Transport phenomena

2. Solving transport equations

mass:

Diffusion

$$\mathbf{j}_d = -D\boldsymbol{\nabla}c$$

Convection

$$\mathbf{j}_c = c\,\mathbf{u}$$

$$\frac{\partial c}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}c = D\,\Delta c + r$$

D: $[m^2.s^{-1}]$

r: source/sink
eg: chemical reaction

heat:

Conduction: Fourrier's law

$$\mathbf{j}_{cond} = -\kappa \boldsymbol{\nabla} T$$

thermal flux [W.m^{-2}]

thermal conductivity [W.m^{-1}.K^{-1}]

Convection

$$\mathbf{j}_{conv} = \rho C_p \, T \, \mathbf{u}$$

density [kg.m^{-3}]

specific heat capacity [J.kg^{-1}.K^{-1}]

Heat equation:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T = \alpha \Delta T + \frac{r}{\rho C_p}$$

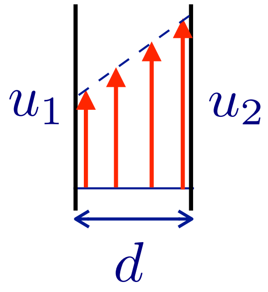
$$\alpha = \kappa / \rho C_p$$

thermal diffusivity [m^2.s^{-1}]

source/sink

eg: Joule effect

momentum:



"Conduction"

$$\boldsymbol{\sigma}_v = \eta(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^{\mathrm{T}})$$

viscous stress [Pa] ~ flux of momentum

dynamic viscosity [Pa.s] ~ momentum "conductivity"

"Convection"

$$\rho\mathbf{u} \otimes \mathbf{u}$$

Navier & Stokes equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \nu\,\Delta\mathbf{u} - \frac{1}{\rho}\boldsymbol{\nabla}p + \mathbf{g}$$

$$\nu = \eta/\rho$$

kinematic viscosity [m^2.s] ~ momentum "diffusion"

boundary conditions: $\mathbf{j}_c.\mathbf{n}$, $\mathbf{j}_T.\mathbf{n}$, $\boldsymbol{\sigma}.\mathbf{n}$, c, T, $\mathbf{u}$ continuous

exception: latent heat

Transport phenomena

2. Solving transport equations

2.1 From mass to heat and viscous stress transport

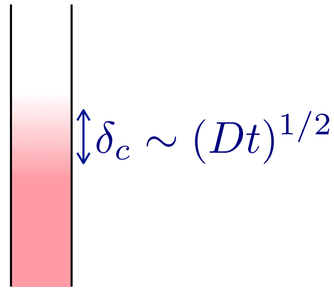
2.2 Homogenization of a cocktail (exact solution?)

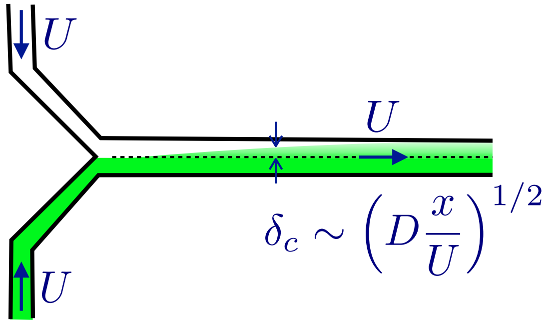
2.3 Measuring a diffusion coefficient ?

2.4 Depth of penetration

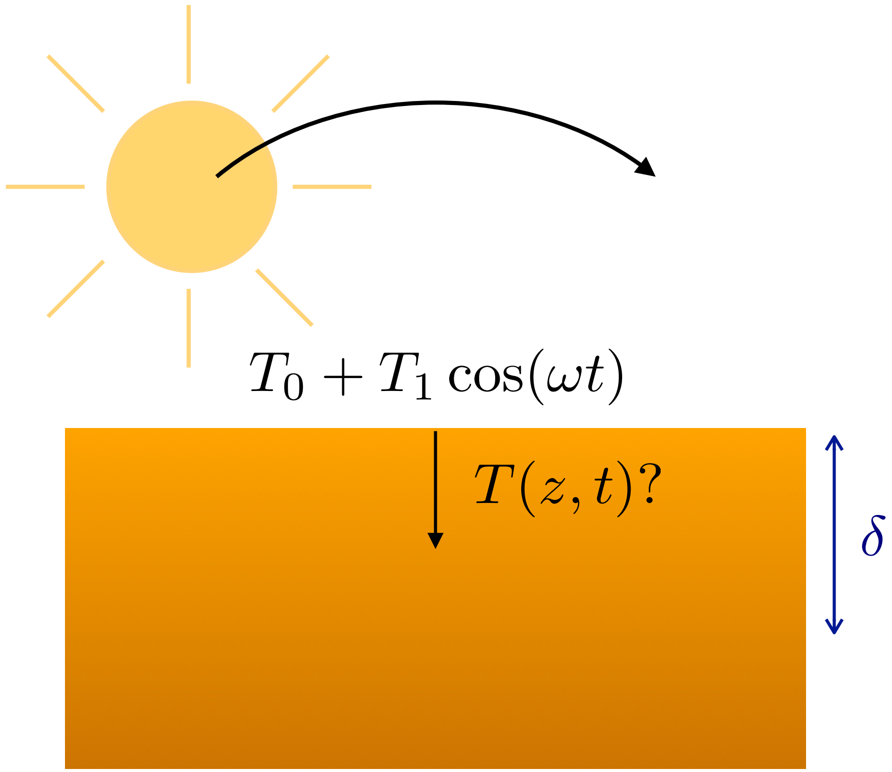
2.5 Contact temperature

2.6 Melting of an ice cube





2.4 Depth of penetration ?



in scaling laws ?

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

$$\omega T \sim \alpha \frac{T}{\delta^2}$$

$$\boxed{\delta \sim (\alpha/\omega)^{1/2}}$$

Detailed solution $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

Solution of the form $T(z,t) = \mathrm{Re}\left(T_0 + T^*(z)e^{i\omega t}\right)$?

$$\frac{\partial T}{\partial t} = i\omega T^*(z)e^{i\omega t} \qquad \frac{\partial^2 T}{\partial z^2} = T^{*\prime\prime}(z)e^{i\omega t}$$

$$T^{*\prime\prime}(z) - i\,\frac{\omega}{\alpha}T^* = 0$$

$$\frac{\omega}{\alpha}e^{i\frac{\pi}{2}}$$

$$\Rightarrow \quad \left(i\,\frac{\omega}{\alpha}\right)^{1/2} = \pm\sqrt{\frac{\omega}{\alpha}}e^{i\frac{\pi}{4}} = \pm\sqrt{\frac{\omega}{2\alpha}}(1+i)$$

$$T^*(z) = A\exp\left(\sqrt{\frac{\omega}{2\alpha}}(1+i)z\right) + B\exp\left(-\sqrt{\frac{\omega}{2\alpha}}(1+i)z\right)$$

$$T^*(z \to \infty) \text{ finite } \Rightarrow A = 0$$

$$T^*(z = 0) = T_1 \;\Rightarrow B = T_1$$

$$T(z,t) = \mathrm{Re}\left[T_0 + T_1 \exp\left(-\sqrt{\frac{\omega}{2\alpha}}\,z\right) e^{i\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}\,z\right)}\right]$$

$$\boxed{T(z,t) = T_0 + T_1 \exp\left(-\sqrt{\frac{\omega}{2\alpha}}\,z\right) \cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}\,z\right)}$$

attenuation   phase lag

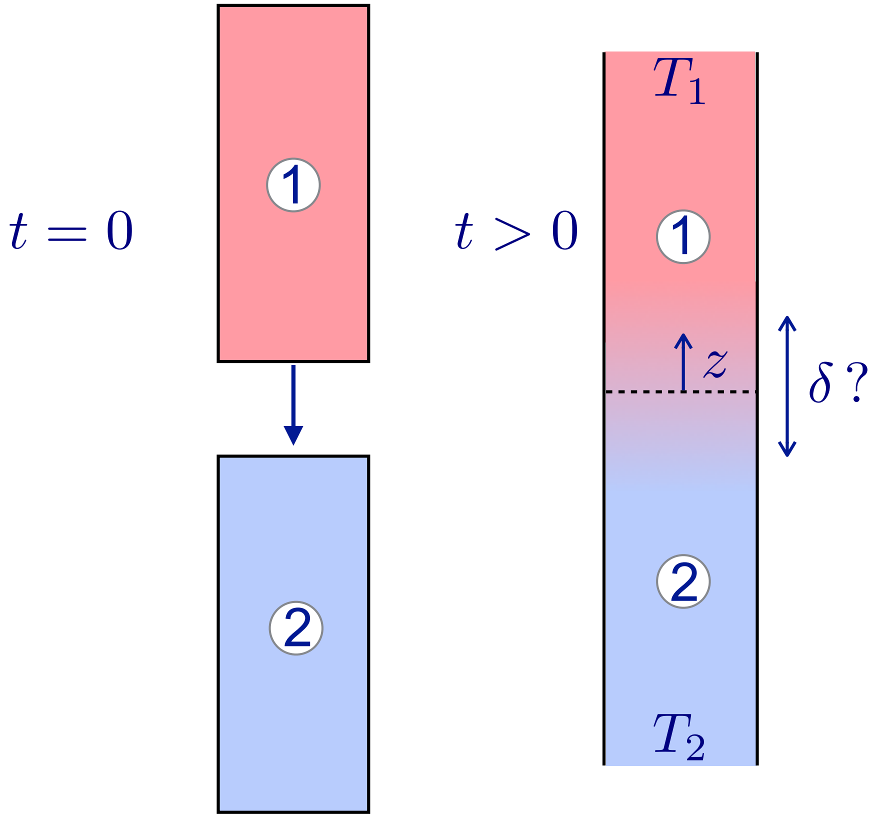
attenuation depth $\boxed{\delta = \sqrt{\frac{\omega}{2\alpha}}}$

numerical estimate: $\alpha \sim 5.10^{-7}$ m²/s

$$\omega_{1\,day} = \frac{2\pi}{24 \times 3600} = 7.3\,10^{-5} \text{ rad/s} \;\Rightarrow\; \delta \sim 10 \text{ cm}$$

$$\omega_{1\,year} = 2.7\,10^{-7} \text{rad/s} \;\Rightarrow\; \delta \sim 2 \text{ m}$$

2.5 Contact temperature

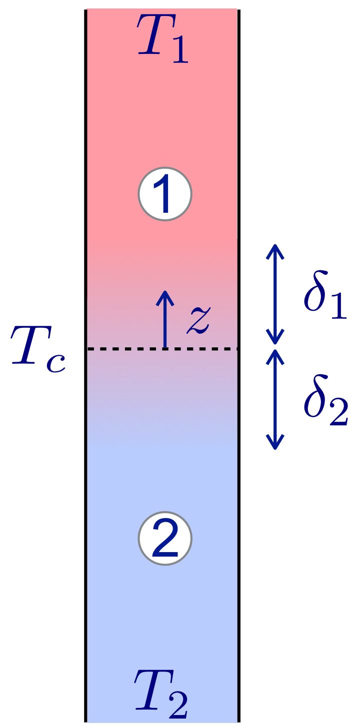


in scaling laws ?

We need to solve:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T = \alpha_i \Delta T + \frac{r_i}{\rho_i C_{p_i}}$$

with T and $\mathbf{j}$ continuous



2 boundary layers:

$$\delta_1 \sim (\alpha_1 t)^{1/2}$$

$$\delta_2 \sim (\alpha_2 t)^{1/2}$$

flux continuity:

$$\kappa_1 \left.\frac{\partial T}{\partial z}\right|_{0^+} = \kappa_2 \left.\frac{\partial T}{\partial z}\right|_{0^-}$$

$$\kappa_1 \frac{T_1 - T_c}{\delta_1} \sim \kappa_2 \frac{T_c - T_2}{\delta_2}$$

contact temperature

$$\alpha_i = \frac{\kappa_i}{\rho_i C_{p_i}}$$

$$\sqrt{\kappa_1 \rho_1 C_{p_1}}\,(T_1 - T_c) \sim \sqrt{\kappa_2 \rho_2 C_{p_2}}\,(T_c - T_2)$$

$$T_c = \frac{\sqrt{\kappa_1 \rho_1 C_{p_1}}\, T_1 + \sqrt{\kappa_2 \rho_2 C_{p_2}}\, T_2}{\sqrt{\kappa_1 \rho_1 C_{p_1}} + \sqrt{\kappa_2 \rho_2 C_{p_2}}}$$

$\sqrt{\kappa \rho C_p}$ effusivity [J.K.m^{-2}.s$^{1/2}$]

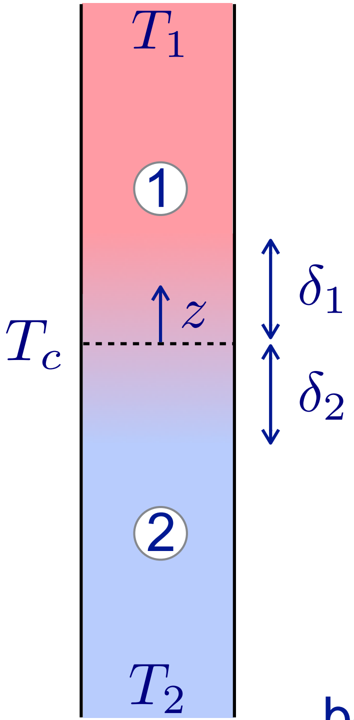
Numerical estimate:

	κ [W.m^{-1}.K^{-1}]	ρ [kg.m^{-3}]	C_p [J.kg^{-1}.K^{-1}]	$\sqrt{\kappa \rho C_p}$ [J.K.m^{-2}.s$^{1/2}$]
skin (~water)	0.6	1000	4180	~ 1500
wood	0.15	600	1700	~ 400
aluminium	100	2700	900	~ 16000
glass	1.8	2200	840	~ 1800

skin at 37ºC + wood at 20ºC $\Rightarrow$ $T_c \sim$ 33ºC

skin at 37ºC + aluminium at 20ºC $\Rightarrow$ $T_c \sim$ 21ºC

Detailed solution



general solution found for the diffusion in the cocktail applied to materials 1 and 2:

material 1

for $z > 0$ $$T(z,t) = A\,\mathrm{erf}\left(\frac{z}{2(\alpha_1 t)^{1/2}}\right) + B$$

material 2

for $z < 0$ $$T(z,t) = C\,\mathrm{erf}\left(\frac{z}{2(\alpha_2 t)^{1/2}}\right) + D$$

boundary conditions

$$T(0^+) = T(0^-) \;\Rightarrow\; B = D = T_c$$

$$T(z \to \infty) = T_1 = A + T_c$$

$$T(z \to -\infty) = T_2 = -C + T_c$$

$$\kappa_1 \left.\frac{\partial T}{\partial z}\right|_{0+} = \kappa_2 \left.\frac{\partial T}{\partial z}\right|_{0-}$$

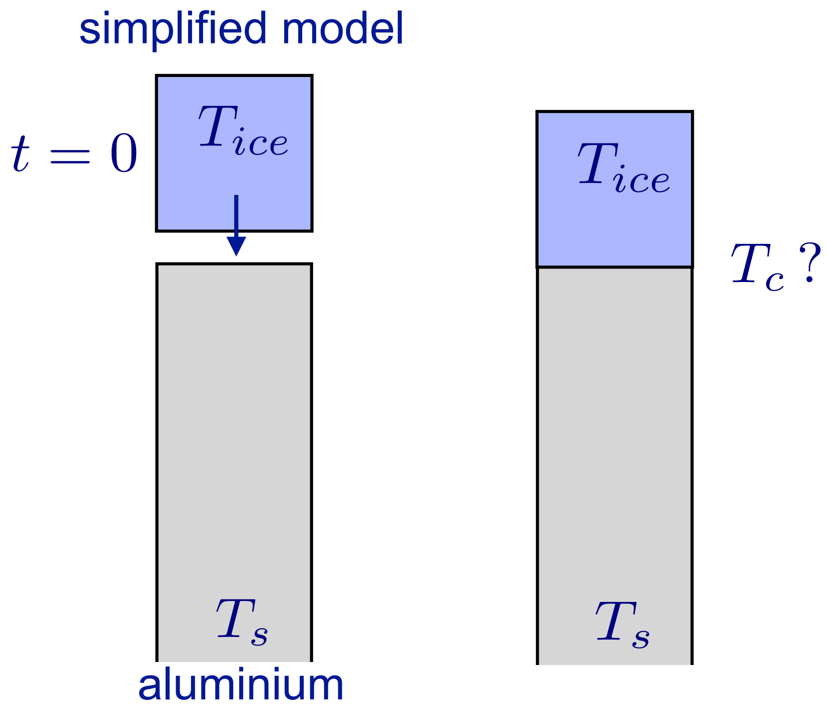
$$\frac{\partial}{\partial z}\mathrm{erf}\left(\frac{z}{2(\alpha t)^{1/2}}\right) = \frac{1}{2(\alpha t)^{1/2}}\frac{2}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4\alpha t}\right)$$

$$\kappa_1 \left.\frac{\partial T}{\partial z}\right|_{0+} = \kappa_2 \left.\frac{\partial T}{\partial z}\right|_{0-} \quad \Rightarrow \quad \frac{\kappa_1}{\sqrt{\alpha_1}}A = \frac{\kappa_2}{\sqrt{\alpha_2}}C$$

$$\frac{\kappa}{\sqrt{\alpha}} = (\kappa\rho C_p)^{1/2} \;\Rightarrow\; T_c = \frac{(\kappa_1\rho_1 C_{p_1})^{1/2}\, T_1 + (\kappa_2\rho_2 C_{p_2})^{1/2}\, T_2}{(\kappa_1\rho_1 C_{p_1})^{1/2} + (\kappa_2\rho_2 C_{p_2})^{1/2}}$$

same expression as scaling law!

2.6 Melting of an ice cube



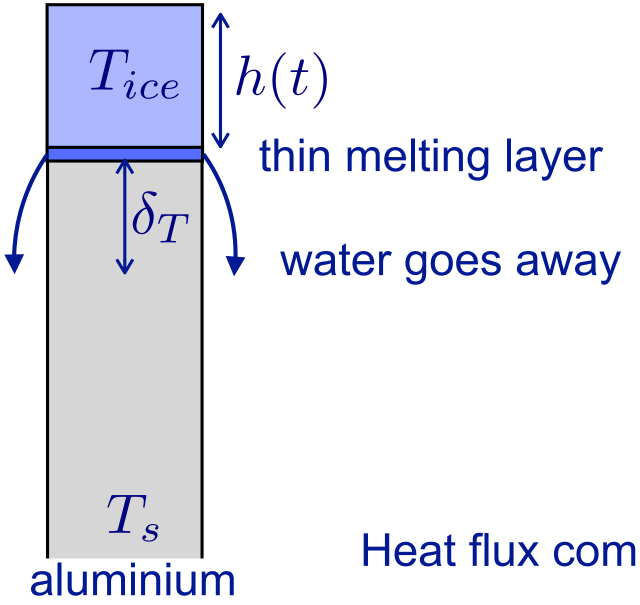
if T_{ice} = -20ºC and T_s = 20ºC

	κ [W.m^{-1}.K^{-1}]	ρ [kg.m^{-3}]	C_p [J.kg^{-1}.K^{-1}]	$\sqrt{\kappa \rho C_p}$ [J.K.m^{-2}.s$^{1/2}$]
ice	2	920	2000	~ 1900
aluminium	100	2700	900	~ 16000

$\Rightarrow$ $T_c \sim$ 15ºC not compatible with T_f = 0ºC …

2.6 Melting of an ice cube

even more simplified model: $T_{ice} = T_f$ = 0°C



$$\delta_T \sim (\alpha t)^{1/2}$$

$$j_T \sim \kappa_s \frac{T_s - T_f}{\delta_T}$$

$$j_T \sim \left(\frac{\kappa_s \rho_s C_p}{t} \right)^{1/2} (T_s - T_f)$$

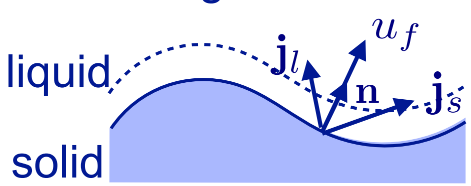
Heat flux compensates for latent heat:

$$j_T S \, dt = -\rho_{ice} L \, S \, dh$$

latent heat [J.kg^{-1}]

latent heat ⇒ jump of heat flux at the boundary

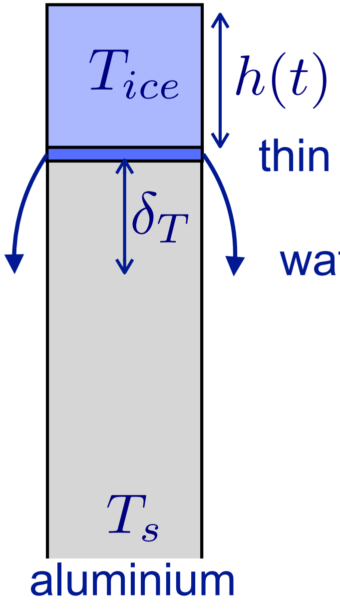
Stefan problem: moving front



$$u_f \rho_s L_f = \mathbf{j}_s . \mathbf{n} - \mathbf{j}_l . \mathbf{n}$$

2.6 Melting of an ice cube

even more simplified model: T_{ice} = 0ºC



$$\frac{dh}{dt} = -\frac{j_T}{\rho_{ice} L} \sim -\left(\frac{\kappa_s \rho_s C_P}{t}\right)^{1/2} \frac{T_s - T_f}{\rho_{ice} L}$$

$$h_0 - h(t) \sim \frac{T_s - T_f}{\rho_{ice} L} (\kappa_s \rho_s C_p t)^{1/2}$$

melting time

$$\tau \sim \left(\frac{\rho_{ice} L\, h_0}{T_s - T_f}\right)^2 \frac{1}{\kappa_s \rho_s C_p}$$

Numerical estimate:

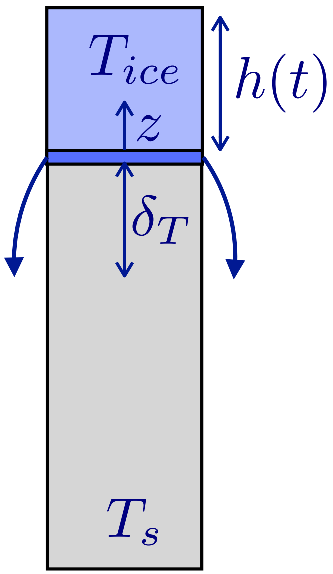
$h_0 \sim$ 3 cm $\rho_{ice} \sim$ 920 kg/m^3 $L \sim$ 334 kJ/kg

aluminium $(\kappa \rho_s C_P)^{1/2} \sim$ 16000 J.K.m^{-2}s$^{1/2}$ $\Rightarrow$ $\tau \sim$ 800s ~ 13 min

wood $(\kappa \rho_s C_P)^{1/2} \sim$ 400 J.K.m^{-2}s$^{1/2}$ $\Rightarrow$ $\tau \sim$ 10^{6}s ~ 15 days !

other heat exchange mechanisms ?

Detailed solution



in the solid: $T(z,t) = A\,\mathrm{erf}\left(\frac{z}{2(\alpha_1 t)^{1/2}}\right) + B$

$$T(z=0) = B = T_f$$

$$T(z \to -\infty) = T_s = -A + B$$

$$T(z,t) = (T_f - T_s)\,\mathrm{erf}\left(\frac{z}{2(\alpha_2 t)^{1/2}}\right) + T_f$$

$$j_T = -\kappa_s \left.\frac{\partial T}{\partial z}\right|_{z=0} = \frac{\kappa_s (T_s - T_f)}{2(\alpha t)^{1/2}} = \left(\frac{\kappa_s \rho_s C_p}{t}\right)^{1/2} \frac{T_s - T_f}{2}$$

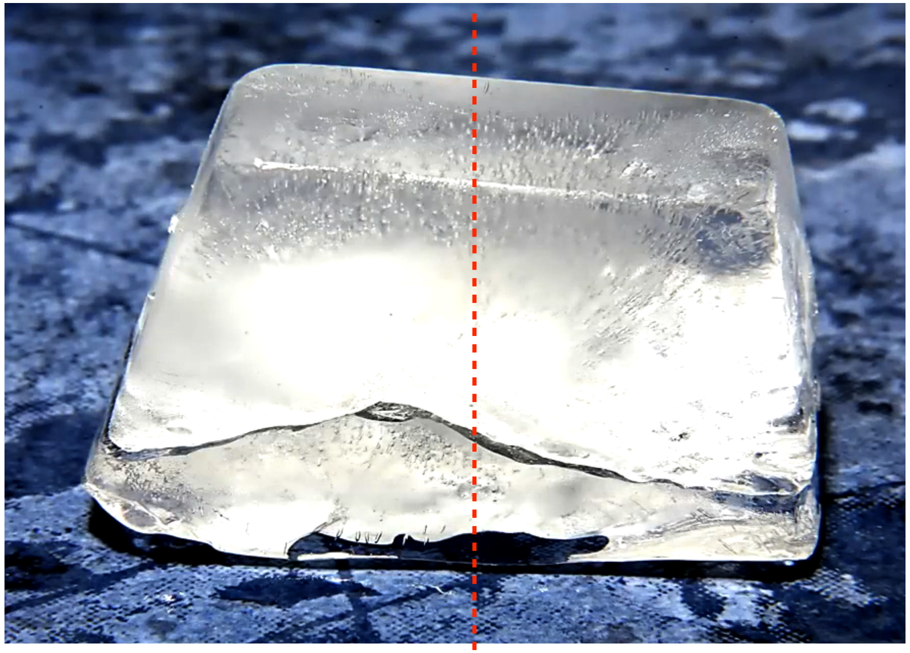
$$\frac{dh}{dt} = -\frac{j_T}{\rho_{ice} L} = -\left(\frac{\kappa_s \rho_s C_p}{t}\right)^{1/2} \frac{T_s - T_f}{2\rho_{ice} L}$$

$$h_0 - h(t) = (\kappa_s \rho_s C_p t)^{1/2} \frac{T_s - T_f}{\rho_{ice} L}$$

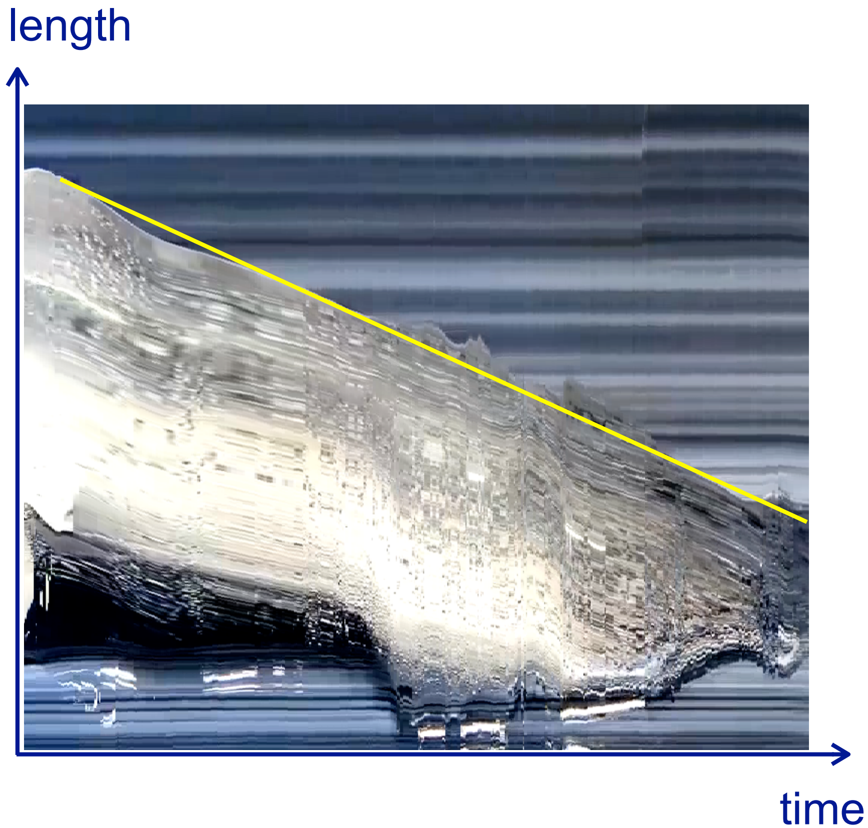
same expression as scaling law!

Space-time diagram

https://youtu.be/WgjksZoznuA?si=WiG9lcc14et5Zhvz



reslice

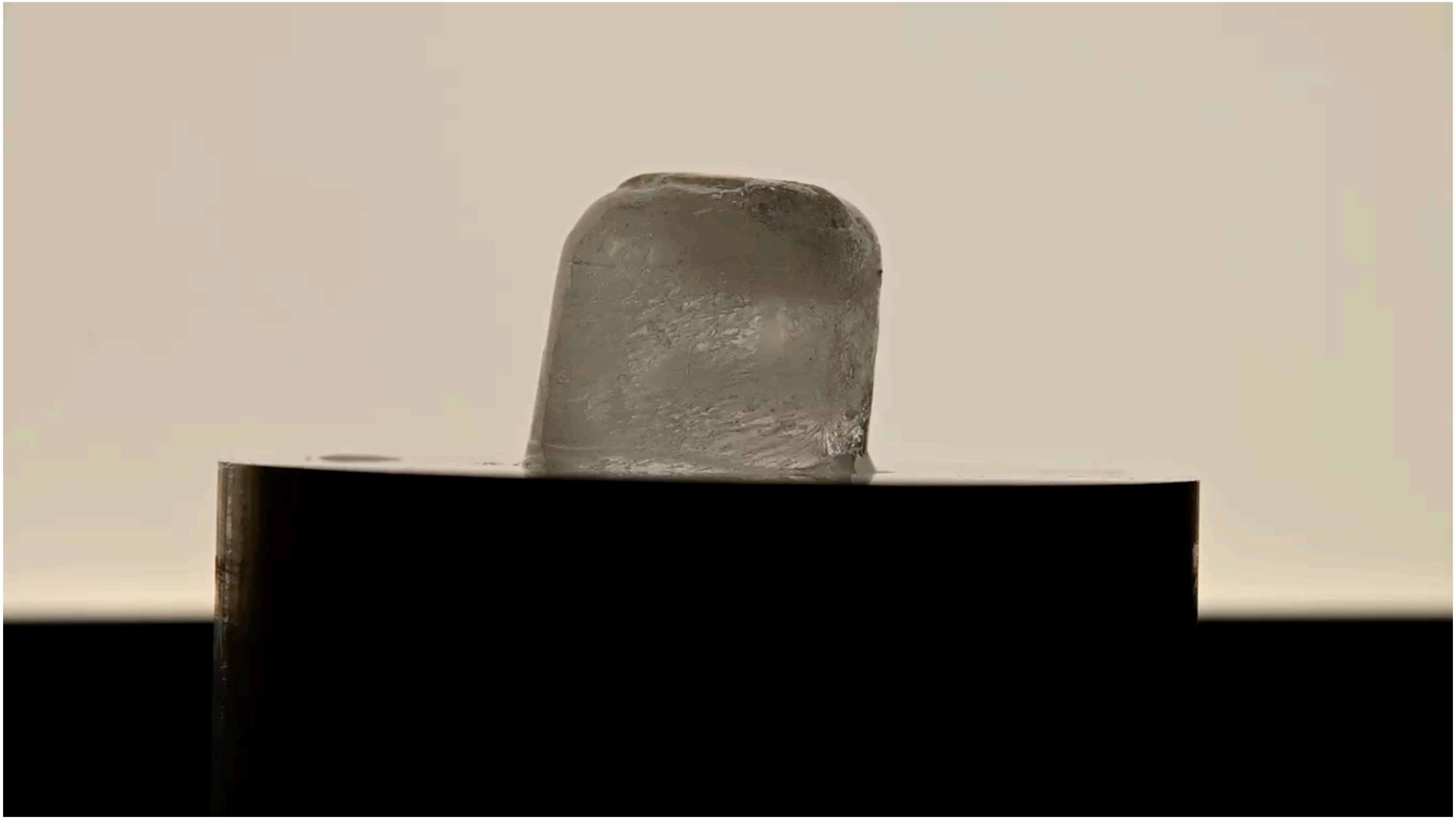


rather $h_0 - h(t) \propto t$ …

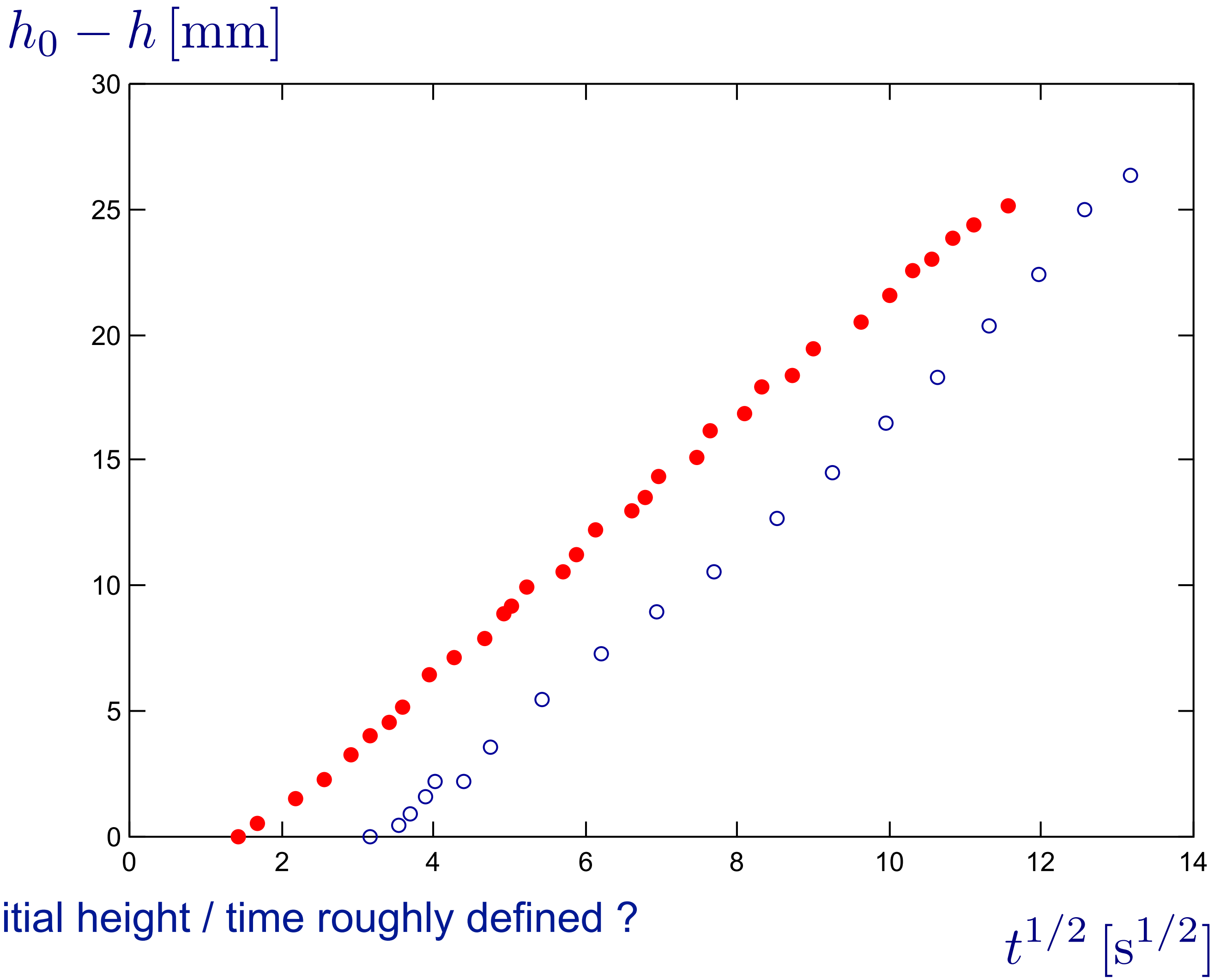
low effusive solid ?

other heat exchange mechanisms ?

Experiment in the lab

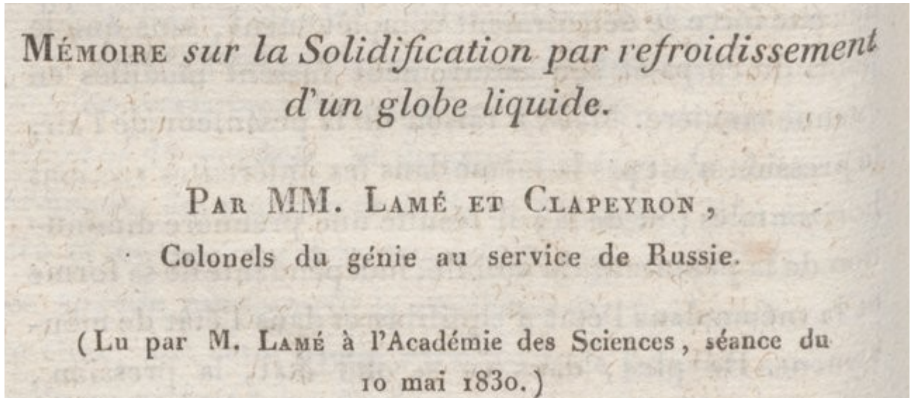


Experiment in the lab



initial height / time roughly defined ?

2.7 Estimating the age of the Earth ?

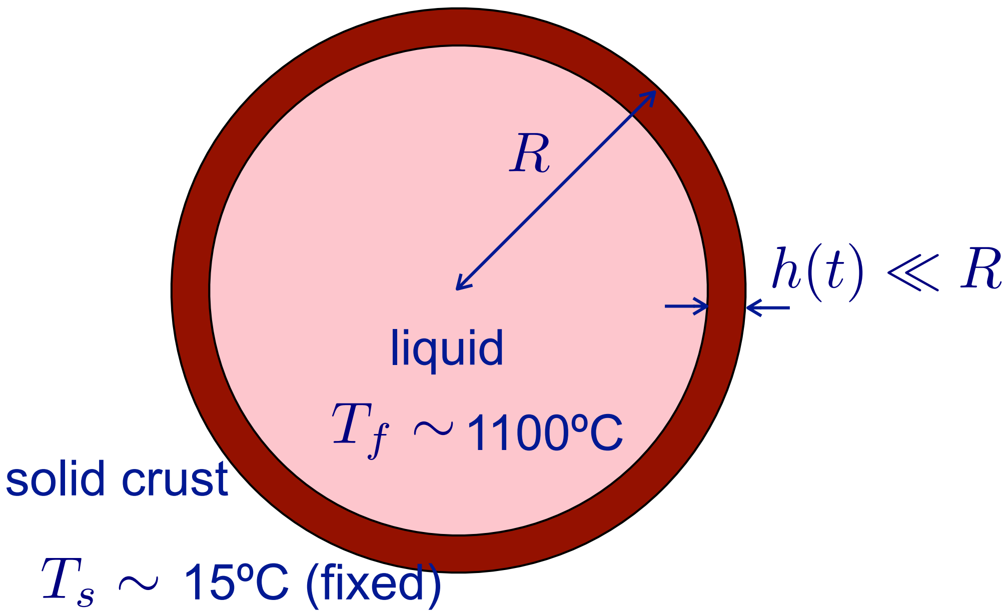

MÉMOIRE *sur la Solidification par refroidissement d'un globe liquide.*

PAR MM. LAMÉ ET CLAPEYRON,

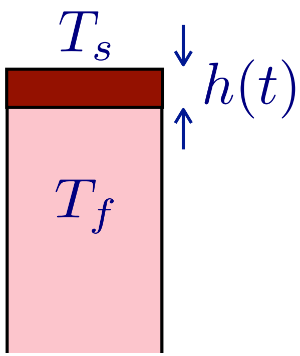
Colonels du génie au service de Russie.

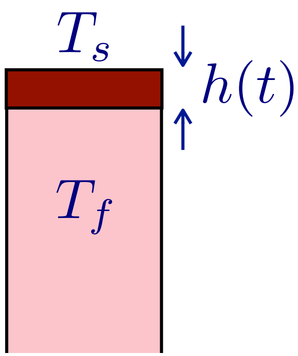
(Lu par M. LAMÉ à l'Académie des Sciences, séance du 10 mai 1830.)

very simplified model



even more simplified model: 1D





uniform T in the liquid ⇒ no heat flux inside the liquid

heat flux through the solid:

$$j_T = \kappa_s \frac{T_f - T_s}{h}$$

heat flux compensates latent heat:

$$\rho_s L \frac{dh}{dt} = j_T = \kappa_s \frac{T_f - T_s}{h}$$

$$h \frac{dh}{dt} = \frac{\kappa_s}{\rho_s L}(T_f - T_s)$$

$$\boxed{h(t) = \left(2 \frac{\kappa_s}{\rho_s L}(T_f - T_s)\, t\right)^{1/2}}$$

Numerical estimate:

$h \sim$ 35 km $\kappa_s \sim$ 2 W.m^{-1}K^{-1} $\rho_s \sim$ 3.10^{3} kg.m^{-3} $L \sim$ 4.10^{5} J.kg^{-1} $T_s \sim$ 15ºC $T_f \sim$ 1100ºC

$\Rightarrow$ $\tau \sim$ 10 million years … model too simple!