Thermal convection

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1 Thermal plumes, convection cells, ...

Thermal convection is the motion of fluids in an acceleration field, created by the density fluctuations linked to the thermal expansion of the fluid. This phenomenon occurs in many different situations, in industrial processes and in the environment. Hot fluid rising from a localized source of heat is called a thermal plume. Three examples of such plumes are shown on fig. 1. Two of them are laminar flows, the third one is turbulent becomes the Reynolds number of the flow is large enough to generate instability and a transition to turbulence.

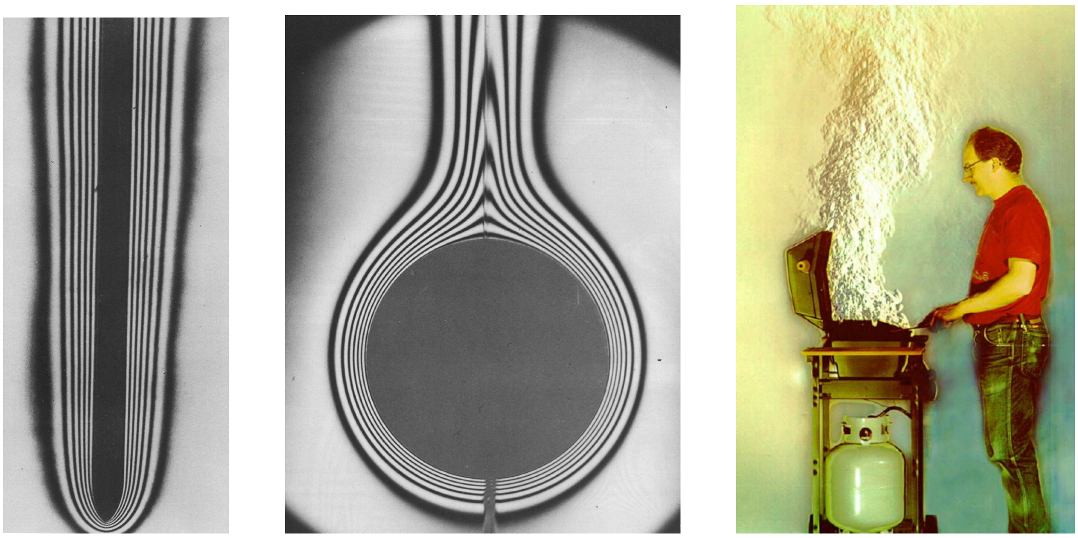

FIGURE 1 – Three examples of thermal convection : left, laminar convection around a heated vertical plate at a Grashof number (see definition in text) of several millions. Middle : laminar convection around a heated horizontal cylinder (Grashoh number = 3×10^4). Visualization of the temperature field by interferometry. Images reproduced from An Album of Fluid Motion by Milton Van Dyke. Right : a turbulent convection plume above a barbecue. Visualization by the Schlieren technique. Image reproduced from Gary Settles, "The Penn State full scale Schlieren system", 11th International symposium on flow visualization (2004).

Another example of thermal convection is shown on fig. 2. This time the flow is confined in a rectangular enclosure and the differential heating of the bottom of the box creates a circulation of the fluid. This experiment was performed as model of large scale circulation in the oceans.

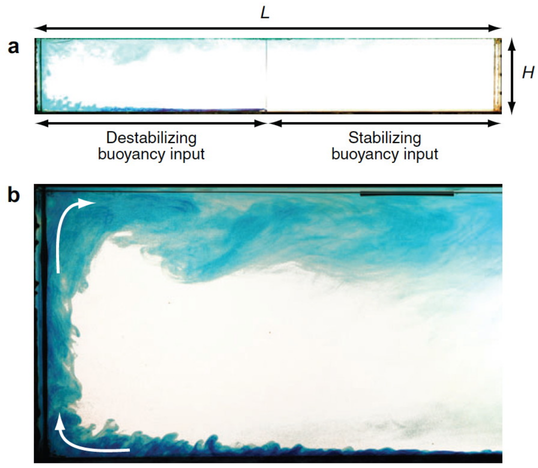

FIGURE 2 – Thermal convection inside a rectangular box. The left part of the bottom wall is heated while the right part is cooled. This differential heating induces a convective circulation within the box. Dye is introduced on the left side of the bottom to visualize the flow. The lower image is a close up of the hot layer of fluid rising along the left wall. The Rayleigh number (see definition in the text) is 2×10^{14}.Image reproduced from G. Hughes & R. Griffiths, "Horizontal convection", Ann. Rev. Fluid Mech. 40, 185 (2008)

2 Boussinesq approximation

In thermal convection problems, the flow is created by the density gradients related to the temperature gradients. As a result the equation for the transport of momentum (the Navier-Stokes equation) is strongly coupled to the heat transport equation and they have to be solved together.

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u}.\nabla \mathbf{u} = -\nabla p + \eta \Delta \mathbf{u} + \rho \mathbf{g}$$
$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p \mathbf{u}.\nabla T = \lambda \Delta T \tag{1}$$

where ρ, η, λ are in general function of temperature. In addition the equation of conservation of mass for the fluid does no longer lead to $\nabla.\mathbf{u} = 0$ since the density is not uniform.

This system of equation can be simplified by using the so-called *Boussinesq's approximation* [1]. If the temperature variation is small enough, the variations in density and transport coefficients can be considered as negligible. The only term in the momentum equation that is retained to take into account the temperature variations is the buoyancy term ρg which drives the flow. We define a reference temperature T_0 at which the density is ρ_0. The local variation of density is given by the coefficient of thermal expansion of the fluid :

$$\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$$

The local density is given by : $\rho = \rho_0 - \alpha \rho_0 (T - T_0)$ and the equation for momentum is :

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho_0 \mathbf{u}.\nabla \mathbf{u} = -\nabla p + \eta \Delta \mathbf{u} + \rho_0 [1 - \alpha (T - T_0)] \mathbf{g} \tag{2}$$

1. Joseph Boussinesq (1842-1929) a french scientist who made major contributions to hydraulics and was the first to introduce the concept of turbulent viscosity

Without flow, the fluid would be in hydrostatic equilibrium with $\nabla p_H = \rho_0 \mathbf{g}$, so if we introduce a modified pressure which is the deviation from hydrostatic equilibrium $p^* = p - p_H$, we get :

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho_0 \mathbf{u}.\nabla \mathbf{u} = -\nabla p^* + \eta \Delta \mathbf{u} - \rho_0 \alpha (T - T_0) \mathbf{g} \tag{3}$$

Dividing by the reference density ρ_0, we get the new set of coupled equations :

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} = -\frac{\nabla p^*}{\rho_0} + \nu \Delta \mathbf{u} - \alpha (T - T_0) \mathbf{g}$$
$$\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \kappa \Delta T. \tag{4}$$

In addition, in the Boussinesq approximation the fluid is considered to be incompressible and the velocity field obeys the continuity equation $\nabla.\mathbf{u} = 0$.

3 A simple problem to introduce the relevant dimensionless parameters

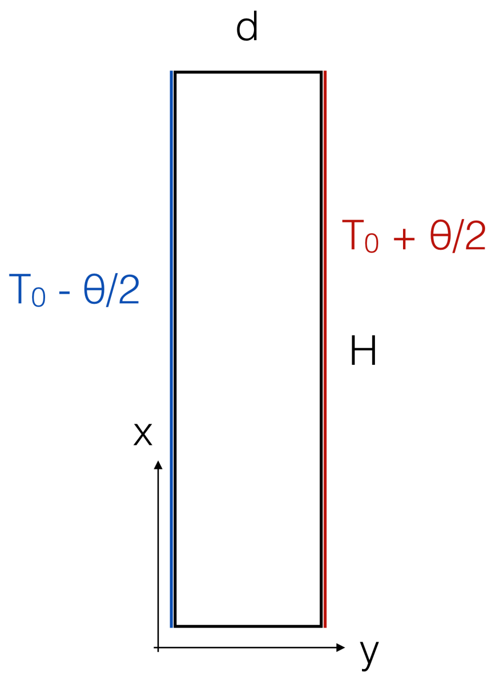

FIGURE 3 – Definition of the problem of convection in a narrow box.

In order to introduce the relevant physical parameters for thermal convection we consider a problem with a simple geometry : a rectangular cavity with height H much larger than the width d (fig. 3). The side walls of the cavity are maintained at constant temperature, the right side being warmer than the left side with a temperature difference θ. The warmer fluid which is less dense rises in the gravity field while the colder fluid sinks. The high aspect ratio H/d suggests to find a one dimensional solution to the problem with a single component of velocity in the vertical direction. If there is a single component of velocity u_x in the vertical direction x (positive x upwards) and if we use Boussinesq's approximation, the Navier-Stokes equation reads :

$$0 = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \nu\frac{\partial^2 u_x}{\partial y^2} + \alpha g(T - T_0) \tag{5}$$

y being the coordinate in the horizontal direction.

The heat transport equation is in steady state :

$$u_x\frac{\partial T}{\partial x} = \kappa\Delta T \tag{6}$$

If we assume that the problem is invariant along x, the heat transport equation is simply $\partial_{yy}T = 0$ and the temperature varies linearly in the horizontal direction : $T = T_0 + \theta y/d$.

The Navier-Stokes equation becomes :

$$0 = \nu\frac{\partial^2 u}{\partial y^2} + \alpha g\frac{\theta y}{d}. \tag{7}$$

Integrating twice with respect to y, we get :

$$u(y) = -\frac{\alpha g\theta}{\nu}\frac{y^3}{6d} + Ay + B \tag{8}$$

The vertical side walls impose a zero velocity condition at $y = \pm d/2$ and the velocity profile is :

$$u(y) = -\frac{\alpha g\theta}{6\nu d}y\left(y^2 - \frac{d^2}{4}\right) \tag{9}$$

The order of magnitude of the velocity is given by $\alpha g\theta d^2/\nu$. This velocity value results from a balance between the buoyancy force and the viscous dissipation.

In this simple approximation to the actual flow, there is a net heat transfer only in the horizontal direction as a result of diffusion. However heat is transported in the vertical direction, upwards on the right side, downwards on the left side. There is no net flux of heat due to convection because the velocity and the temperature do not depend on x ; as a result the divergence of J_{conv} is zero. Nevertheless, comparing the convective flux to the diffusive flux is useful to reveal an essential parameter in thermal convection.

The horizontal diffusive heat flux is :

$$J_{Dy} = -\lambda\frac{\partial T}{\partial y} = -\lambda\frac{\theta}{d}.$$

The vertical convective heat flux, averaged over the right side of the cavity is :

$$J_{Cx} = \frac{2}{d}\int_0^{d/2} u(y)\rho C_p(T - T_0)dy = \frac{2}{d}\int_0^{d/2} \frac{-\alpha g\rho C_p\theta^2}{6\nu d^2}y^2\left(y^2 - \frac{d^2}{4}\right)dy = \frac{\alpha g\rho C_p\theta^2 d^2}{1200\nu}$$

.

The ratio of these two fluxes is :

$$\frac{J_{Cx}}{J_{Dy}} = \frac{\alpha g\rho C_p\theta d^3}{1200\nu\lambda} = \frac{\alpha g\theta d^3}{1200\nu\kappa} \tag{10}$$

This is a dimensionless number which is proportional to the *Rayleigh number* Ra :

$$Ra = \frac{\alpha g\theta d^3}{\nu\kappa} = \frac{\alpha g\theta d^3}{\nu^2}Pr = GrPr. \tag{11}$$

where $Pr = \nu/\kappa$ is the Prandtl number and $Gr = \alpha g\theta d^3/\nu^2$ is the *Grashof number*.

The Rayleigh number can be thought as a Peclet number, that is a ratio of convective transport to diffusive transport, based on the velocity scale $U = \alpha g\theta d^2/\nu$ and on the lengthscale d.

This type of analysis can be generalized in the following way : we consider the system of coupled equations (4) and we rewrite these equations with dimensionless variables. We choose a lengthscale L and, since the velocity u is not known a priori, we choose a velocity scale $U = \nu/L$ (which is somewhat equivalent to introduce a Reynolds number uL/ν). The dimensionless variables are :

$$\hat{\mathbf{r}} = \mathbf{r}/L, \;\; \hat{\mathbf{u}} = \mathbf{u}L/\nu, \;\; \hat{t} = t\nu/L^2, \;\; \hat{T} = (T - T_0)/\delta T, \;\; \hat{p} = p/(\rho\nu^2/L^2)$$

where δT is the scale of the temperature variation within the flow and where we have normalized the pressure by the dynamic pressure ρU^2. With these variables, the system of equation becomes :

$$\begin{aligned} \frac{\nu^2}{L^3}\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \frac{\nu^2}{L^3}\hat{\mathbf{u}}.\hat{\nabla}\hat{\mathbf{u}} &= -\frac{\nu^2}{L^3}\hat{\nabla}\hat{p} + \frac{\nu^2}{L^3}\hat{\Delta}\hat{\mathbf{u}} - \alpha\delta T\hat{T}\mathbf{g} \\ \frac{\nu}{L^2}\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\nu}{L^2}\hat{\mathbf{u}}.\hat{\nabla}\hat{T} &= \frac{\kappa}{L^2}\hat{\Delta}\hat{T}. \end{aligned} \tag{12}$$

where $\hat{\nabla}$ and $\hat{\Delta}$ are spatial derivatives with respect to the dimensionless space coordinates $\hat{\mathbf{r}}$. Multiplying the momentum equation by L^3/ν and the heat transport equation by L^2/ν, we get the dimensionless expression of the governing equations :

$$\begin{aligned} \frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\mathbf{u}}.\hat{\nabla}\hat{\mathbf{u}} &= -\hat{\nabla}\hat{p} + \hat{\Delta}\hat{\mathbf{u}} - \frac{\alpha\delta T\mathbf{g}L^3}{\nu^2}\hat{T} \\ \frac{\partial \hat{T}}{\partial \hat{t}} + \hat{\mathbf{u}}.\hat{\nabla}\hat{T} &= \frac{\kappa}{\nu}\hat{\Delta}\hat{T}. \end{aligned} \tag{13}$$

The system of equation involves only two dimensionless parameters, the Grashof number $\alpha\delta T g L^3/\nu^2$ and the Prandtl number ν/κ. Thermal convection problems, in the Boussinesq approximation can be characterized by these two dimensionless numbers and their product, the Rayleigh number. The Reynolds number of the flow does not appear in the system of equations because of the choice we made for the velocity scale. But in reality, the viscous and inertia terms in the dimensional momentum equation might be of different orders of magnitude. In particular, if the Reynolds number is very large compared to 1, we expect the flow to become unstable and turbulent.

4 Thermal boundary layers.

The solution derived above for the convection in a narrow box assumes that there is no feedback of the buoyancy driven flow on the the transport of heat. So we expect this solution to be valid in the limit of small Peclet (or Rayleigh) numbers. To test this, we can perform numerical solutions of the problem using a finite element program such as FreeFem. The results are shown on fig. 4 (temperature field and flow streamlines) and on fig. 5 (horizontal profiles of velocity and temperature in the plane at $x = H/2$).

When the Rayleigh number is equal to 10, the temperature profile remains linear and the velocity field at midheight of the box shows a cubic profile as predicted. When the Rayleigh number increases, the layer of fluid moving upwards or downwards becomes thinner and thinner and the temperature gradient is localized at the side walls. The flow can then be described as two transport boundary layers, one transporting hot fluid on the right and the other transporting cold fluid on the left. When the Rayleigh number exceeds a critical value, the flow becomes unsteady.

5 Heat transport through the boundary layers. Nusselt number

When the flow can be reduced essentially to two thin boundary layers on each side (this is the case at $Ra = 1000$), it is possible to use the difference in lengthscales in the x direction (height of

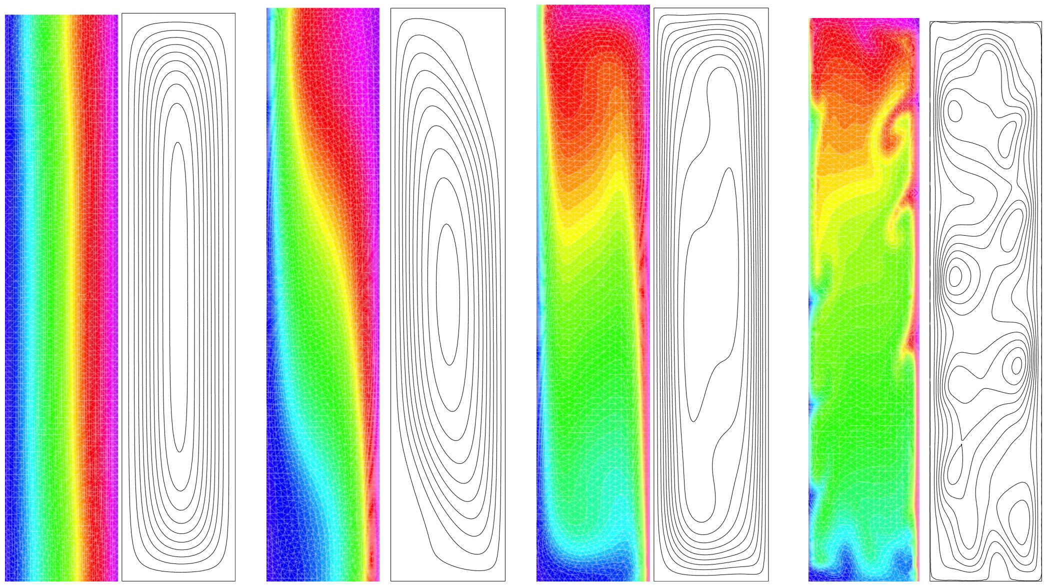

FIGURE 4 – Thermal convection within a high aspect ratio rectangular cavity. Different temperatures are imposed on the right (warmer) and left (colder) sides. The bottom and top walls are thermally insulating (zero heat flux through these walls). The temperature field is shown with false colors (red is hot, blue is cold) and the velocity field is shown as streamlines. From left to right increasing Rayleigh numbers : $Re = 100$, $Pr = 0.1$, $Ra = 10$; $Re = 1000$, $Pr = 0.1$, $Ra = 100$; $Re = 1000$, $Pr = 1$, $Ra = 1000$; $Re = 1000$, $Pr = 10$, $Ra = 10000$.

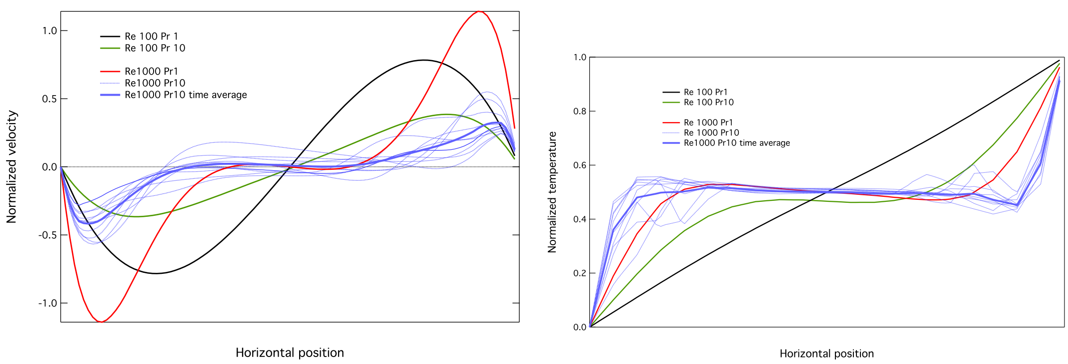

FIGURE 5 – Horizontal profiles of velocity and temperature in the plane at $x = H/2$, corresponding to the fields shown on fig. 4

the box H) and in the y direction (thickness of the momentum and temperature boundary layers δ_ν and δ_T), with δ_ν, $\delta_T \ll H$) to derive scaling laws for the flux of heat towards the side walls. We follow the same type of reasoning used for transport boundary layers when the flow is imposed independently of heat or mass transport.

In the boundary layer approximation, derivatives across the boundary layer (∂_y here) are much larger than derivatives along the boundary layer (∂_x here) and the transport equation is :

$$u_x(y)\frac{\partial T}{\partial x} = \kappa\frac{\partial^2 T}{\partial y^2}. \tag{14}$$

We can find an approximate solution for the velocity profile by assuming that the flow ends at a distance δ_ν from the wall. We have then the following boundary conditions to solve the momentum equation, using now a y coordinate which the *distance from the wall* : $u_x(y = 0) = 0$ (no slip at the wall), $u_x(y = \delta_\nu) = 0$ (no flow outside the boundary layer, $\partial_y u_x(y = \delta_\nu) = 0$ (no shear stress at the edge of the boundary layer) and from the momentum equation $\partial_{yy} u_x(y = 0) = \alpha g\theta/\nu$. Looking for a cubic velocity profile satisfying these boundary conditions we get :

$$u_x(y) = \frac{\alpha g\theta\delta_\nu^2}{4\nu}\left[\frac{y}{\delta_\nu} - 2\left(\frac{y}{\delta_\nu}\right)^2 + \left(\frac{y}{\delta_\nu}\right)^3\right] \tag{15}$$

Since the momentum boundary layer is directly driven by the temperature boundary layer we can assume that δ_ν and δ_T have the same order of magnitude and we can write the following approximate expression for the transport equation :

$$u_x(\delta_T)\frac{\theta}{x} \approx \frac{\alpha g\theta\delta_T^2}{\nu}\frac{\theta}{x} \approx \kappa\frac{\theta}{\delta_T^2}. \tag{16}$$

This leads to the scaling for $\delta_T(x)$:

$$\delta_T(x) \sim \left(\frac{\nu\kappa x}{\alpha g\theta}\right)^{1/4} = H\left(\frac{\nu\kappa}{\alpha g\theta H^3}\right)^{1/4}\left(\frac{x}{H}\right)^{1/4} = H Ra_H^{-1/4}\left(\frac{x}{H}\right)^{1/4} \tag{17}$$

where we introduced Ra_H,a Rayleigh number based on the height of the convection box. We can then compute the heat flux towards the wall, making the approximation $\partial_y T \approx \theta/\delta_T$:

$$J_{y=0} = -\lambda\int_0^H \frac{\partial T}{\partial y}\, dx \approx \lambda\int_0^H \frac{\theta}{\delta_T(x)}\, dx \propto \lambda\theta Ra_H^{1/4} \tag{18}$$

Without convection, the diffusive heat flux integrated over the length H would be in order of magnitude $J_D\lambda\theta/H \times H = \lambda\theta$. The ratio between the effective heat flux and the purely diffusive heat flux is the Nusselt number Nu which evaluates the efficiency of the transport enhancement by convection :

$$Nu = \frac{J_{y=0}}{J_D} = Ra_H^{1/4} \tag{19}$$

This scaling relation is observed in experiments as shown on fig. 6. However, when the Rayleigh number is too large (typically beyond 10^9), the flow becomes turbulent and a different scaling is observed with a larger exponent. The turbulent fluctuations increase the efficiency of heat transport towards the heated vertical plate.

Depending on the particular geometry of the flow, the correlations between the Nusselt number and the Rayleigh number will change and one should refer to the proper curve to determine the heat flux. Such an example of $Nu = f(Ra)$ correlation is shown on fig. 7 for the case of an horizontal cylinder (like the middle image of fig. 1). At large Rayleigh numbers (larger than 10^4), Nu varies as $Ra^{1/4}$ similarly to the case of the plane vertical wall, but for smaller values, Nu does not vary as a simple power law of Ra.

Generally the way to proceed in engineering applications, is to identify the geometry of the problem, find the appropriate Nu/Ra correlation, estimate the value of the Rayleigh number from the physical parameters and deduce the value of the Nusselt number.

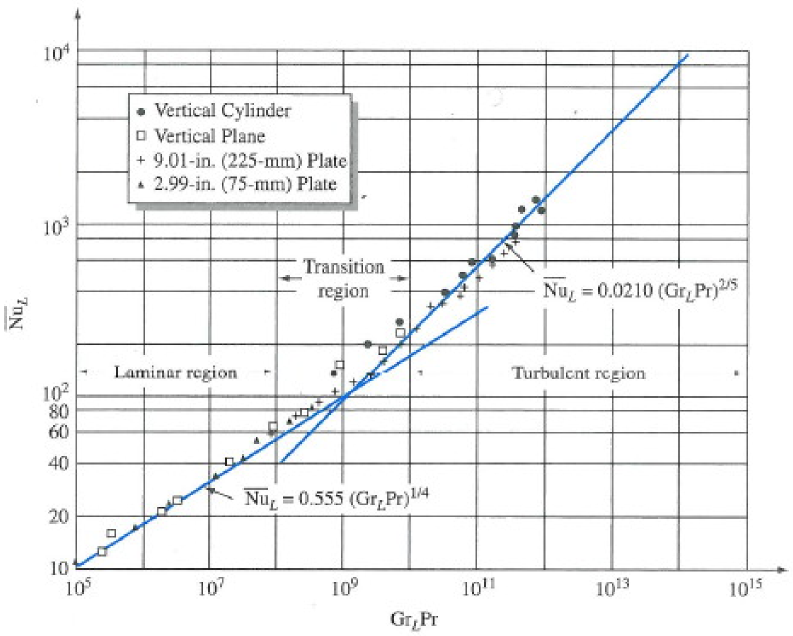

FIGURE 6 – Nusselt number (dimensionless heat flux) as a function of the Rayleigh number for thermal convection along a vertical plate or from an vertical cylinder. Fig. reproduced from Kreith, Manglik et Bohn, "Principles of heat transfer"

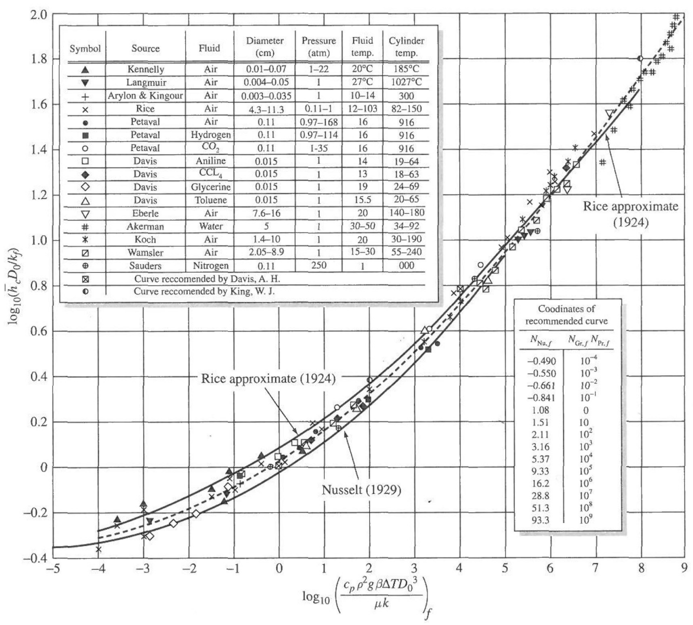

FIGURE 7 – Nusselt number as a function of the Rayleigh number for thermal convection from an horizontal cylinder. Fig. reproduced from Kreith, Manglik et Bohn, "Principles of heat transfer"

6 Summary

- the coupled momentum and heat transfer equations can be simplified by using Boussinesq approximation and retaining only the buoyancy term as coupling between the two equations
- the relevant dimensionless parameters for a thermal convection problem are the Reynolds, Grashof and Prandtl numbers. In some cases the Rayleigh number (product of Grashof and Prandtl) alone determines the properties of the flow
- the effective heat transfer rate is defined by the Nusselt number which is in general a function of Re,Gr and Pr.
- for boundary layer type flows a scaling analysis provides the correlation between Nu and the other dimensionless numbers
- in the general case, one should look for correlations between Nu and Ra appropriate to the problem at hand, estimate Ra and deduce the heat transfer rate