Drop freezing

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For a one-dimensional problem, the transport equations are :

- in the substrate $(z < 0)$: $\partial_t T = \kappa_s \partial_{zz} T$
- in the ice $(0 < z < h(t))$, $\partial_t T = \kappa_i \partial_{zz} T$.

The boundary conditions are the folllowing:

- continuity of temperature and flux at the ice/substrate interface, $z = 0$ $(T = T_0)$,

$$\lambda_i \frac{\partial T_i}{\partial z} = \lambda_s \frac{\partial T_s}{\partial z}$$

- temperature of the substrate away from the surface, $T \to T_s$ when $z \to -\infty$
- Continuity of temperature at the solid liquid interface, $z = h(t)$, $T = T_m$
- The flux of latent heat is removed by diffusion in the solid (Stefan's condition) :

$$\lambda_i \frac{\partial T}{\partial z} = \rho_i L \frac{dh}{dt}$$

The general solutions of the transport equations are : $T = A + B\mathrm{erf}(\zeta/2)$, with $\zeta = z/\sqrt{\kappa t}$. Proof : Making the change in variables, we have :

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa t} \frac{d^2 T}{d\zeta^2}$$

$$\frac{\partial T}{\partial t} = -\frac{1}{2} \frac{\zeta}{t} \frac{dT}{d\zeta}$$

and the diffusion equation becomes an ordinary differential equation:

$$\frac{d^2 T}{d\zeta^2} = -\frac{\zeta}{2} \frac{dT}{d\zeta}$$

Integrating once, we have $dT/d\zeta = C\exp(-\zeta^2/4)$ and, using the error function $\mathrm{erf}(\zeta) = 2/\sqrt{\pi}\int_0^\zeta \exp(-t^2)dt$, $T = A + B\mathrm{erf}(\zeta/2)$.

In the substrate, we have $T = T_0$ at $\zeta = 0$ and $T \to T_s$ when $\zeta \to -\infty$ and

$$T = T_0 + (T_s - T_0)\mathrm{erf}(-z/2\sqrt{\kappa_s t}). \tag{1}$$

In the ice phase, we have also $T = T_0$ at $\zeta = 0$ and the continuity of heat fluxes implies :

$$\lambda_s \frac{T_s - T_0}{\sqrt{\kappa_s t}} = \lambda_i \frac{B_i}{\sqrt{\kappa_i t}}$$

so that :

$$T = T_0 + (e_s/e_i)(T_0 - T_s)\mathrm{erf}(z/2\sqrt{\kappa_i t}), \tag{2}$$

with $e_s = \sqrt{\lambda_s \rho_s C_s}$ and $e_i = \sqrt{\lambda_i \rho_i C_i}$.

The continuity of temperature at the solid/liquid interface gives :

$$T_m - T_0 = (T_0 - T_s)\frac{e_s}{e_i}\mathrm{erf}(h/2\sqrt{\kappa_i t}) \tag{3}$$

The Stefan condition gives :

$$\lambda_i (T_0 - T_s)\frac{e_s}{e_i}\frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{\kappa_i t}}\exp(-h^2/4\kappa_i t) = \rho_i L \frac{dh}{dt} \tag{4}$$

To satisfy eqn. (3) at all times, we take $h(t) = \sqrt{\kappa_e t}$, so that:

$$T_m - T_0 = (T_0 - T_s)\frac{e_s}{e_i}\mathrm{erf}(\sqrt{\kappa_e/\kappa_i}/2) \tag{5}$$

and the Stefan condition writes:

$$\lambda_i (T_0 - T_s)\frac{e_s}{e_i}\frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{\kappa_i t}}\exp(-\kappa_e/4\kappa_i) = \rho_i L \frac{\sqrt{\kappa_e}}{2\sqrt{t}} \tag{6}$$

or:

$$(T_0 - T_s)\frac{e_s}{e_i}\frac{2}{\sqrt{\pi}} C_i \sqrt{\kappa_i/\kappa_e}\exp(-\kappa_e/4\kappa_i) = L \tag{7}$$

or, introducing $\beta = \kappa_e/\kappa_i$ and eliminating T_0, the Stefan number St (which compares the heat required to cool the ice from T_m to T_s to the latent heat of freezing) is:

$$St = \frac{C_i(T_m - T_s)}{L} = \frac{\sqrt{\pi\beta}}{2}\exp(\beta/4)\left(\frac{e_i}{e_s} + \mathrm{erf}(\sqrt{\beta}/2)\right) \tag{8}$$

For a water drop freezing on a copper substrate, $\lambda_s = 400$ W.m^{-1}.K^{-1}, $\rho_s = 9000$ kg/m^3, $C_s = 385$ J.kg^{-1}.K^{-1}, $\lambda_i \approx 0.6$ W.m^{-1}.K^{-1}, $\rho_i = 1000$ kg/m^3, $C_i = 4200$

J.kg^{-1}.K^{-1}, $L = 334$ kJ/kg. For a substrate which is 10 degrees below the melting point, $St \approx 0.1$ and we have $e_i/e_s \approx 0.07$.

The effective diffusion coefficient κ_e is given by the solution of eqn. (8), knowing St and e_i/e_s (fig. 1). Here $\beta \approx 0.15$ and the effective diffusion coefficient is significantly smaller than κ_i.

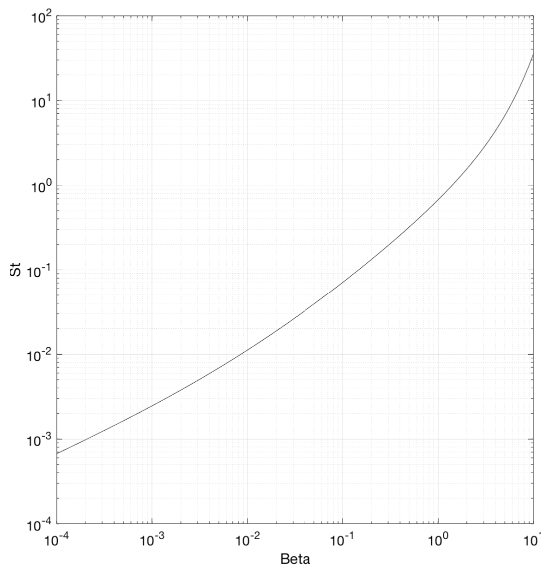

Figure 1: Solution of eqn. (8) Stefan number as a fonction of β for $e_i/e_s = 0.07$.

The temperature at the ice/substrate interface is given by :

$$\frac{T_0 - T_s}{T_m - T_s} = \frac{1}{1 + \frac{e_s}{e_i}\text{erf}(\sqrt{\beta}/2)}$$

With the values chosen above, T_0 is very close to the melting temperature T_m.