Advanced Mechanics: Transport

April 12th, 2023

Duration 1h. Authorized document: lecture notes.

1 Solidification of a lead shot

Lead shots were traditionally manufactured in "shot towers". Molten lead were poured through a copper sieve at the top of the tower. Millimeter size droplets then formed under the effect of surface tension and would solidify as they fall down. A nice exemple of an old 70 m high shot tower can be visited in Couëron, Loire-Atlantique (Fig. 1). We propose to estimate the typical solidification time of a molten sphericule of lead under different heat exchange processes.

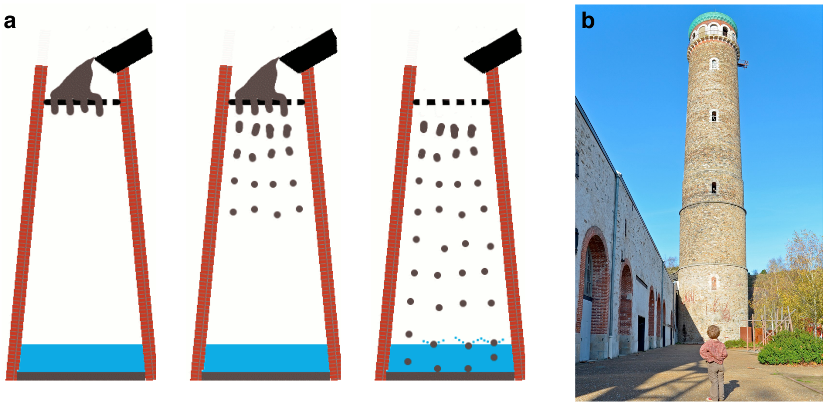

Figure 1: (a) Manufacturing process of lead shots in a "shot tower". (b) *Tour à plomb* in Couëron, Loire-Atlantique (source: Wikipedia).

We consider a sphere of molten lead of radius $R = 1\,\mathrm{mm}$, at the melting temperature $T_f = 327°\mathrm{C}$ and we will assume that the temperature remains uniform through the whole sphere during the solidification process (we will discuss the validity of this assumption later).

Some material properties

- Lead:
emissivity $\varepsilon = 0.1$, density $\rho_{Pb} = 10.5 \cdot 10^3\,\mathrm{kg/m^3}$, latent heat $L_{Pb} = 23\,\mathrm{kJ/kg}$, thermal conductivity $\kappa_{Pb} = 16\,\mathrm{Wm^{-1}K^{-1}}$, thermal diffusivity $\alpha_{Pb} = 10^{-5}\,\mathrm{m^2/s}$,

- Air:
density $\rho_{air} = 1.2\,\mathrm{kg/m^3}$, kinematic viscosity $\nu_{air} = 1.5 \cdot 10^{-5}\,\mathrm{m^2/s}$, thermal conductivity $\kappa_{air} = 0.025\,\mathrm{Wm^{-1}K^{-1}}$, thermal diffusivity $\alpha_{air} = 2 \cdot 10^{-5}\,\mathrm{m^2/s}$, thermal expansion coefficient $\beta_{air} = 3.4 \cdot 10^{-3}\,\mathrm{K^{-1}}$

- Fundamental constants:
Absolute 0: $-273.15°\mathrm{C}$, Stefan-Bolztman constant $\sigma = 5.7 \cdot 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$

1.1 Monitoring the temperature of the droplet (1pt)

If you watch the droplet with an infrared thermal camera, would the software indicate anything close to the correct surface temperature (argue your answer)?

Solution: No, because the estimation of the surface temperature is based on the emission of a black body. Since the emissivity is low, we are far from this approximation. Presumably, the measured temperature will be far below the actual one.

1.2 Droplet in sidereal space (2pt)

Imagine Thomas Pesquet fixes something outside the space station and a droplet of molten lead solder gets ejected.
What is the heat flux through its surface?
How long would it take for the droplet to eventually solidify?

Solution: The only source of heat flux is in this case radiation: $j = \varepsilon\sigma T^4$. Since the temperature is uniform, the surface temperature is equal to $T_f = 600°\mathrm{K}$. The heat to remove from the droplet is the latent heat $\frac{4}{3}\pi R^3 \rho_{Pb} L_{Pb}$, which leads to:

$$\tau_{rad} = \frac{\frac{4}{3}\pi R^3 \rho_{Pb} L_{Pb}}{4\pi R^2 \varepsilon\sigma T^4} = \frac{R\,\rho_{Pb}\,L_{Pb}}{3\varepsilon\sigma T^4} = 110\,\mathrm{s} \sim 2\,\mathrm{min}$$

1.3 Droplet in levitating in the absence of gravity (3pt)

Now imagine that the repair was inside the space station maintained at 20°C. The molten droplet remains standing in the air due to the absence of gravity.
In terms of scaling laws, what is the order of magnitude for the solidification time if we do not account for radiation?

Solution: In the absence of radiation, the remaining heat transfer process is diffusion through the air. The diffusion length scales as $\delta \sim (\alpha_{air} t)^{1/2}$ and the heat flux $j \sim \frac{\kappa_{air}}{\delta}(T_f - T_{amb})$. As a consequence, the solidification time is expected to follow:

$$\tau_{diff} \sim \frac{R\,\rho_{Pb}\,L_{Pb}\,(\alpha_{air}\tau_{diff})^{1/2}}{3\kappa_{air}(T_f - T_{amb})}$$

$$\tau_{diff} \sim \frac{\alpha_{air}}{9}\left(\frac{R\,\rho_{Pb}\,L_{Pb}}{\kappa_{air}(T_f - T_{amb})}\right)^2 = 2200\,\mathrm{s} \sim 40\,\mathrm{min}$$

As a consequence, radiation is significantly more efficient than pure diffusion through air of low conductivity.

1.4 Sessile droplet on Earth (3pt)

We now come back to Earth gravity and assume that the droplet as been deposited on a thermally insulated surface such as a glass plate. We again neglect radiative processes.
What is now the order of magnitude of the solidification time?

Solution: Under the action of gravity, air heated up by the sphere become less dense and moves up resulting into a natural convection process. In that case, the heat flux is expected to follow:

$$j \sim \kappa_{air} \frac{Nu}{R}(T_f - T_{amb}), \quad \text{with} \quad Nu \sim Ra^{1/4} \sim \left(\frac{\beta_{air} g R^3 (T_f - T_{amb})}{\nu_{air} \alpha_{air}} \right)^{1/4}$$

which leads to

$$\tau_{natural} \sim \frac{R^2 \rho_{Pb} L_{Pb}}{3\kappa_{air} (T_f - T_{amb})} \frac{1}{Ra^{1/4}}$$

$$Ra \sim 35 \quad \tau_{natural} \sim 4\,\mathrm{s}$$

Radiative effects are negligible in this case.

1.5 Falling droplet (4pt)

Now consider the case of a droplet falling from the tower. What is its terminal velocity U? Just a quick recap from last year: drag force on a sphere: $6\pi\eta RU$, for $Re \ll 1$ and $\frac{1}{2}\rho U^2 \pi S C_d$, for $Re \gg 1$ with a drag coefficient $C_d \sim 0.5$ for a sphere.
What is now the order of magnitude of the solidification time?
Does the hight of the Couëron tower seem appropriate?

Solution: We are presumably at high Re, take for instance $U = 1\,\mathrm{m/s}$, we obtain $Re \sim UR/\nu_{air} \sim 67$. If we balance the drag force with the weight of the droplet, we get:

$$\frac{4}{3}\pi R^3 \rho_{Pb}\, g = \frac{1}{2}\rho_{air} U^2 \pi R^2 C_d$$

$$U = \left(\frac{8}{3} \frac{\rho_{Pb}}{\rho_{air}} \frac{R\,g}{C_d} \right)^{1/2} = 23\,\mathrm{m/s}$$

The high Re hypothesis is definitely valid.
Heat exchange now corresponds to forced convection. We thus expect:

$$\tau_{forced} \sim \frac{R^2 \rho_{Pb} L_{Pb}}{3\kappa_{air} (T_f - T_{amb})} \frac{1}{Nu} \quad \text{with} \quad Nu \sim Re^{1/2} Pr^{1/3} \sim \left(\frac{UR}{\nu_{air}} \right)^{1/2} \left(\frac{\nu_{air}}{\alpha_{air}} \right)^{1/3}$$

We finally obtain $\tau_{forced} \sim 0.3\,\mathrm{s}$. Much more efficient than the other processes. The travelled distance is on the order of $7\,\mathrm{m}$. The hight of the tower is of the good order of magnitude (we did not account for the acceleration phase of the droplet nor for the fact that the initial temperature of lead may be higher than T_f, which will increase the solidification time).

2 Freezing fronts

Estimating the age of the Earth has been the subject of vivid debates during centuries, even between scientists. In the 1830's Lamé and Clapeyron proposed an estimate based on the propagation of a solidification front. We propose to follow their steps.

2.1 Propagation of a solidification front (3pt)

We consider a semi-infinite column of liquid at its melting temperature T_f in contact with a solid at a fixed temperature $T_s < T_f$. As a consequence a freezing front of thickness h progressively grows over the cold surface (Fig. 2a).
What is the heat flux through the solidified layer?
What is the time evolution of the thickness of the freezing front?

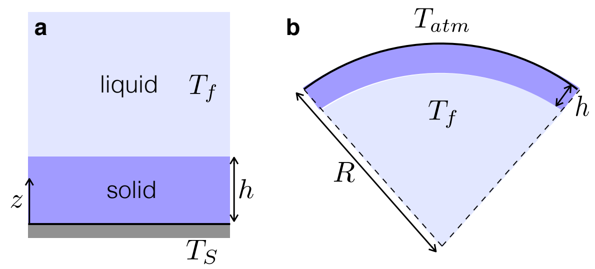

Figure 2: (a) Solidification front of a liquid on a cold surface ($T_s < T_f$). (b) Application to the estimation of the age of the Earth proposed by Lamé & Clapeyron.

Solution: We have T_f in one side and T_s on the opposite side, so the gradient is simply $(T_f - T_s)/h$. As a consequence, the heat flux is given by:

$$j = \kappa_s \frac{T_f - T_s}{h}$$

During a time dt the heat lost by the liquid side is $j\,dt$ and balances the latent heat corresponding to a growth of dh of the thickness:

$$\rho_s L_s\, dh = \kappa_s \frac{T_f - T_s}{h}\, dt$$

which leads to:

$$h\frac{dh}{dt} = \frac{\kappa_s}{\rho_s L_s}\frac{T_f - T_s}{h}$$

After integration, we obtain:

$$h(t) = \left(2\,\frac{\kappa_s}{\rho_s L_s}(T_f - T_s)\,t\right)^{1/2}$$

2.2 Estimating the age of the Earth (2pt)

In order to estimate the age of the Earth, Lamé & Clapeyron though the Earth was formed from a initial mass of liquid lava of temperature $T_f \sim 1100°\text{C}$ that progressively solidified from the surface towards the center of the planet. The surface temperature of the Earth was assume to be constant and equal to the ambient temperature we experience today ($T_s \sim 15°\text{C}$). They knew the current thickness of the continental crust of the Earth is on the order of 35 km, the latent heat of silica (the may component of the crust) $L_s \sim 400\,\text{kJ/kg}$, its density $\rho_s \sim 3 \cdot 10^3\,\text{kg/m}^3$ and thermal conductivity $\kappa_s \sim 2\,\text{Wm}^{-1}K^{-1}$.
What would be the age of the Earth following their approach?

Solution: We just need to invert the previous expression:

$$\tau = \frac{\rho_s L_s}{2\kappa_s} \frac{h^2}{T_f - T_s} = 3 \cdot 10^{14}\,\text{s} \simeq 10\,\text{Ma}$$

This is obviously far below the actual age, but already a conceptual revolution for that time. The crust is actually not floating on liquid lava as it was though at that time and natural radioactivity was not discovered yet.

2.3 Surface temperature of the solidifying lead shot (2pt)

In the estimate of the solidification of the lead shot we assumed the temperature of the droplet to be uniform. In reality, a solidified crust also separates a hot liquid side of temperature T_f, from a colder surface of temperature T_s.
Considering the same heat flux as in the case of the *falling droplet*, give an expression for the surface temperature (as a simplifying approximation consider a linear configuration as sketched in Figure 2a).
Was our approximation of uniform temperature valid?

Solution: We simply balance the heat flux through the crust with the heat flux through the air:

$$j = \frac{\kappa_s}{h}(T_f - T_s) \sim \frac{\kappa_{air} Nu}{R}(T_s - T_{air})$$

which leads to:

$$T_s = \frac{\frac{\kappa_s}{h} T_f + \frac{\kappa_{air} Nu}{R} T_{amb}}{\frac{\kappa_s}{h} + \frac{\kappa_{air} Nu}{R}}$$

In the falling droplet configuration, we have $Nu \sim 36$ and thus $\kappa_{air} Nu/R \sim 900\,\text{Wm}^{-2}\text{K}^{-1}$ to be compared with κ_s/h. The lowest value of this heat transfer coefficient corresponds to $\kappa_s/R \sim 16000\,\text{Wm}^{-2}\text{K}^{-1}$. As a consequence T_s is very close to T_f and it was reasonable to assume that the temperature was uniform through the droplet.