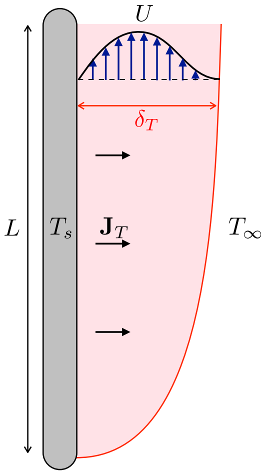
Transport Phenomena:

6. Natural convection & Convective instabilities

Take home message:



Thermal expansion coefficient:

$$\rho = \rho_0\left(1-\beta(T-T_0)\right)$$

Rayleigh number:

$$\mathrm{Ra} = \frac{\beta g L^3 \Delta T}{\nu\alpha}$$

Heat transfer coefficient:

$$\mathbf{J}_T = -h_T(T_\infty - T_s)$$

Nusselt number:

$$\mathrm{Nu} = \frac{h_T L}{\kappa} \sim \mathrm{Ra}^{1/4} \quad \text{(vertical plate, } \mathrm{Re}_x \text{>>1)}$$

different expressions for other configurations

same formalism for mass transport

1 Dissolution of a lollipop

How long does it take to dissolve a lollipop in a bath of water? About 200 min, following an experiment conducted with a lollipop of initial radius $R_0 = 3\,\mathrm{cm}$ (Fig. 1). The dissolution mechanism relies on a self-induced convective flow. A lollipop is basically composed of sugar. In contact with water, sugar tends to dissolve and reaches a saturating (mass) concentration c_{sat} in the vicinity of the lollipop. As a consequence, the solution is denser close to the lollipop than far away, which results in a convective flow.

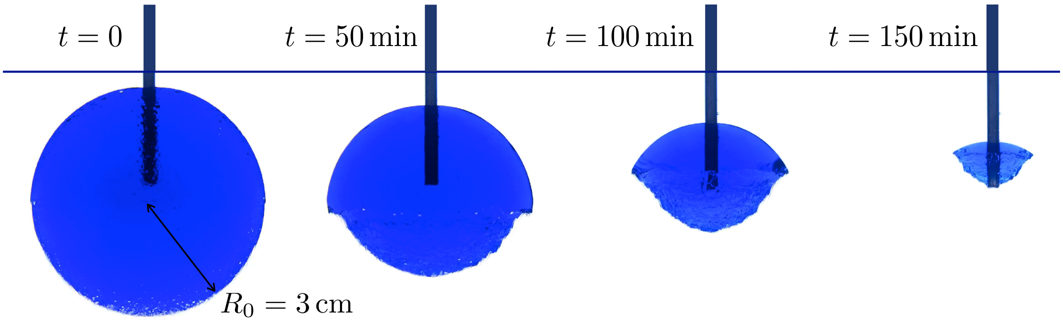


Figure 1: Successive snapshots of the dissolution of a lollipop in a bath of water (from M.S. Davies Wykes, *Physical Review Fluids*, **3**, 043801 (2018)).

The liquid density varies linearly with the sugar mass concentration c: $\rho = \rho_0 + c$. What is the analogous of Rayleigh number in the problem? Estimate the mass flux through the surface of the lollipop.

Using mass conservation, predict the time evolution of the radius of the lollipop $R(t)$ (assuming that the lollipop remains roughly spherical). Estimate the dissolution time of the lollipop t_f. Is the prediction for the time evolution in agreement with the experiment (Fig. 2)?

Numerical data
$R_0 = 3\,\mathrm{cm}$, $c_{sat} = 0.3\,\mathrm{g/cm^3}$, lollipop density $\rho_S = 1.43\,\mathrm{g/cm^3}$, $D = 4.3 \cdot 10^{-10}\,\mathrm{m^2/s}$ (sucrose in pure water, lower in concentrated sucrose solution), $\nu_{water} = 10^{-6}\,\mathrm{m^2/s}$ (for saturated sucrose $\nu_s = 8 \cdot 10^{-4}\,\mathrm{m^2/s}$).

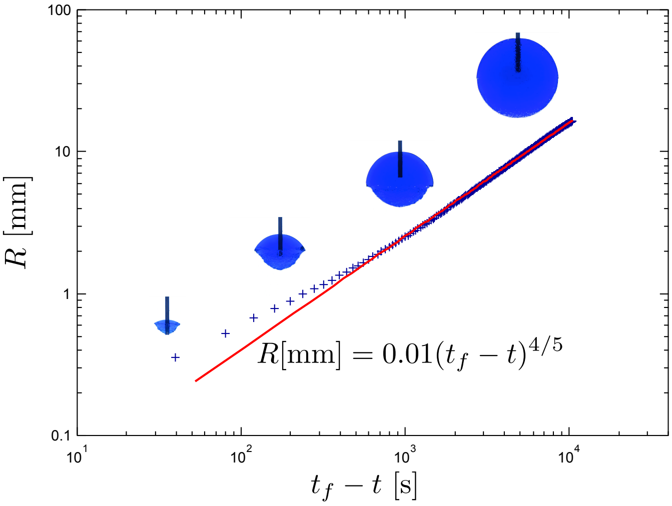


Figure 2: Evolution of the effective radius (estimated from the area of the lollipop in the pictures $\mathcal{A}$ as $R = \mathcal{A}/\pi$) as a function of $t_f - t$ and fit by a power law.

2 Convective instabilities

2.1 Rayleigh-Bénard instability

Here is an experiment you can try at home: pour a millimetric layer of oil in a frying pan, sprinkle it with some fine powder such as flour and place it on a hotplate (Fig. 3a). If the heating temperature is low, nothing happens. However, you should observe the particles moving beyond a critical temperature and gradually forming cellular patterns (Fig. 3c). A pedagogical video describing the phenomenon is available on YouTube: https://youtu.be/Eud7uG5JHng This instability was first described by Rayleigh and Bénard although the exact origin of the observed patterns have been the subject of scientific debates for almost 50 years.

We fist consider the situation where the liquid remains still. If the difference in temperature between bottom and top is θ, what is the temperature profile across the layer of thickness a? What is the resulting profile in density for a fluid of a thermal expansion coefficient β?

Imagine that due to some perturbation, a bubble of fluid of radius R moves by a quantity δz (Fig. 3b). We will first assume that the sphere is thermally isolated from the surrounding liquid. What is the resulting buoyancy force acting on the sphere? Is this situation stable?

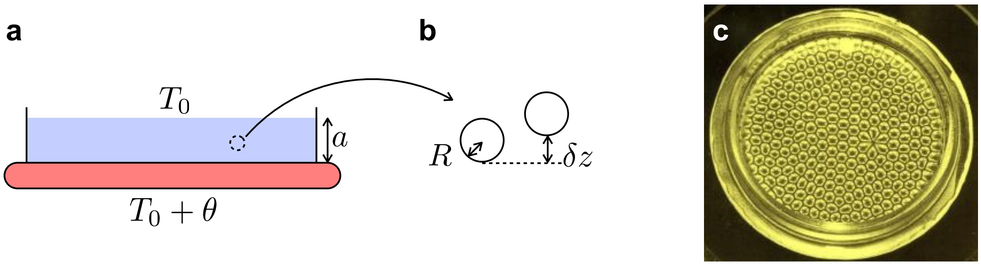


Figure 3: **a.** Liquid layer heated from underneath. **b.** What does happen to sphere of fluid of radius R whose vertical position is perturbed by a quantity δz? **c.** Convection cells in a millimetric layer of viscous oil (image E. Koschmieder).

If we ignore inertial effects, what term balances this force? Deduce an expression of the velocity of the bubble. Integrate the velocity to obtain the position of the bubble as a function of time. What is the characteristic convection time for the bubble?

In reality, the bubble can exchange heat with the surrounding fluid. What would be the diffusion time required to equilibrate the inner temperature of the bubble with the outer temperature?

In which case is the situation unstable? What is the most unstable situation in terms of bubble size?

2.2 A controversy for 50 years: Bénard-Marangoni instability

Although the mechanism proposed by Rayleigh describes correctly the experiments from Bénard in the case of closed cells, it generally overestimates the threshold for the observation of cellular patterns in the case of free interfaces. It took 50 years of scientific discussions to put in evidence a different scenario involving gradients in surface tension.
Surface tension of liquids tends to slightly decrease with temperature, leading to a variation of the form:

$$\gamma(T) = \gamma(T_0)\,(1 - \beta_T(T - T_0))$$

Imagine that due to some perturbation, a spot a the surface of the liquid gets warmer than the surrounding fluid. What would be the effect this perturbation to the distribution of surface tension at the surface of the fluid? What type of flow would the perturbation generate at the surface? across the thickness of the layer? Does the situation seems stable in the absence of thermal diffusion?

In the presence of a convection roll of size a (Fig. 4), what is the typical temperature gradient across the roll? What is the corresponding surface stress? What is the flow velocity when this surface stress is balanced by viscous stress? Estimate a typical convection time.

Again, thermal diffusion tends to smooth down temperature heterogeneities. What is the diffusion time at the scale of a roll? What is the condition for the roll to persist?

This second mechanism referred to as Bénard-Marangoni instability is crucial in coating

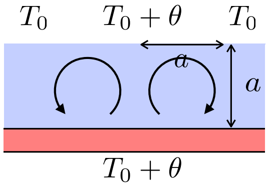


Figure 4: Convection roll across the thickness of the layer.

processes as it may lead to uneven surfaces. If we compare the dependance of the thresholds for both mechanism as a function of the thickness of the layer, which type of instability is likely to appear first?