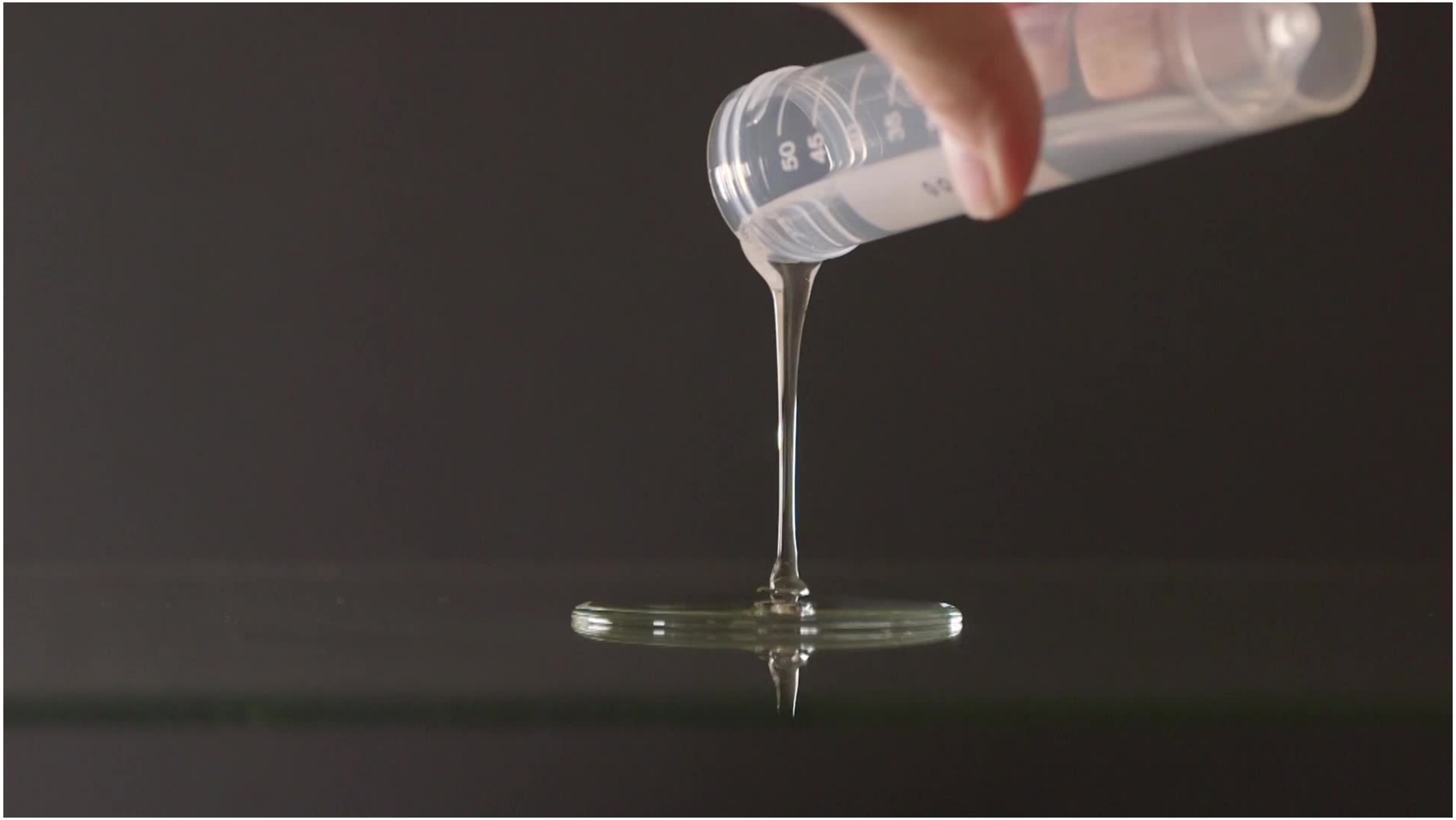

Figure 3. Rayleigh–Taylor Instability in a thin viscous film (Experiment : Christophe Clanet)

Figure 4. Rayleigh–Plateau instability on a thin rod

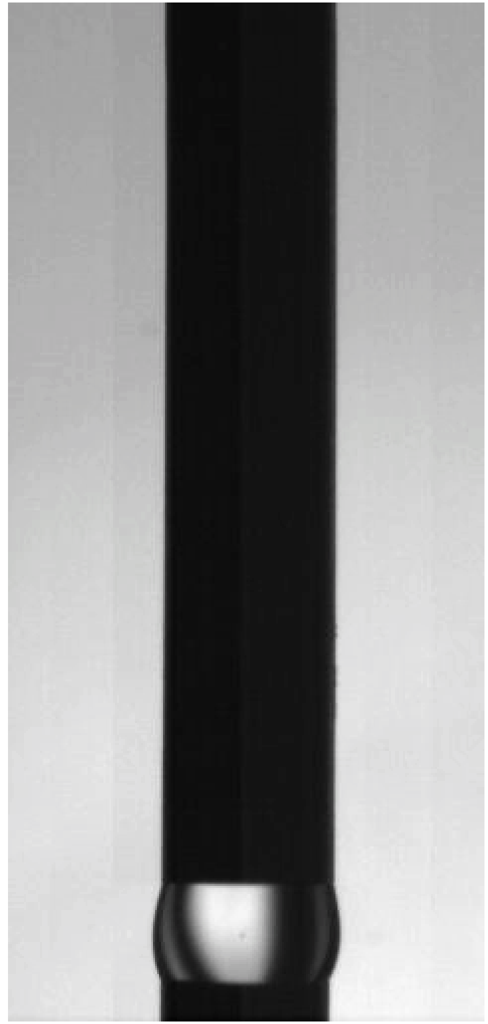

Figure 5. Stretching of a millimetric liquid ligament (Lionel Vincent, PhD Thesis, 2013).

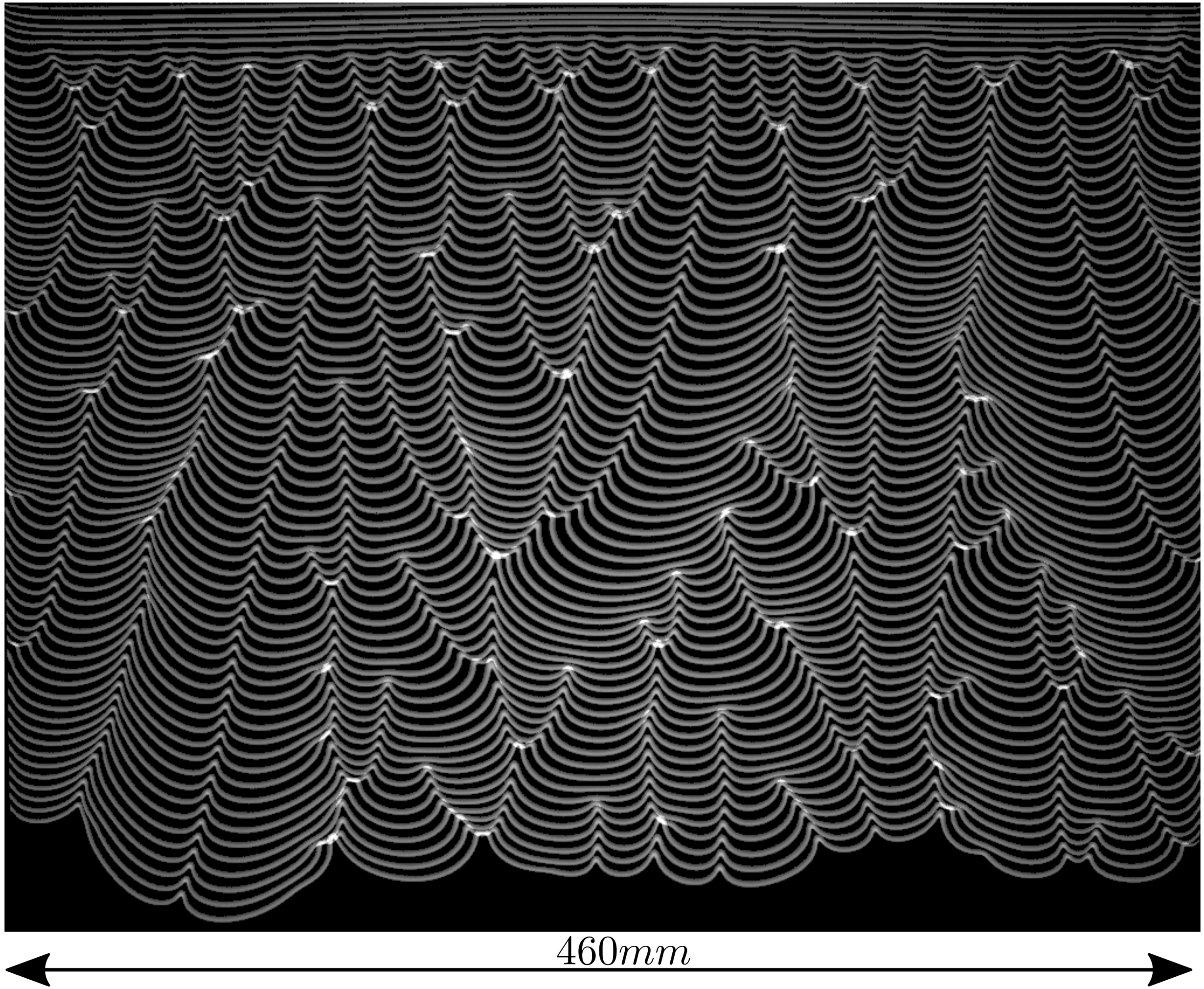

Figure 6. Instability of a premixed flame in a Hele–Shaw cell (Basile Radisson, PhD Thesis, 2019)

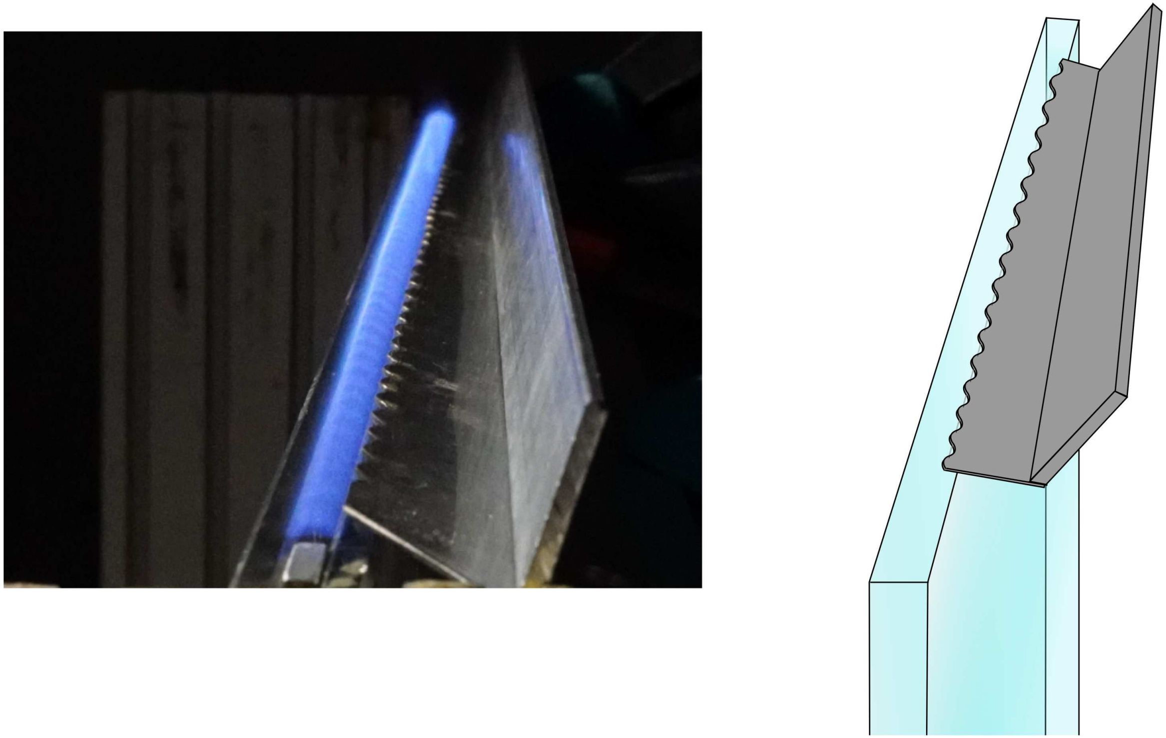

Figure 7. An initial condition for the flame front is imposed with a profiled steel plate at the top

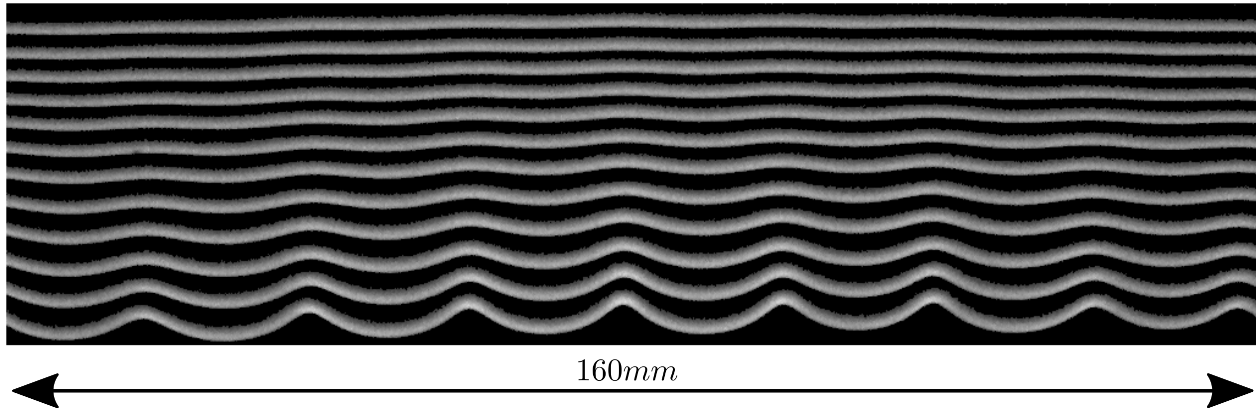

Figure 8. Short-time dynamics

A system is linearly stable if it is stable regarding “small” perturbations.

BUT it can be unstable regarding “large” perturbations.

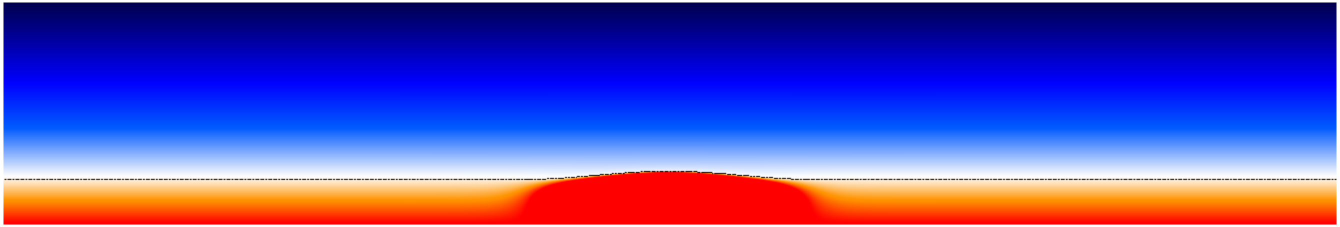

Figure 9. Rayleigh–Bénard instability with a melting boundary.

Fundamental concepts

Navier–Stokes equations

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} &= -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\Delta\boldsymbol{v} + \boldsymbol{f} \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0\end{aligned}$$

where $\boldsymbol{v} = u(x,y,z,t)\boldsymbol{e_x} + v(x,y,z,t)\boldsymbol{e_y} + w(x,y,z,t)\boldsymbol{e_z}$,

$p(x,y,z,t)$ is the pressure,

ρ the density,

$\nu = \mu/\rho$ the kinematic viscosity,

$\boldsymbol{f}$ an external force,

$\boldsymbol{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$, and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in cartesian coordinates.

\+ BOUNDARY CONDITIONS

Basic state and small perturbation

We suppose that there exists a stationary state $(\boldsymbol{v_0}, p_0)$, called the basic state, solution of the Navier–Stokes equations (case $\boldsymbol{f} = \boldsymbol{0}$):

$$\begin{aligned} (\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} &= -\frac{1}{\rho}\boldsymbol{\nabla} p_0 + \nu \Delta \boldsymbol{v_0} \\ \boldsymbol{\nabla} \cdot \boldsymbol{v_0} &= 0 \end{aligned}$$

+ Must satisfy boundary conditions.

In a linear stability analysis, we study the dynamics of a small perturbation $(\boldsymbol{v'}, p')$ around the basic state.

Hypothesis : $\rho = C^{\text{ste}}$.

We look for a solution in the form : $\boldsymbol{v} = \boldsymbol{v_0} + \varepsilon \boldsymbol{v'}, p = p_0 + \varepsilon p'$, where $\varepsilon \ll 1$.

Plugging these expressions into Navier–Stokes equations, we get :

$$\begin{aligned} \varepsilon \frac{\partial \boldsymbol{v'}}{\partial t} + \varepsilon(\boldsymbol{v'} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} + \varepsilon(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v'} + \varepsilon^2(\boldsymbol{v'} \cdot \boldsymbol{\nabla})\boldsymbol{v'} &= -\frac{\varepsilon}{\rho}\boldsymbol{\nabla} p' + \varepsilon \nu \Delta \boldsymbol{v'} \\ \varepsilon \boldsymbol{\nabla} \cdot \boldsymbol{v'} &= 0 \end{aligned}$$

Basic state and small perturbation

$$
\begin{aligned}
\varepsilon\frac{\partial \boldsymbol{v'}}{\partial t}+\varepsilon(\boldsymbol{v'}\cdot\boldsymbol{\nabla})\boldsymbol{v_0}+\varepsilon(\boldsymbol{v_0}\cdot\boldsymbol{\nabla})\boldsymbol{v'}+\varepsilon^2(\boldsymbol{v'}\cdot\boldsymbol{\nabla})\boldsymbol{v'} &= -\frac{\varepsilon}{\rho}\boldsymbol{\nabla}p'+\varepsilon\nu\Delta\boldsymbol{v'} \\
\varepsilon\boldsymbol{\nabla}\cdot\boldsymbol{v'} &= 0
\end{aligned}
$$

Linearizing the equations for $\varepsilon \longrightarrow 0$, we get :

$$
\begin{aligned}
\frac{\partial \boldsymbol{v'}}{\partial t}+(\boldsymbol{v'}\cdot\boldsymbol{\nabla})\boldsymbol{v_0}+(\boldsymbol{v_0}\cdot\boldsymbol{\nabla})\boldsymbol{v'} &= -\frac{1}{\rho}\boldsymbol{\nabla}p'+\nu\Delta\boldsymbol{v'} \\
\boldsymbol{\nabla}\cdot\boldsymbol{v'} &= 0
\end{aligned}
$$

The solution of these equations is defined up to a multiplicative prefactor.

If such a solution increases in time, the basic state is said to be linearly unstable, otherwise it is linearly stable. The linear system can be written :

$$
\frac{\partial \boldsymbol{X}}{\partial t}=\boldsymbol{\mathcal{D}}[\boldsymbol{X}],
$$

where $\boldsymbol{\mathcal{D}}$ is a differential operator and

$\boldsymbol{X}$ is a vector of 4 unknown functions $u'(x,y,z,t), v'(x,y,z,t), w'(x,y,z,t), p'(x,y,z,t)$

Example : 2D Poiseuille flow

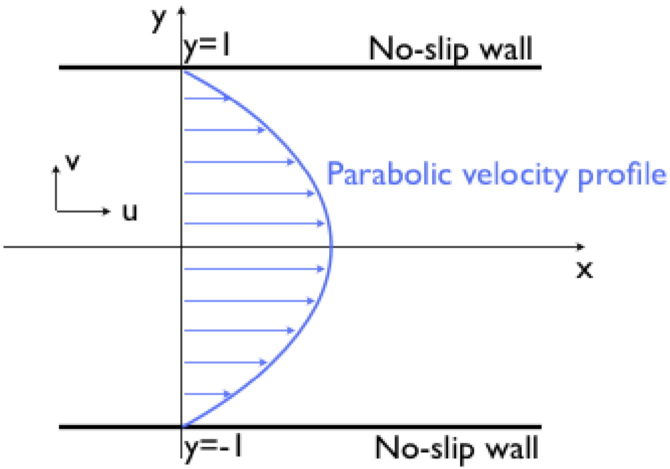

Figure 10. Poiseuille flow between two plates, driven by a constant pressure gradient

Have a look at http://basilisk.fr/sandbox/easystab/poiseuille_uvp.m

The parallel flow solution of the Navier–Stokes equations reads :

$$u(y) = -\frac{a\,h^2}{2\rho\nu}\left(1-\left(\frac{y}{h}\right)^2\right), \qquad v = 0, \qquad w = 0,$$

where $a = \frac{\partial p}{\partial x} < 0$.

Example : 2D Poiseuille flow

Navier–Stokes equations

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} &= -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\Delta\boldsymbol{v}\\ \boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0\end{aligned}$$

with $u = v = w = 0$ in $y = -h$ and $y = h$.

The parallel flow solution reads :

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where $a = \frac{\partial p}{\partial x} < 0$.

Example : 2D Poiseuille flow

$$u(y) = -\frac{a\,h^2}{2\rho\nu}\left(1-\left(\frac{y}{h}\right)^2\right), \qquad v=0, \qquad w=0, \quad a=\frac{\partial p}{\partial x}<0$$

Maximum velocity : $U_0 = u(y=0) = -\frac{a\,h^2}{2\rho\nu}$. Pressure : $P_0(x) = p_{\mathrm{ref}} + a\,x$

Dimensionless equations : we choose U_0 as a characteristic velocity, h as a characteristic length, and ρU_0^2 as a characteristic pressure.

$$[x, y, z] = h\,[x^\star, y^\star, z^\star], [u, v, w] = U_0\,[u^\star, v^\star, w^\star], t = \frac{h\,t^\star}{U_0}, p = p^\star \rho U_0^2$$

$$\begin{aligned} \frac{\partial \boldsymbol{v}^\star}{\partial t^\star} + (\boldsymbol{v}^\star \cdot \boldsymbol{\nabla}^\star)\boldsymbol{v}^\star &= -\boldsymbol{\nabla}^\star p^\star + \frac{1}{\mathrm{Re}}\Delta^\star \boldsymbol{v}^\star \\ \boldsymbol{\nabla}^\star \cdot \boldsymbol{v}^\star &= 0 \end{aligned}$$

with $u^\star = v^\star = w^\star = 0$ in $y^\star = -1$ and $y^\star = 1$, $\mathrm{Re} = \frac{U_0 h}{\nu}$, and $a^\star = \frac{a\,h}{\rho U_0^2} = -\frac{2\,\rho\nu U_0}{h^2}\,\frac{h}{\rho U_0^2} = -\frac{2}{\mathrm{Re}}$.

Parallel flow solution : $\boldsymbol{v_0} = (1 - {y^\star}^2)\boldsymbol{e_x} \equiv U(y)\boldsymbol{e_x}$