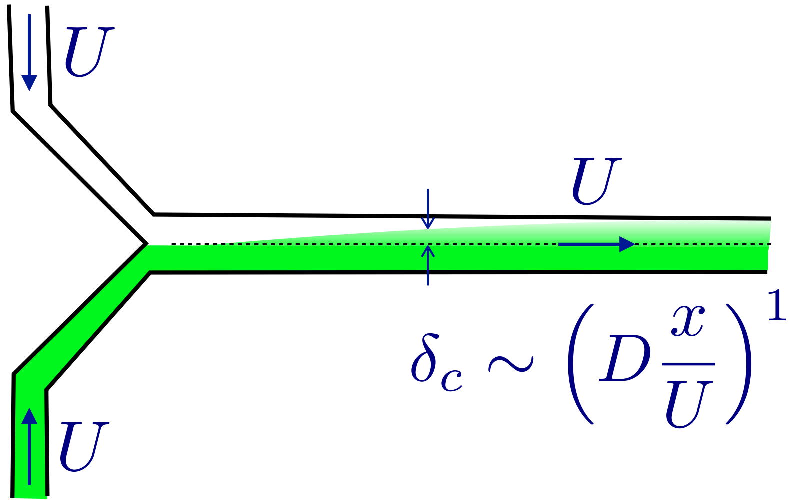
Transport phenomena

3. Forced convection

3.0 Coupling flow with heat/mass exchange

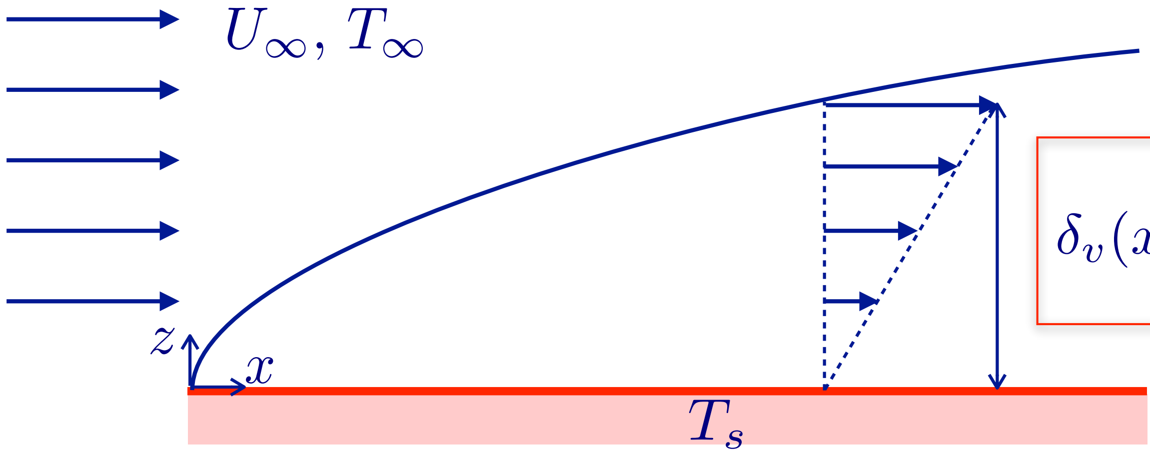
$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} c = D\,\Delta c + r$$

Simple configuration: U uniform



$$\delta_c \sim \left(D\frac{x}{U}\right)^{1/2}$$

3.1 Cooling flow over a plate



$$\delta_v(x) \sim \left(\nu \frac{x}{U_\infty}\right)^{1/2}$$

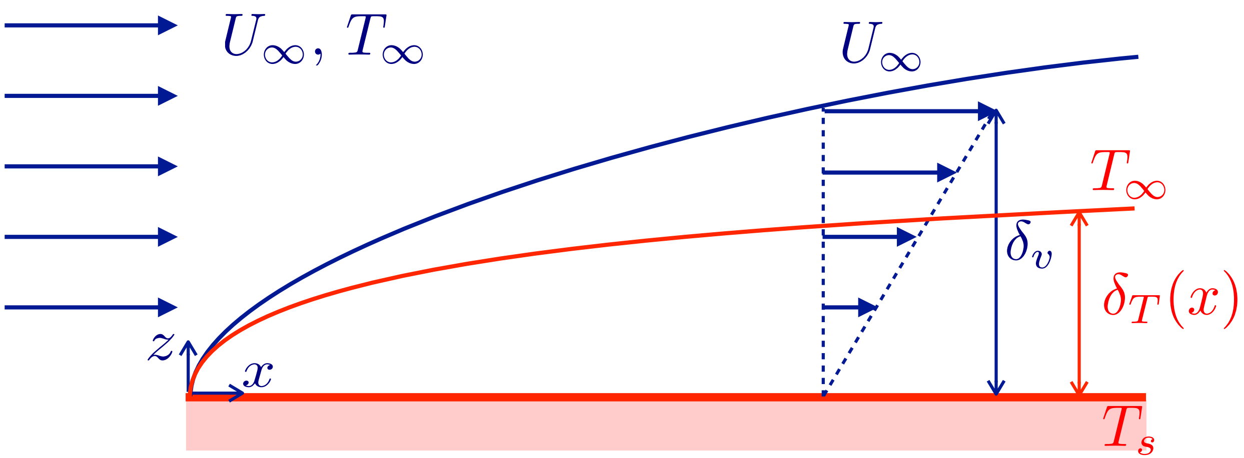
incompressible flow: $\boldsymbol{\nabla} \cdot \mathbf{u} = 0 = \dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_z}{\partial z} \quad \Rightarrow \quad \dfrac{U_\infty}{x} \sim \dfrac{u_z}{\delta_v}$

$$u_z \sim \frac{\delta_v}{x} U_\infty \ll U_\infty \quad \text{for } x \gg \delta_v$$

velocity profile inside the boundary layer:

$$u_x(x, z) \sim \frac{U_\infty}{\delta_v(x)} \, z$$

3.1.1 Thermal boundary layer (hypothesis: $\delta_T < \delta_v$)



heat equation:

$$\cancel{\frac{\partial T}{\partial t}} + \mathbf{u} \cdot \nabla T = \alpha \Delta T + \cancel{\frac{r}{\rho C_p}}$$

steady

$$\mathbf{u} \cdot \nabla T \simeq u_x \frac{\partial T}{\partial x}$$

in scaling laws:

$$\frac{U_\infty}{(\nu\, x / U_\infty)^{1/2}} \delta_T \frac{T}{x} \sim \alpha \frac{T}{\delta_T^2}$$

$$\Rightarrow \boxed{\delta_T(x) \sim \alpha^{1/3}\nu^{1/6}\left(\frac{x}{U_\infty}\right)^{1/2} \sim \delta_v(x)\left(\frac{\alpha}{\nu}\right)^{1/3} = \delta_v(x)\,\mathrm{Pr}^{-1/3}}$$

Prandtl number $\boxed{\mathrm{Pr} = \frac{\nu}{\alpha}}$

water ~ 7

air ~ 1

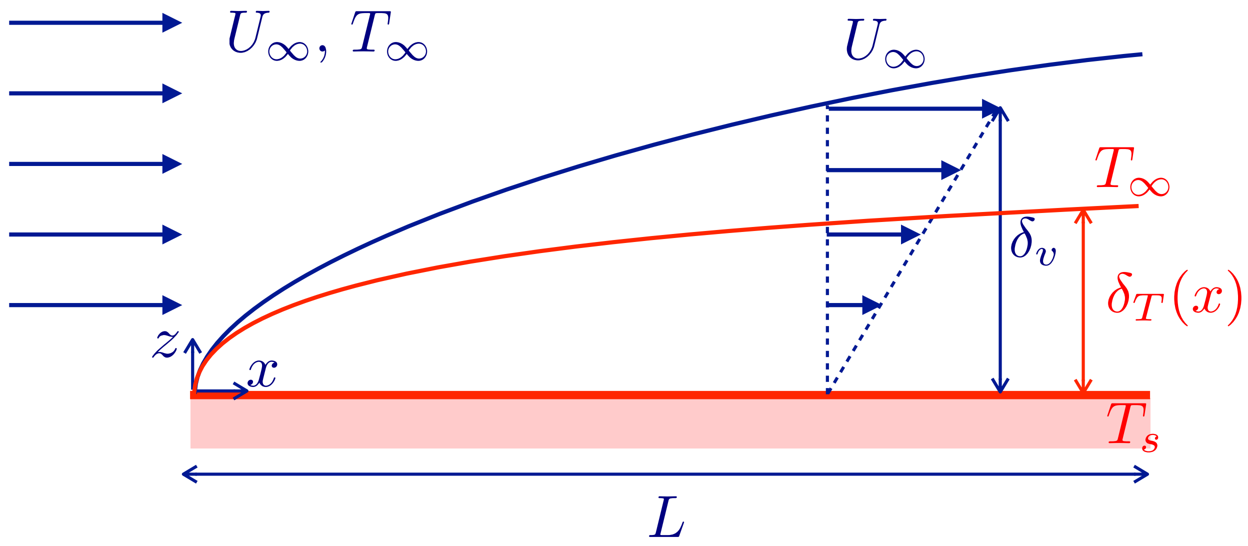
Hg ~ 0.02 ⇒ different behavior

glycerin ~ 1000

$\delta_T < \delta_v$ ~ OK, if $\mathrm{Pr} > 0.6$

Schmidt number for mass transport $\boxed{\mathrm{Sc} = \frac{\nu}{D}}$

3.1.2 Heat transport across the thermal boundary layer



Heat flux $$J_T = -\kappa \frac{\partial T}{\partial z} \sim \kappa \frac{T_\infty - T_s}{\delta_T(x)}$$

Average flux $$< J_T > \sim -\frac{1}{L} \int_0^L \kappa (T_\infty - T_s) \frac{U_\infty^{1/2}}{\alpha^{1/3} \nu^{1/6} x^{1/2}} dx$$

$$< J_T > \sim -\kappa \frac{T_\infty - T_S}{L} \left(\frac{U_\infty L}{\nu} \right)^{1/2} \left(\frac{\nu}{\alpha} \right)^{1/3} = \kappa \frac{T_\infty - T_S}{L} \mathrm{Re}^{1/2} \mathrm{Pr}^{1/3}$$

Heat transfer coefficient $J_T = -h_T(T_\infty - T_s)$ $h_T \sim \frac{\kappa}{\delta_T}$

Nusselt number $\mathrm{Nu} = \frac{h_T L}{\kappa} \sim \frac{L}{\delta_T}$

Forced convection over a plate $\mathrm{Nu} \simeq 0.33\,\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}$

Other expressions for different geometries $\mathrm{Nu} = f(\mathrm{Re},\,\mathrm{Pr})$

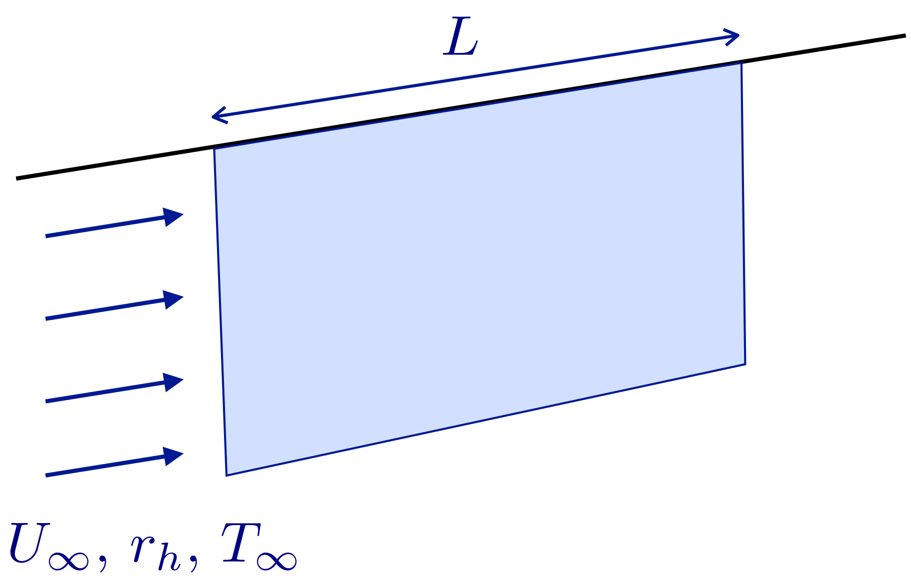
Same formalism for mass transport:

Mass transfer coefficient $J_m = -h_m(c_\infty - c_s)$ $h_m \sim \frac{\kappa}{\delta_c}$

Sherwood number $\mathrm{Sh} = \frac{h_m L}{D}$

3.2 Practical exemples

3.2.1 Drying a sheet in the wind

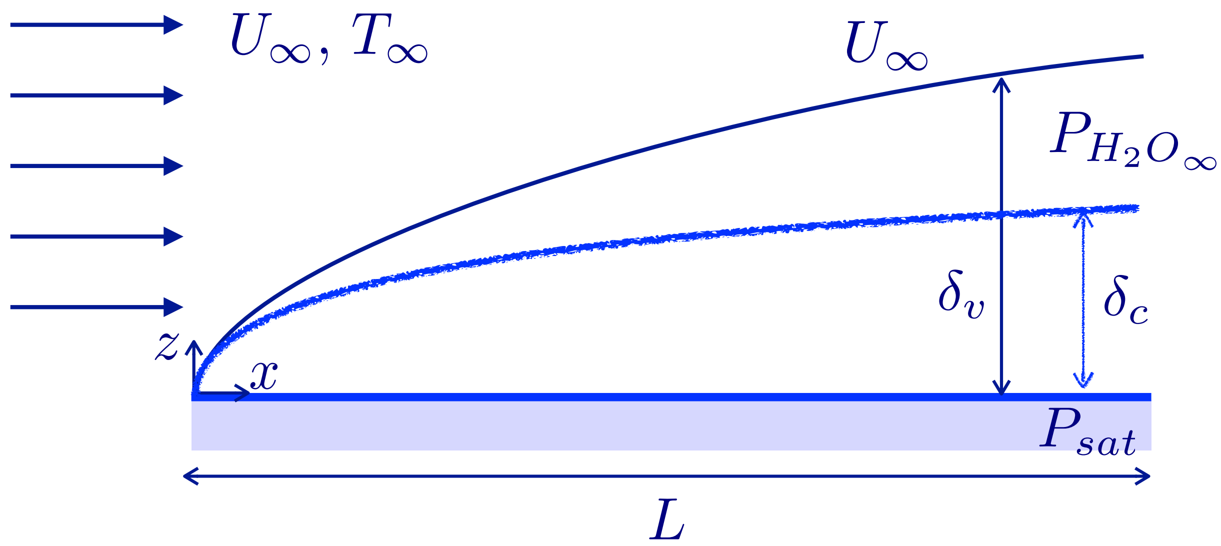


Simplified model

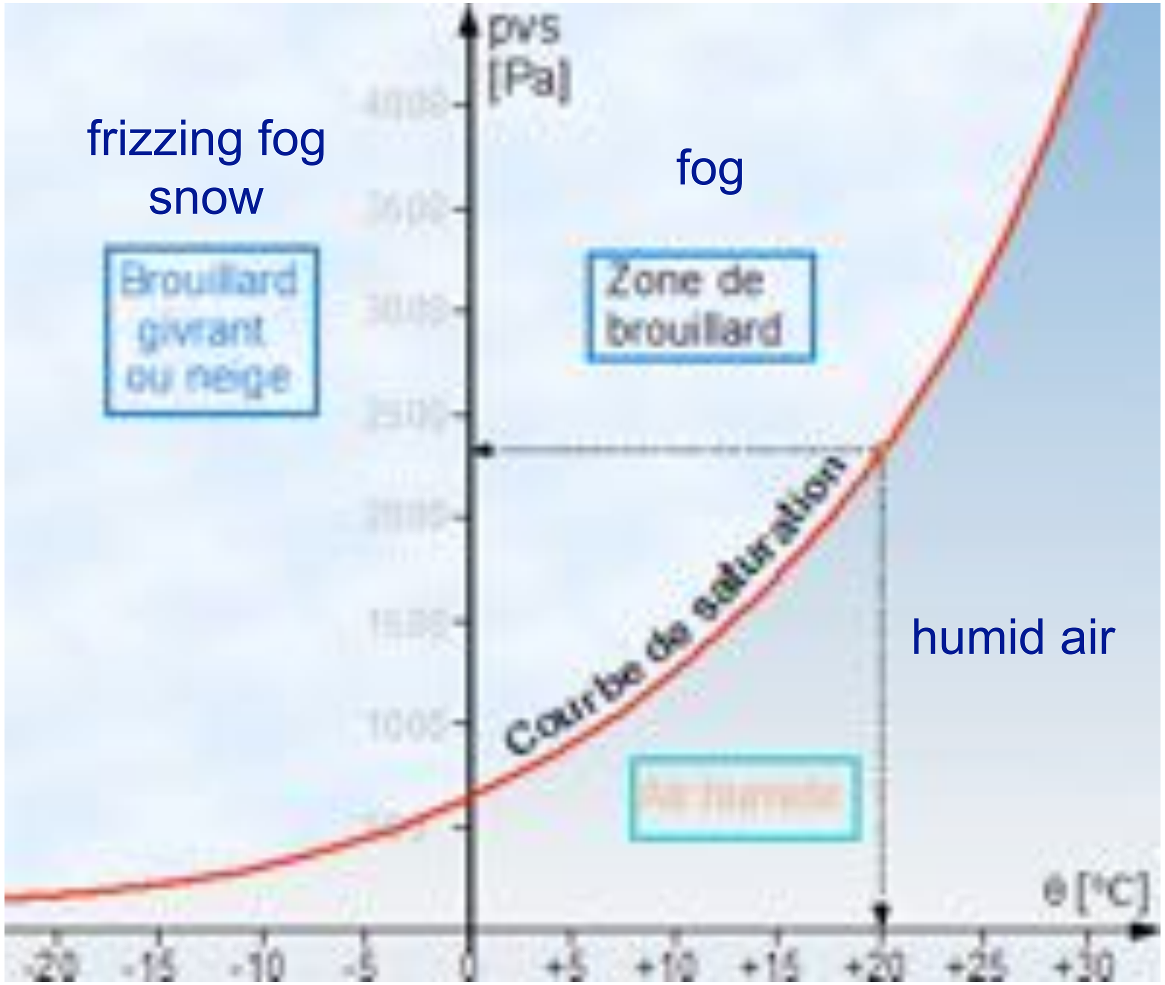
- uniform temperature
- radiative exchanges neglected
- P_{H_2O} at the surface = P_{sat}

relative humidity r_h

$$P_{H_2O_\infty} = r_h \, P_{sat}$$



Vapor pressure



source: dimclim.fr

$$P_{H_2O} \leftrightarrow c_{H_2O}? \qquad PV = nRT \Rightarrow c_{H_2O} = \frac{P_{H_2O}}{RT}$$

mol.m^{-3}

H$_2$O molar flux: $J_{H_2O} = h\, c_{sat}(1 - r_h)$

$$\frac{D}{L}\mathrm{Sh} \sim \frac{D}{L}\left(\frac{U_\infty L}{\nu}\right)^{1/2}\left(\frac{\nu}{D}\right)^{1/3}$$

fine to call it "Nu" !

$$J_{H_2O} \sim \frac{D}{L}\left(\frac{U_\infty L}{\nu}\right)^{1/2}\left(\frac{\nu}{D}\right)^{1/3} c_{sat}(1 - r_h) \qquad [\mathrm{mol.m^{-2}.s^{-1}}]$$

if r_h fixed, $T \nearrow \Rightarrow c_{sat} \nearrow \Rightarrow$ higher evaporation

mass flux: $J_{m_{H_2O}} = M_{H_2O} J_{H_2O}$ [kg.m^{-2}.s^{-1}]

volumic flux: $J_{v_{H_2O}} = \frac{1}{\rho_{H_2O}} J_{m_{H_2O}}$ [m.s^{-1}]

Numerical estimate: L = 1m U_∞~ 5 m/s

$D_{H_2O/air}$ = 2,6.10^{-5} m^2/s ν_{air} = 1,5.10^{-5} m^2/s

P_{sat} = 2.3 kPa $\Rightarrow$ c_{sat} = 0.9 mol/m^3

$$J_{H_2O} \sim 0.03\,(1 - r_h) \text{ [mol.m}^{-2}\text{.s}^{-1}]$$

(with prefactor 0.33 for Sh)

$$J_{m_{H_2O}} \sim 7.10^{-5}(1 - r_h) \text{ [kg.m}^{-2}\text{.s}^{-1}]$$

$$J_{v_{H_2O}} \sim 0.07\,(1 - r_h) \text{ [µm/s]}$$

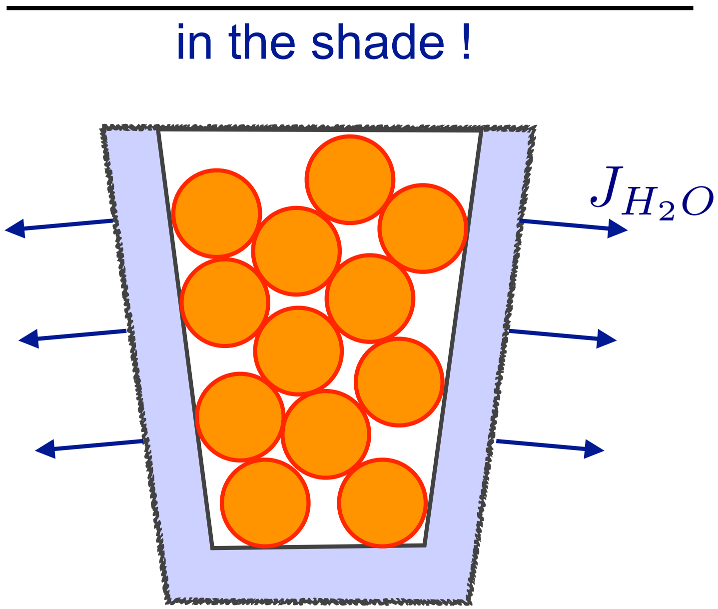
to evaporate 0.5 kg/m^2 of water (i.e. 0.5 mm) with r_h = 50%

$$\tau_{evap} \sim \frac{0.5 \cdot 10^{-3}}{2 \times 0.07 \cdot 10^{-6} \times 0.5} \sim 2\text{h}$$

2 faces

probably underestimated: evaporation ⇒ cooling down ⇒ $c_{sat} \searrow \Rightarrow J_{H_2O} \searrow$

3.2.2 Clay pot cooling



evaporation ⇒ latent heat

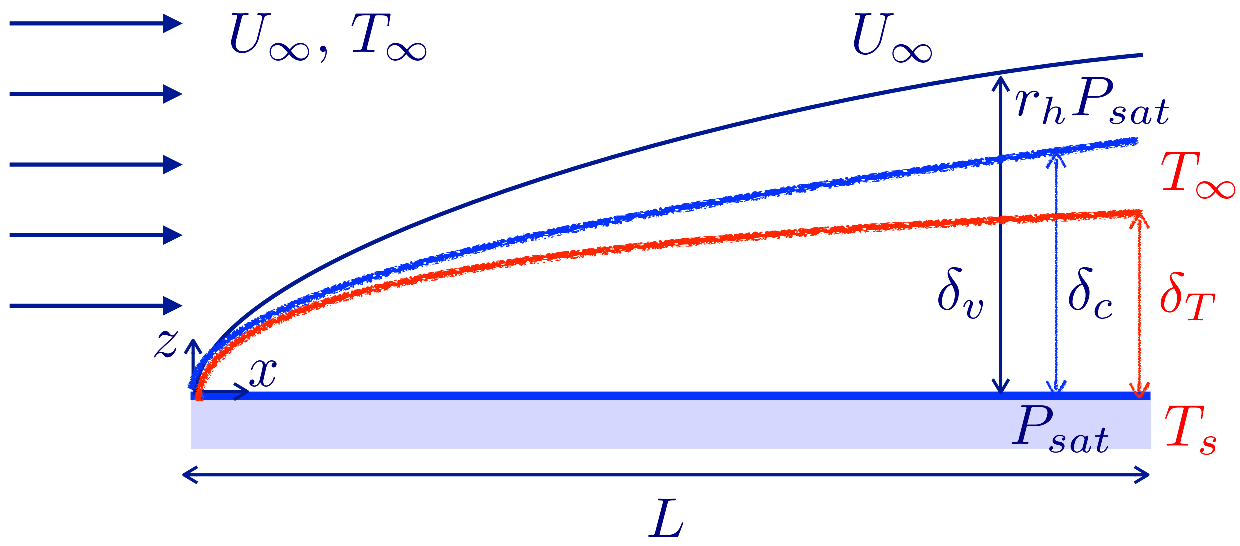
heat flux at the surface

$$J_{latent} = L_{vap}\, J_{m_{H_2O}}$$

$$J_{latent} \sim 150\,(1 - r_h)\text{W/m}^2$$

standard fridge ~ 30 to 50 W

3.2.3 Thermal balance



heat flux from the fluid to the surface

$$J_T \sim \frac{\kappa}{L}\left(\frac{U_\infty L}{\nu}\right)^{1/2}\left(\frac{\nu}{\alpha}\right)^{1/3}(T_\infty - T_s)$$

heat flux from latent heat

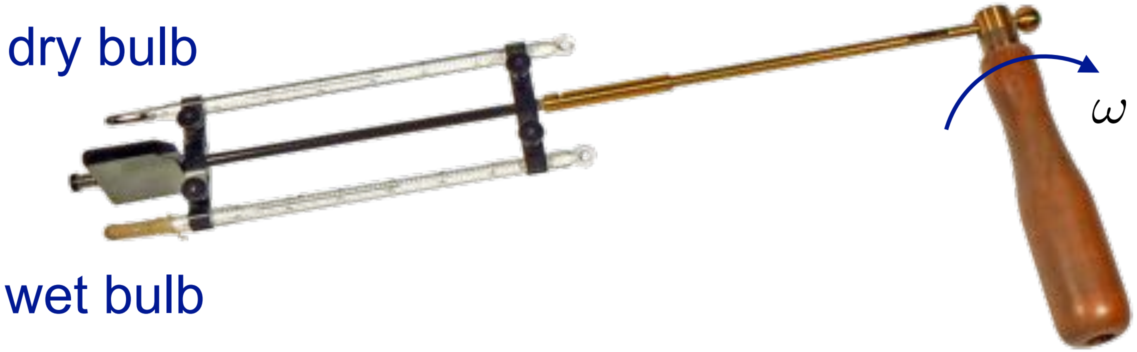
$$J_{latent} \sim L_{vap} M_{H_2O}\,\frac{D}{L}\left(\frac{U_\infty L}{\nu}\right)^{1/2}\left(\frac{\nu}{\alpha}\right)^{1/3} c_{sat}(1 - r_h)$$

$$\boxed{\Delta T \sim L_{vap} M_{H_2O}\,\frac{D}{\kappa}\left(\frac{\alpha}{D}\right)^{1/3} c_{sat}(1 - r_h)}$$ independent of U_∞

Measuring relative humidity

Numerical estimate: κ_{air} = 0.03 J.m^{-1}.K^{-1} α_{air} = 1.9 10^{-5} m^2/s

psychrometer



measure $\Delta T \Rightarrow r_h$

$$\Delta T \simeq 27^\circ\mathrm{C}(1 - r_h)$$

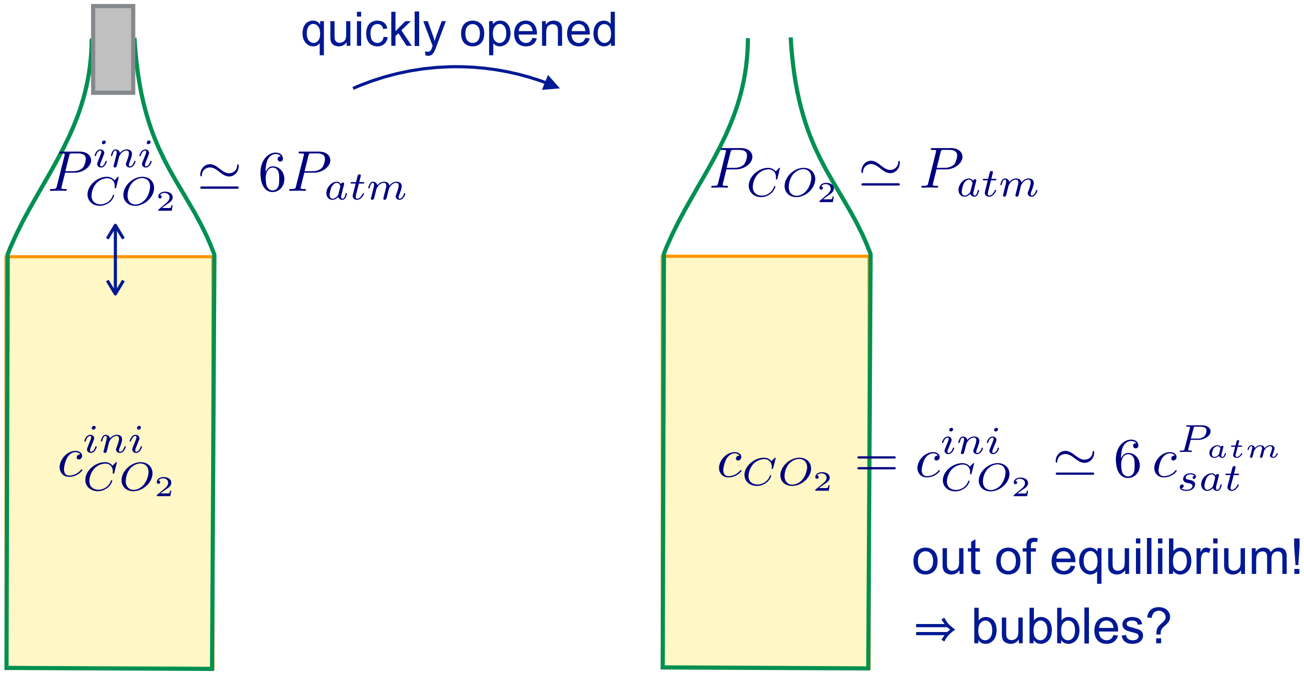
3.3 Bubbles in a champagne bottle

1294 *Langmuir* **2002,** *18,* 1294−1301

Source:

Kinetics of Gas Discharging in a Glass of Champagne: The Role of Nucleation Sites

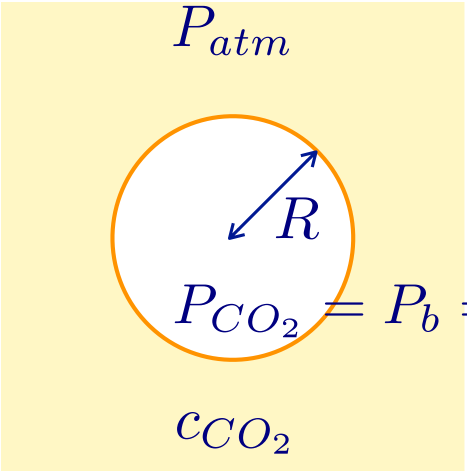
Gérard Liger-Belair,[*,†] Michèle Vignes-Adler,[‡] Cédric Voisin,[†]
Bertrand Robillard,[§] and Philippe Jeandet[†]



Henry's law: $c_{CO_2} = H P_{CO_2}$ at equilibrium

↳ constant depending on the system

3.3.1 Bubbles nucleation



$$P_{CO_2} = P_b = P_{atm} + 2\gamma/R$$

Homogeneous nucleation ?

bubble grows if $c_{CO_2} > c_{sat}^{P_b} = c_{sat}^{P_{atm}} \dfrac{P_b}{P_{atm}}$

ie: $$\frac{P_{CO_2}^{ini}}{P_{atm}} c_{sat}^{P_{atm}} > \left(1 + \frac{2\gamma}{RP_{atm}}\right) c_{sat}^{P_{atm}}$$

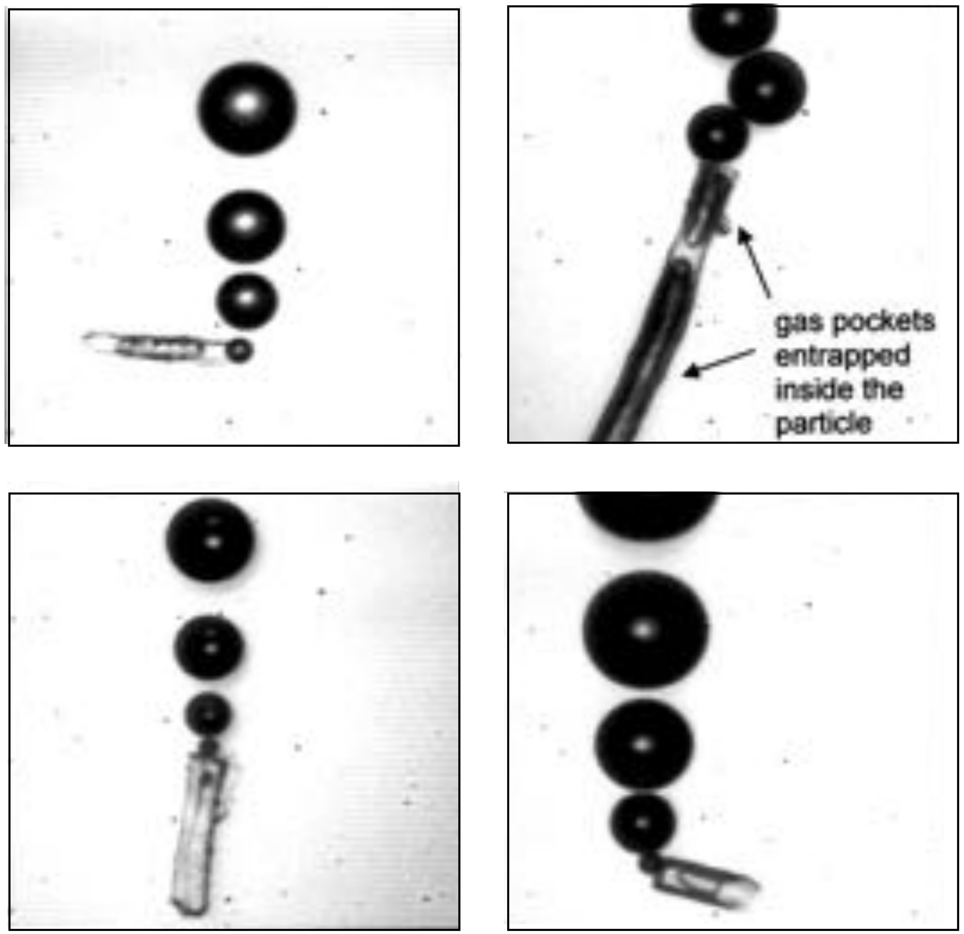
$$\boxed{R > \frac{2\gamma}{P_{CO_2}^{ini} - P_{atm}}}$$

numerical estimate: $\gamma \simeq$ 50 mN/m $\Rightarrow R_{min} \sim$ 0.5 µm

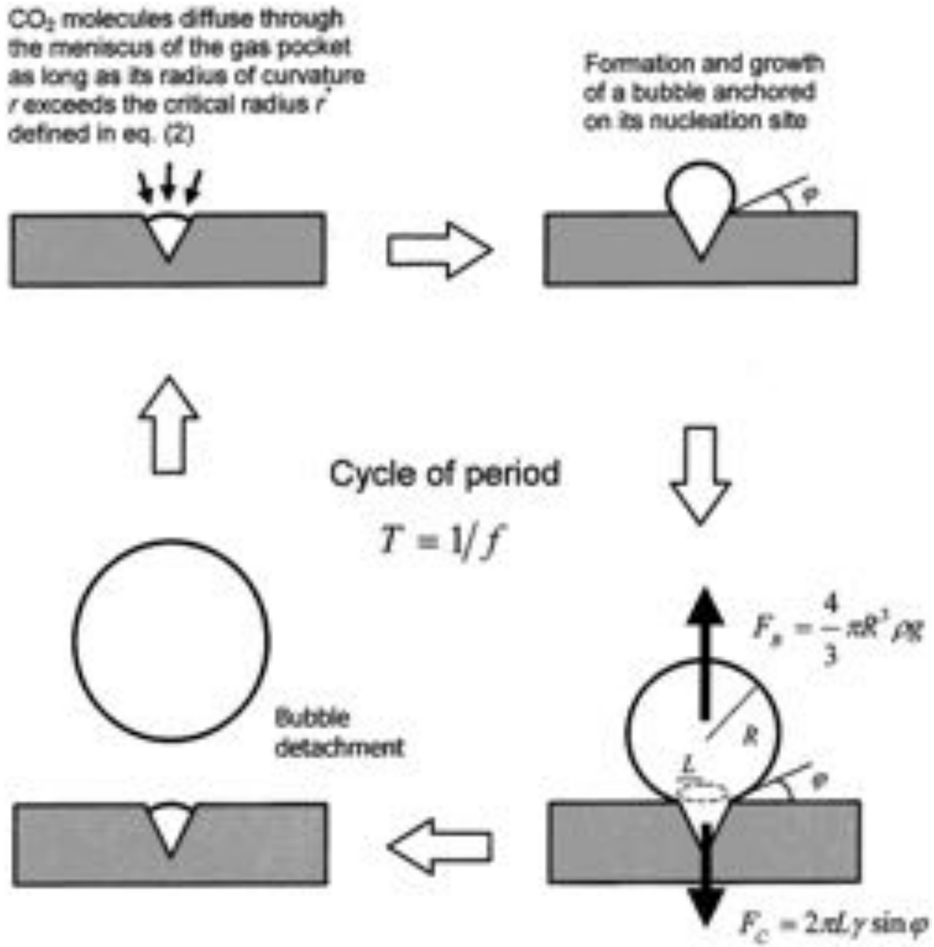
⇒ heterogenous nucleation

Heterogenous nucleation

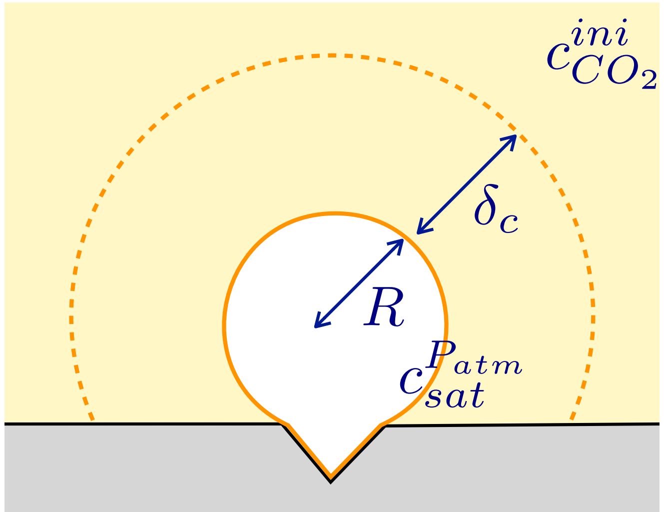
$$R_{init} \gg R_{min}$$



100 µm



3.3.2 Growth rate



Diffusion process:

$$\delta_c \sim (Dt)^{1/2}$$

$$J_{CO_2} \sim \frac{D}{\delta_c}\left(c_{CO_2}^{ini} - c_{sat}^{P_{atm}}\right)$$

[molecules/m³] in article

$$\frac{d\mathcal{V}}{dt} = 4\pi R^2 \frac{dR}{dt} = J_{CO_2} 4\pi R^2 \frac{1}{\mathcal{N}_a}\frac{RT}{P}$$

volume 1 mole of gas

$$\frac{dR}{dt} \sim \left(\frac{D}{t}\right)^{1/2}\left(c_{CO_2}^{ini} - c_{sat}^{P_{atm}}\right)\frac{1}{\mathcal{N}_a}\frac{RT}{P}$$

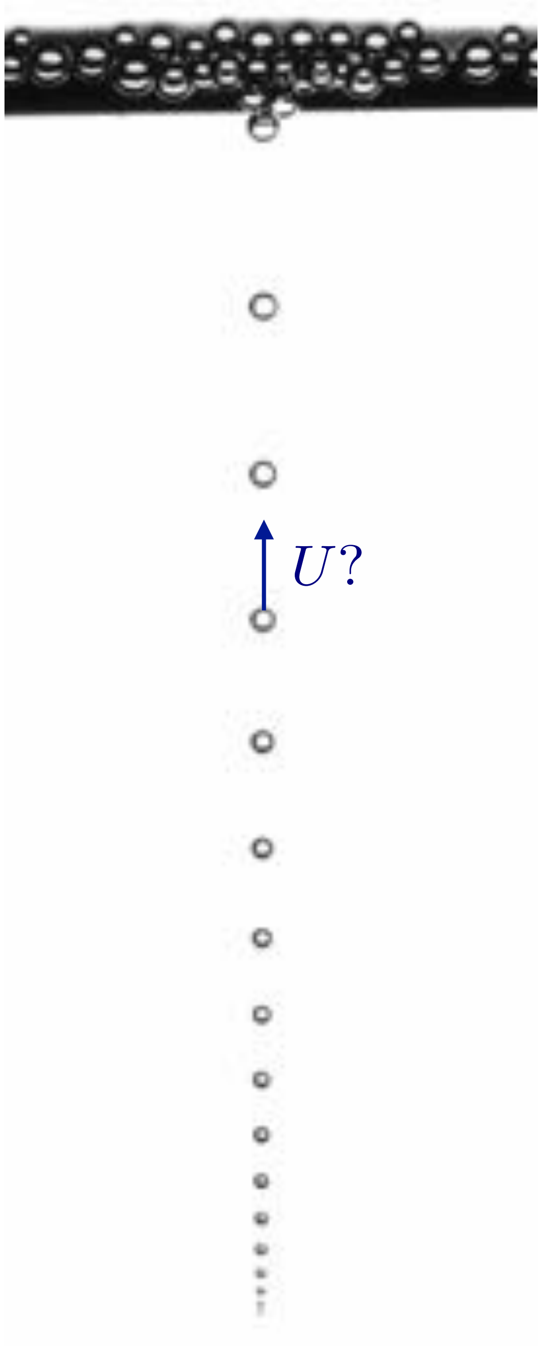
$$R(t) - R_{ini} \sim \left(c_{CO_2}^{ini} - c_{sat}^{P_{atm}}\right)\frac{1}{\mathcal{N}_a}\frac{RT}{P}(Dt)^{1/2}$$

numerical estimate:

$c_{CO_2}^{ini} \simeq$ 1,6.10^{26} molecules/m^3 $c_{sat}^{P_{atm}} \simeq$ 2,7.10^{25} molecules/m^3 $D_{CO_2} \simeq$ 1,4.10^{-9} m^2/s

$R_{detach} \sim$ 20 µm $\Rightarrow \tau_{growth} \sim$ 10 ms a bit overestimated: $\tau_{exp} \sim$ 40 ms

3.3.2 Rising velocity



3.3.2 Rising velocity

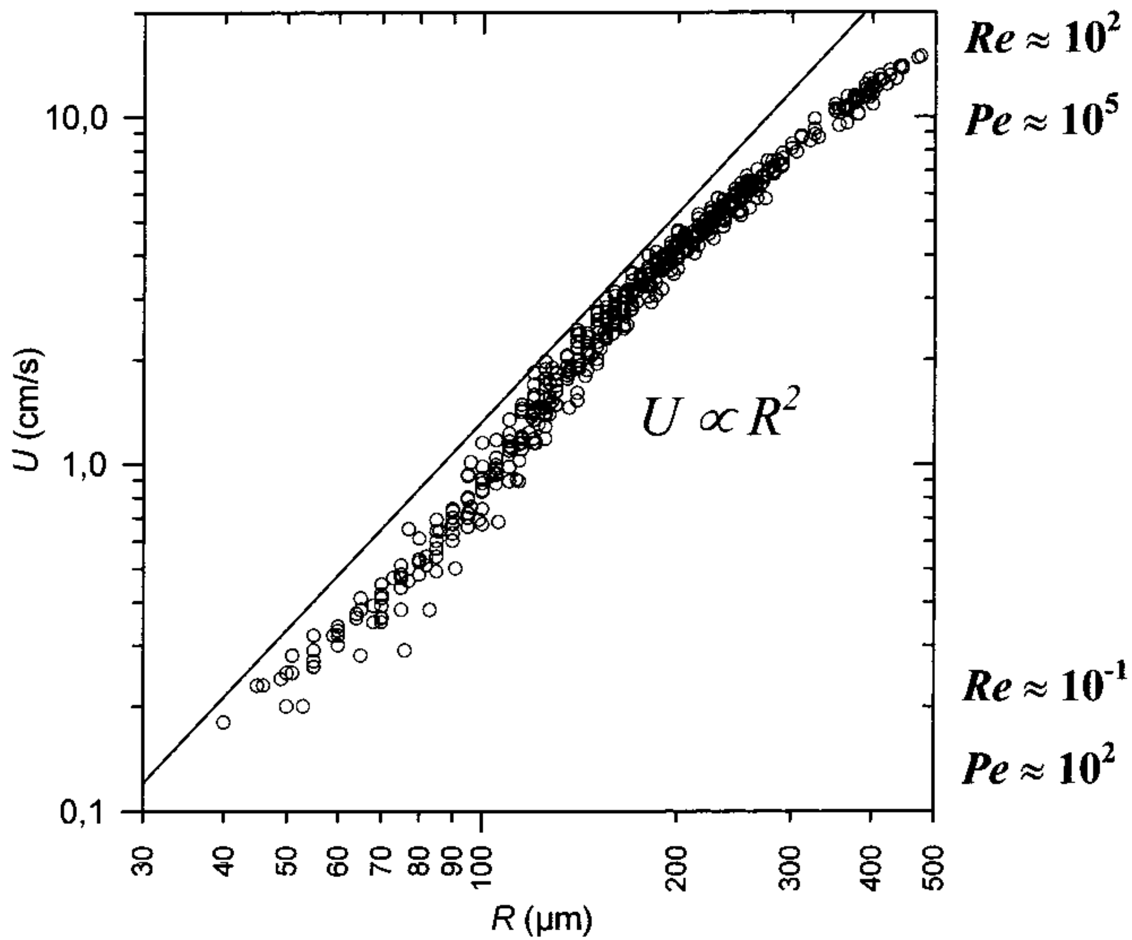
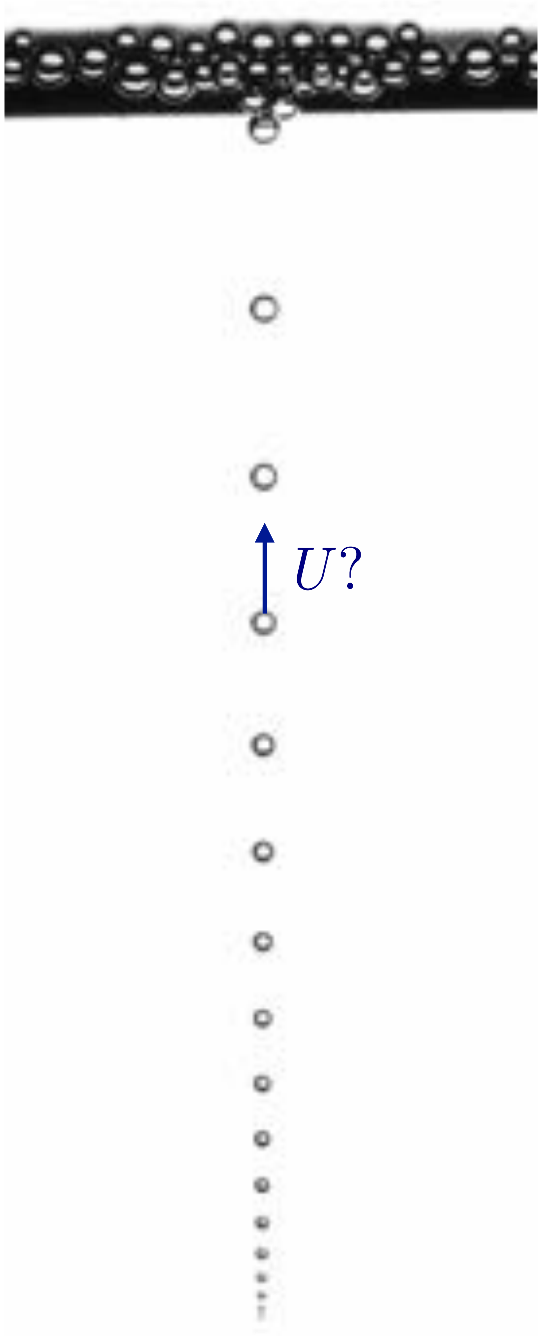


Figure 5. Velocity of ascending champagne bubbles $U(R)$ (○), compared with the Stokes velocity (—). Re and Pe are respectively the Reynolds and Peclet numbers associated with a rising bubble.

3.3.2 Rising velocity



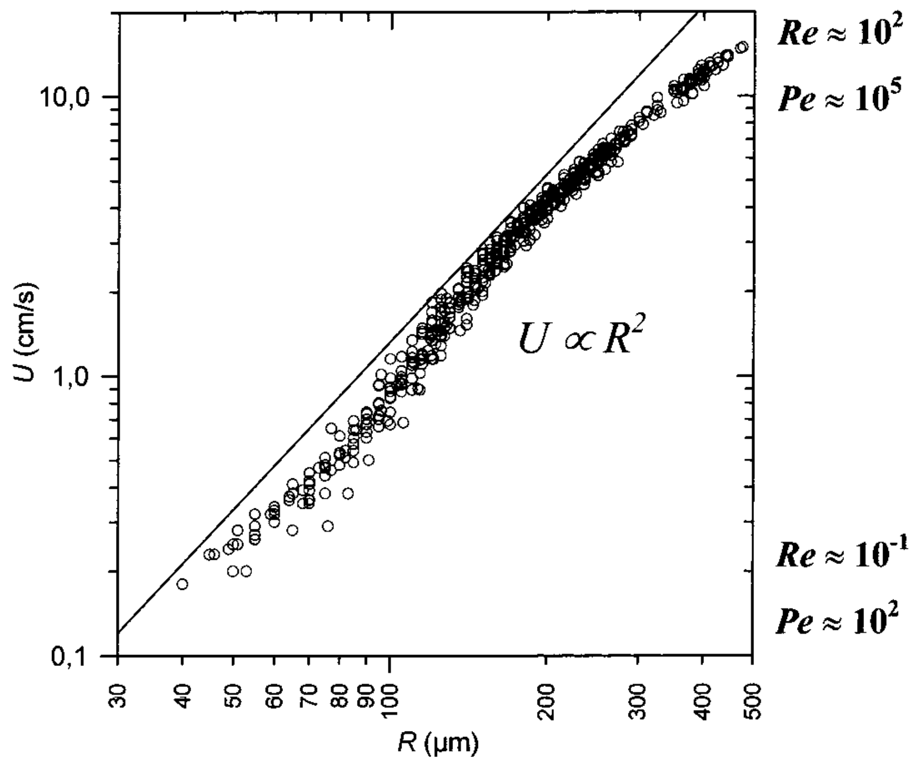


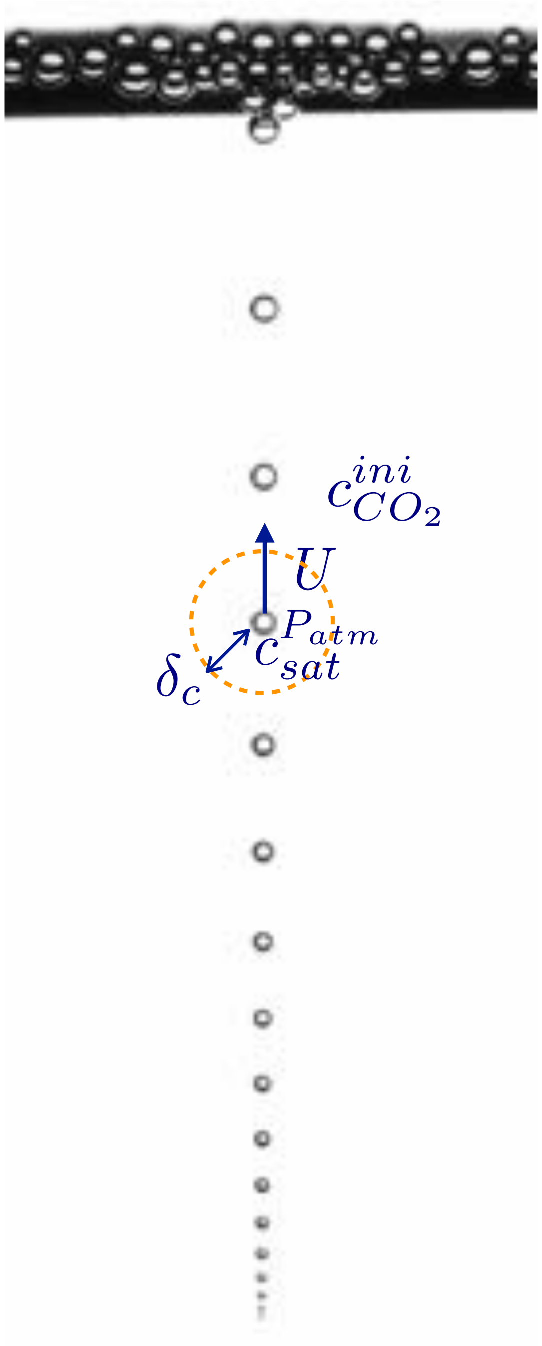
Figure 5. Velocity of ascending champagne bubbles $U(R)$ (○), compared with the Stokes velocity (—). Re and Pe are respectively the Reynolds and Peclet numbers associated with a rising bubble.

Limit low Reynolds number (~OK at the beginning)

$$\frac{4}{3}R^3\Delta\rho g = 6\pi\eta R U$$

$$U \sim \frac{\rho_{liq} g R^2}{\eta}$$

3.3.3 Bubble growth



Mass transfer coefficient?

Flow low *Re* ≠ boundary layer over a plate

in article:

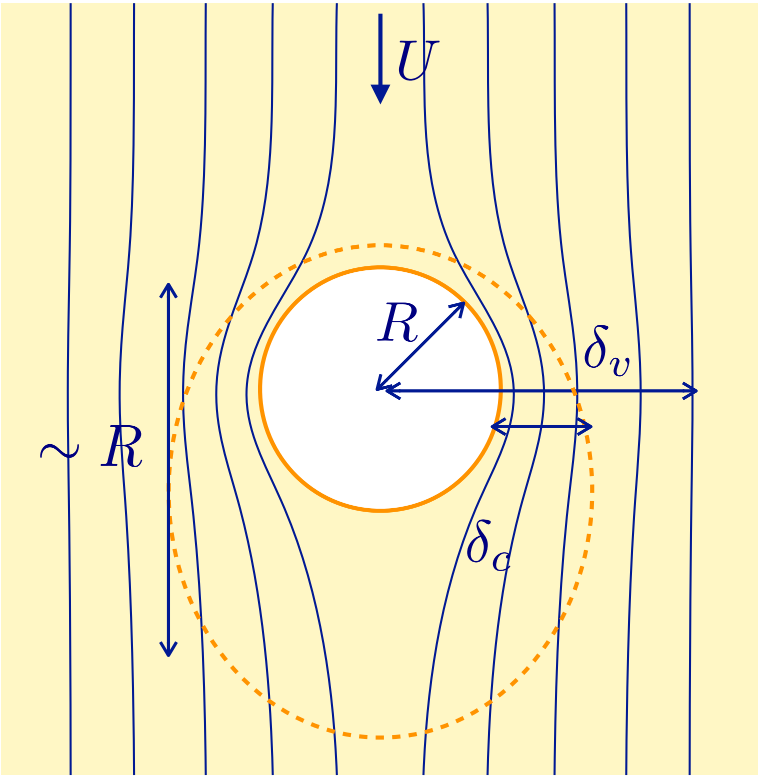
$$\mathrm{Sh} \sim \mathrm{Pe}^{1/3} = \left(\frac{UR}{D}\right)^{1/3} = \left(\frac{UR}{\nu}\right)^{1/3}\left(\frac{\nu}{D}\right)^{1/3}$$

$$\mathrm{Sh} \sim \mathrm{Re}^{1/3}\mathrm{Sc}^{1/3}$$

for heat exchange, this would be:

$$\mathrm{Nu} \sim \mathrm{Re}^{1/3}\mathrm{Pr}^{1/3}$$

Can we justify this form?



Length scale for the flow: $\delta_v \sim R$

Length scale for mass diffusion ?

Analogy derivation boundary layer over a plate

$$\frac{U_\infty}{\delta_v(x)} z \frac{\partial T}{\partial x} \simeq \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

$$\frac{U_\infty}{\cancel{(\nu x / U_\infty)^{1/2}}} \quad \delta_T(x) \frac{T}{x} \qquad \frac{T}{x^2} \qquad \frac{T}{\delta_T^2}$$

$\sim R$ $\qquad x \leftrightarrow R$

with $\delta_T \ll R$

$$\frac{U}{R} \delta_T \frac{T}{R} \sim \alpha \frac{T}{\delta_T^2}$$

$$\delta_T \sim \left(\frac{\alpha R^2}{U} \right)^{1/3} \Rightarrow \mathrm{Nu} \equiv \frac{R}{\delta_T} \sim \left(\frac{UR}{\alpha} \right)^{1/3}$$

for mass transport:

$$\mathrm{Sh} \equiv \frac{R}{\delta_c} \sim \left(\frac{UR}{D} \right)^{1/3}$$

3.3.3 Coupling rising dynamics and mass transport

$$J_{CO_2} \sim h_c \left(c_{CO_2}^{ini} - c_{sat}^{P_{atm}} \right) = \frac{D}{R} \mathrm{Sh} \left(c_{CO_2}^{ini} - c_{sat}^{P_{atm}} \right)$$

$$J_{CO_2} \sim \frac{D}{R} \left(\frac{UR}{D} \right)^{1/3} \left(c_{CO_2}^{ini} - c_{sat}^{P_{atm}} \right)$$

$$\frac{d\mathcal{V}}{dt} = 4\pi R^2 \frac{dR}{dt} = J_{CO_2} 4\pi R^2 \frac{1}{\mathcal{N}_a} \frac{RT}{P}$$

$$\frac{dR}{dt} \sim \frac{D}{R} \left(\frac{\rho g R^3}{\eta D} \right)^{1/3} \left(c_{CO_2}^{init} - c_{sat}^{P_{atm}} \right) \frac{1}{\mathcal{N}_a} \frac{RT}{P}$$

$$\boxed{\frac{dR}{dt} \sim \left(\frac{D^2 g}{\nu} \right)^{1/3} \left(c_{CO_2}^{init} - c_{sat}^{P_{atm}} \right) \frac{1}{\mathcal{N}_a} \frac{RT}{P}}$$

numerical estimate:

$\frac{dR}{dt} \sim$ 1.5 mm/s   experiment ~ 0.2 mm/s (but after pouring in a glass $\Rightarrow c_{CO_2}^{init} \searrow$)

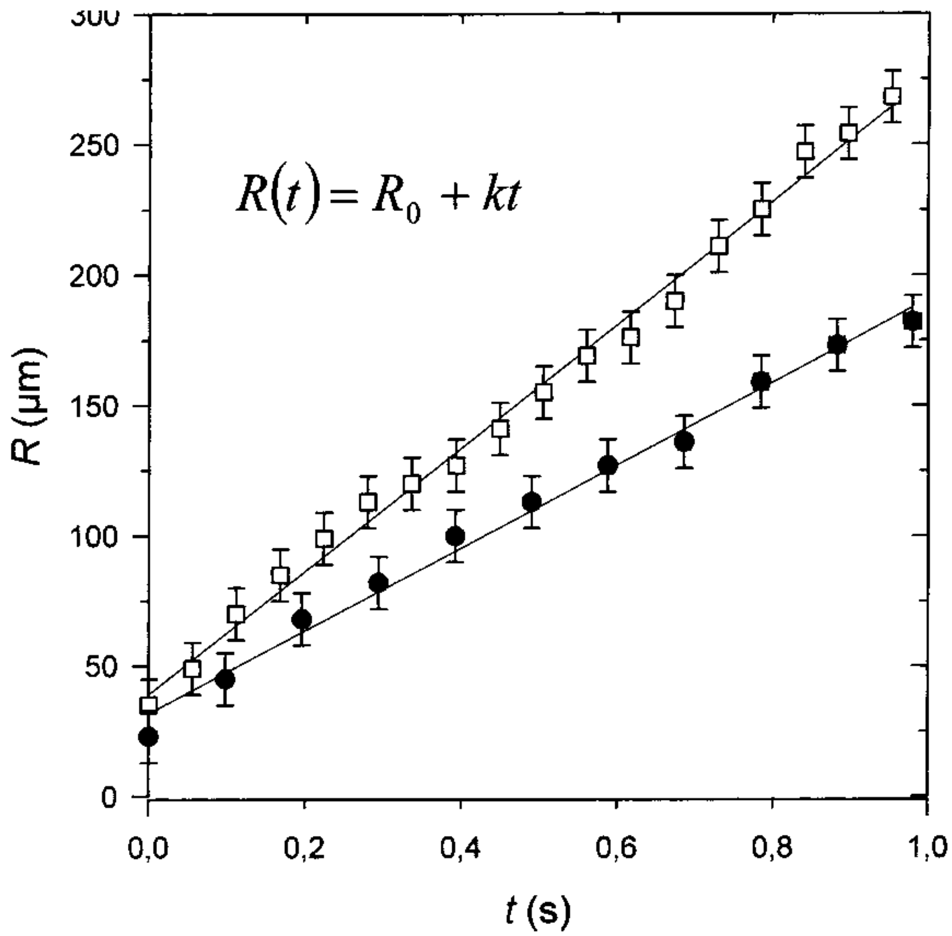


Figure 6. Bubble radius increase vs time for a bubble rising toward the liquid surface. Two typical bubble trains at different steps of gas discharging are considered.